Homogenization of elastic structures leading to second gradient models

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The goal

Designing second gradient materials

- Second gradient materials (or strain gradient materials) : their elastic energy depends on the second gradient of the displacement

$$E(u) = \int_\Omega W(\nabla\nabla u) = \int_\Omega \tilde{W}(\nabla(e(u)))$$

  - very simple from the variationnal point of view.

    Equilibrium equations, natural boundary conditions.
  - controversial from the physical point of view.

    Compatibility with second principle of thermodynamics ? with continuum mechanics theory ? : the point is that boundary actions which go beyond a simple surface density of forces are hardly "understood".
  - Well known in the case of beams, plates, shells but the general belief is that low dimension is essential. Boundary actions are density of torques.
  - A few rigorous homogenization results lead to second gradient limit models (Pideri & P.S. 1997, Bellieud & Bouchitt'e 2002, Briane & Camar-Eddine 2007, Alibert & Della Corte 2015) Matrix reinforced with hard thin cylinders or hard thin plates.
  - Almost all the few known results give energies involving only the part $\nabla(\nabla^{skew} u)$ of $\nabla(\nabla u)$. ("incomplete" second gradient models)

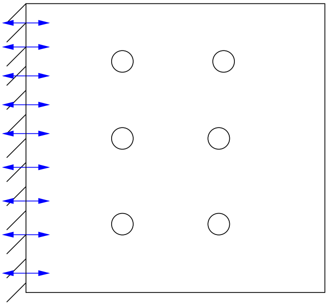
    In which case boundary actions are density of torques.

The goal

Why are complete second gradient material so special ?

Assume an energy $E(u) = \int_\Omega (\partial_1 \partial_1 u_1)^2$, no applied bulk forces. We just change the boundary conditions on the left hand side

$$\text{from} \quad u = 0, \; \partial_1 u_1 = 0$$



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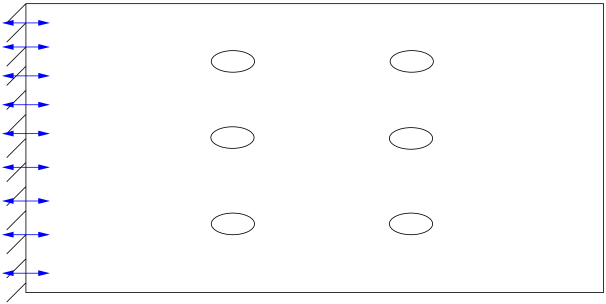
$$\text{to} \quad u = 0,\ \partial_1 u_1 = 1$$



The goal

The context

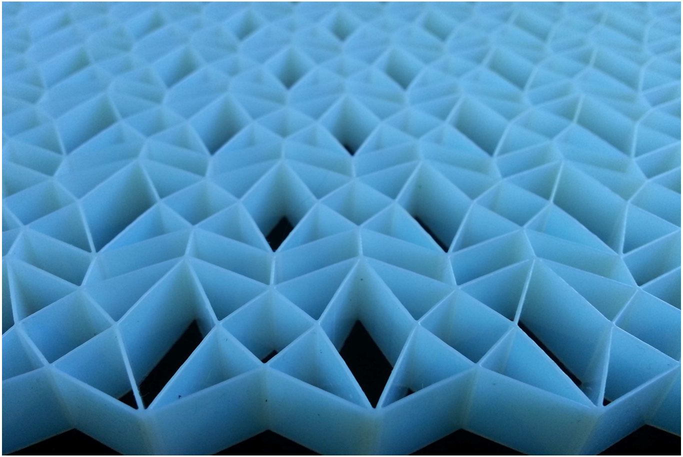
- Many researchers use the model which has the advantage to regularize singularities (interfaces, plasticity, fractures, . . .)
- Nobody has directly measured second gradient effects nor design materials with specific second gradient properties nor experimented the associated special boundary conditions. Do these materials actually exist ?
- A closure theorem (Camar-Eddine and P.S. 2003) states that any quadratic l.s.c. and objective functional can be obtained from classical linear elasticity through an homogenization process.

  A very indirect method. The result is quite different when homogenizing conductivity problems (Camar-Eddine et P.S. 2002).

Thin walled structures

The printed structure

We want to describe the homogenized elastic behavior of this structure



Thin walled structures

Assumptions

- Homogeneous linear elastic material (no micro buckling effect).
- Plane strain elasticity conditions : the problem reduces to a linear elastic problem set in a thickened periodic planar graph.
- The size ℓ of the period of the graph is small compared to the characteristic size L of the domain Ω (standard asymptotic homogenization assumption) $\varepsilon = \ell / L << 1$.
- The thickness e of the walls is small compared with ℓ : $\delta = e/\ell << 1$. In the sequel $\delta = \varepsilon^\alpha$ with $\alpha > 1$. (actually $\alpha = 1$ is the most interesting case).
- Lamé coefficients (μ, λ) of the material tend to infinity like $\delta^{-1}\varepsilon^{-2}$:

$$\mu = \frac{\mu_0}{\delta\varepsilon^2}, \qquad \lambda = \frac{\lambda_0}{\delta\varepsilon^2}$$

- The material is fixed on some part of the domain to ensure some coercivity.

Thin walled structures

Geometry

- a prototype cell Y made of K nodes at points y_s, $s \in \{1, \dots, K\}$,
- two vectors t_1, t_2 for translating the cell.

Notation : $y_{l,s} = y_s + it_1 + jt_2$, $l = (i,j) \in \mathbb{N}^2$. $\mathcal{I}_\varepsilon := \{(i,j) : \forall (p,s), \, \varepsilon y_{l+p,s} \in \Omega\}$.

- five matrices a^p defining the edges between the nodes of cells Y_l and Y_{l+p}, $p \in \mathcal{P} := \{(0,0),(0,1),(1,0),(1,1),(1,-1)\}$. An edge links $y_{l,s}$ and $y_{l+p,s'}$ if $a^p_{s,s'} > 0$.

Notation : $\mathbf{p} = p_1 t_1 + p_2 t_2$, $\ell_{p,s,s'} := \|y_{l+p,s'} - y_{l,s}\|$, $\tau_{p,s,s'} := (y_{l+p,s'} - y_{l,s})/\ell_{p,s,s'}$.

$$\mathcal{A} := \{(p,s,s'); \; p \in \mathcal{P}, \; 1 \le s \le K, \; 1 \le s' \le K, \; a^p_{s,s'} > 0\}$$

$$G := \bigcup_{(l,p,s,s') \in \mathcal{I}_\varepsilon \times \mathcal{A}} [y_{l,s}, y_{l+p,s'}]$$

- Considered 2D domain : $G_\varepsilon := \left\{ x \in \mathbb{R}^2; \; d(\frac{x}{\varepsilon}, G) < \delta \right\}$.

Examples of periodic trusses

Example 1 : the most standard truss

Geometry : $K = 1$ node, $t_1 = (1,0)$, $t_2 = (-0.5, 0.866)$, $a^{(0,0)} = (-)$, $a^{(1,0)} = (1)$, $a^{(0,1)} = (1)$, $a^{(1,1)} = (1)$, $a^{(1,-1)} = (0)$.

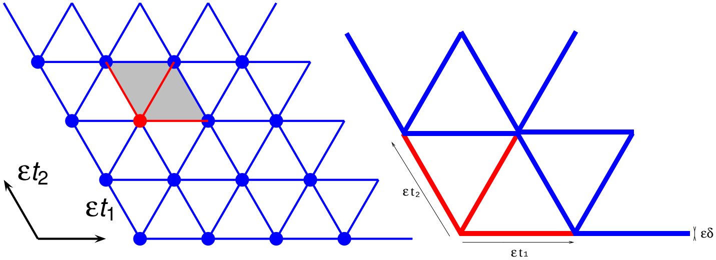


FIGURE – Regular triangle truss

Examples of periodic trusses

Example 2 : A degenerated truss

We just modify $t_2 = (0, 1)$, $a^{(1,1)} = (0)$ in the previous example.

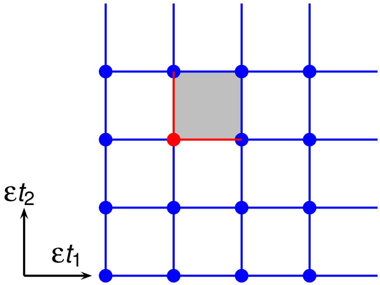


FIGURE – The regular square truss and its free shear deformation

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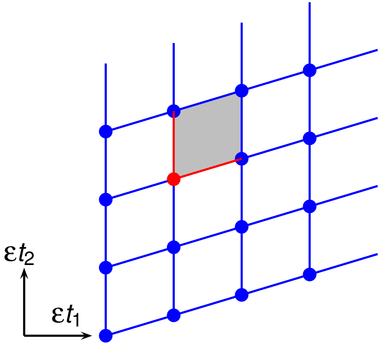


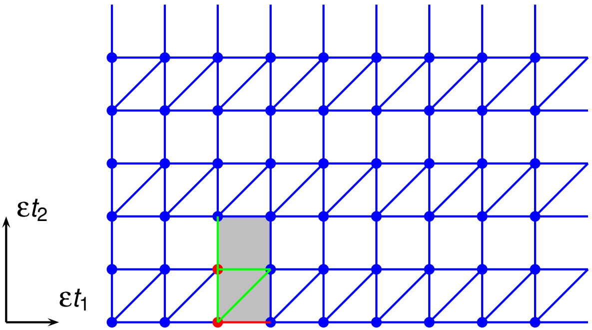
FIGURE – The regular square truss and its free shear deformation

Examples of periodic trusses

Example 3 : another structure with free shear

Geometry : $K = 2$ nodes, $t_1 = (1,0)$, $t_2 = (0,2)$, $a^{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$a^{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $a^{(0,1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $a^{(1,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a^{(1,-1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.



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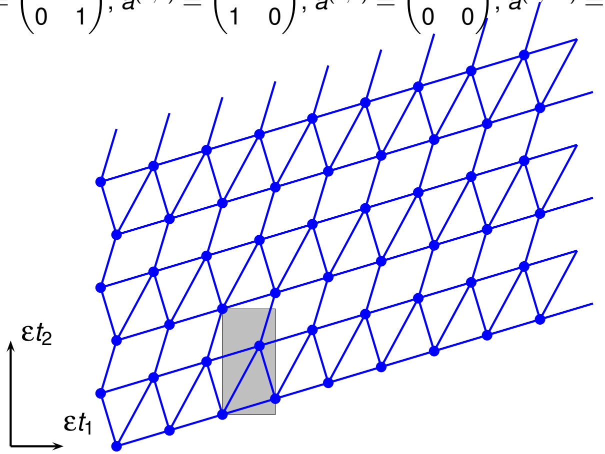

Examples of periodic trusses

Example 4 : the truss we have printed

Geometry : $K = 6$ nodes, $t_1 = (4,0)$, $t_2 = (-2,4)$, ...

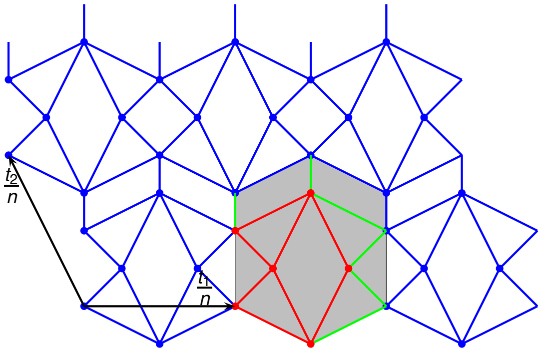

FIGURE – Pantographic truss

Examples of periodic trusses

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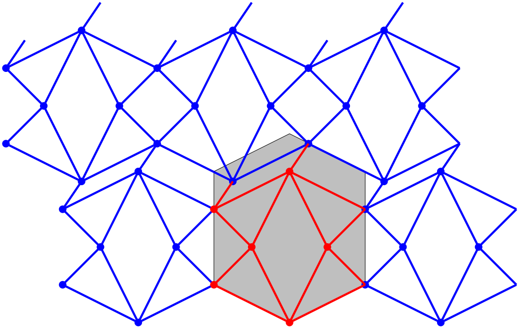

FIGURE – Free shear of the pantographic truss

Examples of periodic trusses

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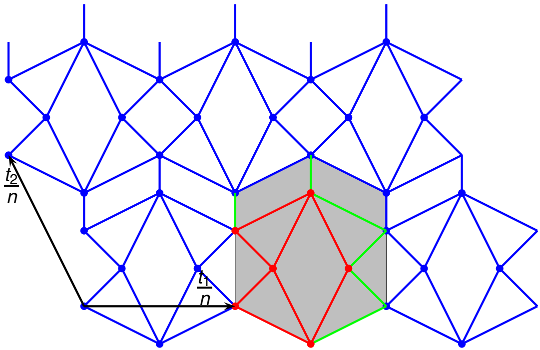

FIGURE – Pantographic truss

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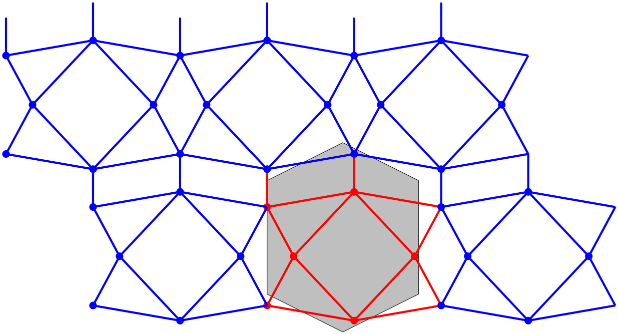


FIGURE – Free horizontal dilatation of the pantographic truss

Reduction to a discrete problem

Energies

Initial energy :

$$\boxed{\mathscr{E}_\varepsilon(u) := \int_{\Omega_\varepsilon} \left(\frac{\mu_0}{\delta\varepsilon^2} \|e(u)\|^2 + \frac{\lambda_0}{2\delta\varepsilon^2} \mathrm{tr}(e(u))^2 \right) dx}$$

($\mu_0 > 0,\ \lambda_0 + \mu_0 > 0,\ Y_0 := \frac{4\mu_0(\mu_0+\lambda_0)}{2\mu_0+\lambda_0}$, $\nu := \nu_0 := \frac{\lambda_0}{2\mu_0+\lambda_0}$). We fix

$$(a^p)_{s,s'} = \frac{2Y_0}{\ell_{p,s,s'}}.$$

and to any discrete vector field U, we associate the discrete energy

$$\boxed{E_\varepsilon(U) := \frac{1}{\varepsilon^2} \sum_{(I,p,s,s') \in \mathcal{I}_\varepsilon \times \mathcal{A}} \frac{a^p_{s,s'}}{2} ((U_{I+p,s'} - U_{I,s}) \cdot \tau_{p,s,s'})^2}$$

Reduction to a discrete problem

Convergences

- We say that a sequence of families of vectors (Z^ε) converges to z ($Z^\varepsilon \rightharpoonup z$), when

$$\varepsilon^2 \sum_{I \in \mathcal{I}_\varepsilon} \frac{1}{K} \sum_{s=1}^{K} Z^\varepsilon_{I,s} \delta_{\varepsilon y_{I,s}} \overset{*}{\rightharpoonup} z(x)\, dx$$

- We say that a sequence of families of vectors (Z^ε) double-scale converges to z ($Z^\varepsilon \rightharpoonup\rightharpoonup z$), when

$$\forall s \in \{1, \dots, K\}, \quad \varepsilon^2 \sum_{I \in \mathcal{I}_\varepsilon} Z^\varepsilon_{I,s} \delta_{\varepsilon y_{I,s}} \overset{*}{\rightharpoonup} z(x,s)\, dx$$

- To any field $u \in \mathrm{L}^2(\Omega_\varepsilon; \mathbb{R}^2)$ we associate the family

$$\bar{u}_{I,s} := \fint_{B(\varepsilon y_{I,s}, \varepsilon\delta)} u(x)\, dx$$

We say that the sequence of functions (u^ε) converges to u when $\bar{u}^\varepsilon \rightharpoonup u$.

Reduction to a discrete problem

Result

Theorem

The Γ-limits of $(\mathscr{E}_\varepsilon)$ and (E_ε) are identical : for any u, we have

$$(i) \quad \inf\{\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u^\varepsilon) \ : \ u^\varepsilon \rightharpoonup u\} \geq \inf\{\liminf_{\varepsilon\to 0} E_\varepsilon(U^\varepsilon); \ U^\varepsilon \rightharpoonup u\}$$

$$(ii) \quad \inf\{\limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u^\varepsilon) \ : \ u^\varepsilon \rightharpoonup u\} \leq \inf\{\limsup_{\varepsilon\to 0} E_\varepsilon(U^\varepsilon); \ U^\varepsilon \rightharpoonup u\}$$

From now on we focus on (E_ε)

Study of the discrete problem

Double-scale limits

We set

$$m_I := \sum_{s=1}^{K} U_{I,s}, \quad v_{I,s} := \frac{1}{\varepsilon}(U_{I,s} - m_I) \quad \text{and} \quad \chi_{p,I} := \frac{1}{\varepsilon}(m_{I+p} - m_I).$$

So that

$$E_\varepsilon(U) = \bar{E}_\varepsilon(v,\chi) := \sum_{(I,p,s,s')\in \mathcal{I}_\varepsilon \times \mathcal{A}} \frac{a^p_{s,s'}}{2}((v_{I+p,s'} - v_{I,s} + \chi_{I,p}) \cdot \tau_{p,s,s'})^2$$

We assume that the sequences

$$\varepsilon^2 \sum_{I\in \mathcal{I}_\varepsilon} \|m^\varepsilon_I\|^2, \qquad \varepsilon^2 \sum_{I\in \mathcal{I}_\varepsilon} \|v^\varepsilon_{I,s}\|^2 \quad \text{and} \quad \varepsilon^2 \sum_{I\in \mathcal{I}_\varepsilon} \|\chi^\varepsilon_{I,p}\|^2$$

are bounded. Thus there exist u, v and χ in L^2 such that, up to a sub-sequence,

$$m^\varepsilon \rightharpoonup u, \qquad v^\varepsilon \rightharpoonup\rightharpoonup v \quad \text{and} \qquad \chi^\varepsilon_p \rightharpoonup \chi_p.$$

Study of the discrete problem

Double-scale limits

We also have

$$U^{\varepsilon} \rightharpoonup\rightharpoonup u, \qquad \sum_{s=1}^{K} v(x,s) = 0 \quad \text{and} \qquad \chi_p = \nabla u \cdot \mathbf{p}$$

To check that $\chi_p = \nabla u \cdot \mathbf{p}$, it is enough to notice that, for any smooth test field φ,

$$\int_\Omega \chi_p(x)\cdot\varphi(x) = \lim \varepsilon^2 \sum_{l \in I_\varepsilon} \varepsilon^{-1}(m^\varepsilon_{l+p} - m^\varepsilon_l)\cdot\varphi(\varepsilon y_l) = \lim \varepsilon^2 \sum_{l\in I_\varepsilon} m^\varepsilon_l \cdot \varepsilon^{-1}(\varphi(\varepsilon y_{l-p}) - \varphi(\varepsilon y_l))$$

$$= \lim \varepsilon^2 \sum_{l \in I_\varepsilon} m^\varepsilon_l \cdot (-\nabla\varphi(\varepsilon y_l)\cdot\mathbf{p}) + O(\varepsilon) = -\int_\Omega u(x)\cdot(\nabla\varphi(x)\cdot\mathbf{p}) = \int_\Omega (\nabla u(x)\cdot\mathbf{p})\cdot\varphi(x).$$

Study of the discrete problem

First homogenization result

$$\varepsilon^2 \bar{E}_\varepsilon(v^\varepsilon,\chi^\varepsilon) := \varepsilon^2 \sum_{l\in I_\varepsilon} \sum_{(p,s,s')\in\mathcal{A}} \frac{a^p_{s,s'}}{2}((v^\varepsilon_{l+p,s'} - v^\varepsilon_{l,s} + \chi^\varepsilon_{l,p})\cdot\tau_{p,s,s'})^2$$

$$\bar{E}(v,\eta_u) := \int_\Omega \sum_{(p,s,s')\in\mathcal{A}} \left(\frac{a^p_{s,s'}}{2}((v_{s'}(x) - v_s(x) + (\eta_u)_{p,s'}(x))\cdot\tau_{p,s,s'})^2\right) dx$$

with $(\eta_u)_{p,s}(x) := \nabla u(x)\cdot\mathbf{p}$.

Theorem

The sequence $(\varepsilon^2 E_\varepsilon)$ Γ*-converges to the function* $E := \inf_v \bar{E}(v,\eta_u)$ *:*

(i) For all sequence U^ε *such that* $U^\varepsilon \rightharpoonup u$, $\liminf \varepsilon^2 E_\varepsilon(U^\varepsilon) \geq E(u)$.

(ii) $\forall u\in \mathrm{L}^2(\Omega)$, $\exists U^\varepsilon$ *such that* $U^\varepsilon \rightharpoonup u$ *and* $\limsup \varepsilon^2 E_\varepsilon(U^\varepsilon) \leq E(u)$.

Study of the discrete problem

Main homogenization result

Consider a sequence such that $\bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon) < M$.

- Previous theorem implies $\bar{E}(v, \eta_u) = 0$ and so

$$(v_{s'}(x) - v_s(x) + \nabla u \cdot \mathbf{p}) \cdot \tau_{p,s,s'} = 0.$$

- $\omega^\varepsilon_{l,p,s,s'} := \varepsilon^{-2}(U^\varepsilon_{l+p,s'} - U^\varepsilon_{l,s}) \cdot \tau_{p,s,s'} \rightharpoonup \omega_{p,s,s'}$.
- Set $\mathcal{D}_A := \left\{(w_{s'} - w_s + \nabla\lambda \cdot p) \cdot \tau_{p,s,s'} :\ w_s, \lambda \in \mathrm{L}^2(\mathbb{R}^2, \mathbb{R}^2)\right\}$
  $\phi \in \mathcal{D}_A{}^\perp$ means

$$\sum_{(p,s,s')\in\mathcal{A}} (\nabla\phi_{p,s,s'} \cdot \mathbf{p})\,\tau_{p,s,s'} = 0$$

and

$$\sum_{(p,s,s')\in\mathcal{P}\times\{1,\ldots,K\}^2} \tau_{p,s,s'}\,\phi_{p,s,s'} - \tau_{p,s',s}\,\phi_{p,s',s} = 0.$$

Study of the discrete problem

Main homogenization result

For such functions we have

$$\int_\Omega \sum_{(p,s,s')} \omega_{p,s,s'}(x)\phi_{p,s,s'}(x) = \lim \varepsilon^2 \sum_{(l,p,s,s')} \varepsilon^{-2}(U^\varepsilon_{l+p,s'} - U^\varepsilon_{l,s}) \cdot (\phi_{p,s,s'}(\varepsilon y_l)\tau_{p,s,s'})$$

$$= \lim \varepsilon^2 \sum_{(l,p,s,s')} (\varepsilon^{-1}((v^\varepsilon_{l+p,s'} - v^\varepsilon_{l,s'}) + \cancel{(v^\varepsilon_{l,s'} - v^\varepsilon_{l,s})}) + \varepsilon^{-2}(m^\varepsilon_{l+p} - m^\varepsilon_l)) \cdot (\phi_{p,s,s'}(\varepsilon y_l)\tau_{p,s,s'})$$

Assuming, moreover, that they are smooth we get for the first addend :

$$\lim \varepsilon^2 \sum_{(l,p,s,s')} \varepsilon^{-1}(v^\varepsilon_{l+p,s'} - v^\varepsilon_{l,s'}) \cdot (\phi_{p,s,s'}(\varepsilon y_l)\tau_{p,s,s'})$$

$$= \lim \varepsilon^2 \sum_{(l,p,s,s')} v^\varepsilon_{l,s'} \cdot (\varepsilon^{-1}(\phi_{p,s,s'}(\varepsilon y_{l-p}) - \phi_{p,s,s'}(\varepsilon y_l))\tau_{p,s,s'})$$

$$= \lim \varepsilon^2 \sum_{(l,p,s,s')} v^\varepsilon_{l,s'} \cdot ((-\nabla\phi_{p,s,s'}(\varepsilon y_l) \cdot \mathbf{p})\tau_{p,s,s'}) + O(\varepsilon)$$

$$= -\int_\Omega \sum_{(p,s,s')} v_{s'}(x) \cdot ((\nabla\phi_{p,s,s'}(x) \cdot \mathbf{p})\tau_{p,s,s'})$$

$$= \left\langle \sum_{(p,s,s')} (\nabla v_{s'}(x) \cdot \mathbf{p}) \cdot (\phi_{p,s,s'}(x)\tau_{p,s,s'}) \right\rangle,$$

Study of the discrete problem

Main homogenization result

and for the second addend

$$\lim \varepsilon^2 \sum_{(l,p,s,s')} (\varepsilon^{-2}(m^{\varepsilon}_{l+p} - m^{\varepsilon}_{l})) \cdot (\phi_{p,s,s'}(\varepsilon y_l)\tau_{p,s,s'})$$

$$= \lim \varepsilon^2 \sum_{(l,p,s,s')} \varepsilon^{-2} m^{\varepsilon}_{l} \cdot ((\phi_{p,s,s'}(\varepsilon y_{l-p}) - \phi_{p,s,s'}(\varepsilon y_l))\tau_{p,s,s'})$$

$$= \lim \varepsilon^2 \sum_{(l,p,s,s')} m^{\varepsilon}_{l} \cdot ((\cancel{-\varepsilon^{-1}\nabla\phi_{p,s,s'}(\varepsilon y_l) \cdot \mathbf{p}} + \frac{1}{2}\nabla\nabla\phi_{p,s,s'}(\varepsilon y_l) \cdot \mathbf{p} \cdot \mathbf{p})\tau_{p,s,s'}) + O(\varepsilon)$$

$$= \int_{\Omega} \sum_{(p,s,s')} u(x) \cdot ((\frac{1}{2}\nabla\nabla\phi_{p,s,s'}(x) \cdot \mathbf{p} \cdot \mathbf{p})\tau_{p,s,s'})$$

$$= \left\langle \sum_{(p,s,s')} \frac{1}{2}(\nabla\nabla u(x) \cdot \mathbf{p} \cdot \mathbf{p}) \cdot (\phi_{p,s,s'}(x)\tau_{p,s,s'}) \right\rangle .$$

Collecting these results we obtain that the distribution

$$\omega_{p,s,s'}(x) - \left(\nabla v_{k'}(x) \cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u \cdot \mathbf{p} \cdot \mathbf{p}\right) \cdot \tau_{p,s,s'}$$

is orthogonal to all smooth functions in ${\mathcal{D}_A}^{\perp}$.

Study of the discrete problem

Main homogenization result

There exist some fields w_s and λ in $\mathrm{L}^2(\mathbb{R}^2)$ such that, for any $(p,s,s') \in \mathcal{A}$,

$$\omega_{p,s,s'}(x) - \left(\nabla v_{s'}(x)\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p}\right)\cdot\tau_{p,s,s'} = \left(\nabla\lambda(x)\cdot\mathbf{p} + w_{s'}(x) - w_s(x)\right)\cdot\tau_{p,s,s'}$$

$$\omega_{p,s,s'}(x) = \left(\nabla(v_{s'}+\lambda)(x)\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p} + w_{s'}(x) - w_s(x)\right)\cdot\tau_{p,s,s'}.$$

$$\liminf E_\varepsilon(U) = \liminf \varepsilon^2 \sum_{(l,p,s,s')} (\omega^\varepsilon_{l,p,s,s'})^2 \geq \int_\Omega (\omega_{p,s,s'}(x))^2 dx = \bar{E}(w, \xi_{u,v+\lambda})$$

with

$$(\xi_{u,v})_{p,s} = \nabla v_s\cdot\mathbf{p} + \frac{1}{2}\nabla\nabla u\cdot\mathbf{p}\cdot\mathbf{p}.$$

Study of the discrete problem

Main homogenization result

$$\boxed{\mathscr{E}(u) := \inf_{w,v}\left\{\bar{E}(w,\xi_{u,v}); \bar{E}(v,\eta_u) = 0\right\}}$$

Theorem

E_ε Γ-converges to $\mathscr{E}$:
(i) For all sequence (U^ε) such that $U^\varepsilon \rightharpoonup u$, $\liminf E_\varepsilon(U^\varepsilon) \geq \mathscr{E}(u)$.
(ii) $\forall u$, there exists a sequence $U^\varepsilon \rightharpoonup u$ and $\limsup E_\varepsilon(U^\varepsilon) \leq \mathscr{E}(u)$.

The approximating sequence is built by setting

$$U^\varepsilon_{l,s} := u(\varepsilon y_l) + \varepsilon v_s(\varepsilon y_l) + \varepsilon^2 w_s(\varepsilon y_l)$$

where (v,w) are such that $\mathscr{E}(u) = \bar{E}(w,\xi_{u,v})$ and $\bar{E}(v,\eta_u) = 0$.

Study of the discrete problem

Explicit computation of the limit energy

We can easily rewrite the integrand of $\bar{E}$:

$$\sum_{p,s,s'} \frac{a^p_{s,s'}}{2}((v_{s'} - v_s + \eta_{p,s'})\cdot\tau_{p,s,s'})^2$$

under the form

$$\frac{1}{2} v \cdot A \cdot v + v \cdot B \cdot \eta + \frac{1}{2} \eta \cdot C \cdot \eta.$$

(identifying v with the $2K$-dimensional vector $(v_1, v_2, \ldots, v_K)$ and η with the $10 \times K$ vector $(\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{5,K})$)

A solution of the minimization problem $\inf_v \bar{E}(v,\eta)$ is $\bar{v} := -A^+ \cdot B \cdot \eta$ and the minimal value is $\frac{1}{2}\eta \cdot D \cdot \eta$ where $D := C - B^t \cdot A^+ \cdot B$.

Let L (identified to a $10K \times 4$ matrix) be the the linear operator such that, for any 2×2 matrix M and any (p,s), $(L \cdot M)_{p,s} = M \cdot \mathbf{p}$.

$$E(u) = \int_\Omega \frac{1}{2} \nabla u(x) \cdot L^t \cdot D \cdot L \cdot \nabla u(x)\, dx$$

$$R = L^t \cdot (C - B^t \cdot A^+ \cdot B) \cdot L.$$

Study of the discrete problem

Explicit computation of the limit energy

The minimal value of $\inf_w \bar{E}(w, \xi_{u,v})$ is again $\int_\Omega \frac{1}{2}\xi_{u,v} \cdot D \cdot \xi_{u,v}$.
Remind that v has to satisfy $\bar{E}(v, \eta_u) = 0$. Therefore $v = -A^+ \cdot B \cdot L \cdot \nabla u + \tilde{v}$ with $\tilde{v}$ in the kernel of A. ($\tilde{v} \to \mu$ in A. Abdulle, $\to \hat{u}_1$ in V. Vinoles)

$$(\xi_{u,v})_{p,s}(x) = -(A^+ \cdot B \cdot L \cdot \nabla\nabla u(x) \cdot \mathbf{p})_s + \nabla \tilde{v}_s(x) \cdot \mathbf{p} + \frac{1}{2}\nabla\nabla u(x) \cdot \mathbf{p} \cdot \mathbf{p}$$

which we can rewrite under the form $\xi_{u,v}(x) = \bar{L} \cdot \nabla\nabla u(x) + \tilde{L} \cdot \nabla \tilde{v}(x)$.
Setting $G := \bar{L}^t \cdot D \cdot \bar{L}$, $H := \bar{L}^t \cdot D \cdot \tilde{L}$, $J := \tilde{L}^t \cdot D \cdot \tilde{L}$

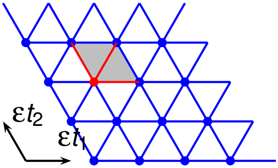
$$\frac{1}{2}\xi_{u,v} \cdot D \cdot \xi_{u,v} = \frac{1}{2}\nabla\nabla u \cdot G \cdot \nabla\nabla u + \nabla\nabla u \cdot H \cdot \nabla \tilde{v} + \frac{1}{2}\nabla \tilde{v} \cdot J \cdot \nabla \tilde{v}.$$

- Either stop there : the model is a a second gradient model coupled with an extra kinematic variable.
- Or (when possible) use $\tilde{v}$ such that $\nabla \tilde{v} = -J^+ \cdot H^t \cdot \nabla\nabla u$. In which case the model is a pure second gradient one :

$$\mathcal{E}(u) = \inf_v \int_\Omega \frac{1}{2}\nabla\nabla u \cdot \mathcal{R} \cdot \nabla\nabla u \, dx \quad \text{where } \mathcal{R} := G - H \cdot J^+ \cdot H^t.$$

Results for the different examples

The regular triangular truss



R is given in the basis corresponding to $(e_{11}, e_{22}, e_{21}, \cancel{\partial_2 u_1 - \partial_1 u_2})$ instead of $(\partial_1 u_1, \partial_2 u_1, \partial_1 u_2, \partial_2 u_2)$. We get

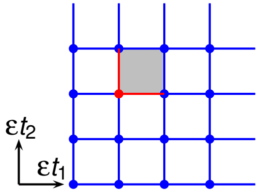
$$\check{R} = a\frac{\sqrt{3}}{2}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

($\mu = \lambda = a\sqrt{3}$ and its Poisson ratio is $\nu = 1/3$)

As E is non degenerated, only rigid motions are admissible in $\mathscr{E}$.

Results for the different examples

The square truss



For this example, our procedures lead to

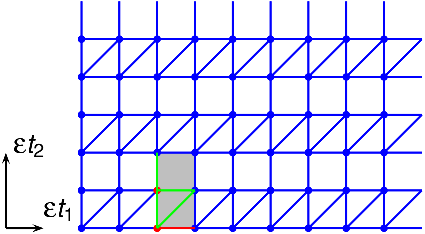
$$\check{R} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{R} = 0.$$

As expected the truss shows a degeneracy with respect to shear. As $\mathcal{R} = 0$ this truss does not present any second gradient effect.

A free global degree of mobility is not a sufficient condition for observing second gradient effects.

Results for the different examples

The second structure with free shear



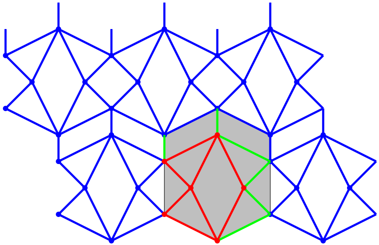
We get $\quad \check{R} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad$ and $\quad \check{\mathcal{R}} = \frac{a}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

The kernel of $\check{R}$ has dimension 1 and direction e_{12}. The matrix $\check{\mathcal{R}}$ is thus written in the basis corresponding to $(\partial_1 e_{12}, \partial_2 e_{12})$. The limit energy reads

$$\mathscr{E}(u) = \int_\Omega \frac{a}{8} (\partial_1 e_{12}(u))^2 \, dx, \;\; \text{if } e_{11}(u) = e_{22}(u) = 0$$

Examples of periodic trusses

The truss we have printed



We get $\check{R} = \frac{12}{17} a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The kernel of $\check{R}$ has dimension 2 and directions (e_{11}, e_{12}). The limit model is a coupled one, but when restricting to displacement fields of the type $u(x) = u_1(x_1)$ the energy reduces to

$$\mathscr{E}(u) = \int_\Omega 0.0656\, a (\partial^2_{11} u_1)^2$$

Conclusion

- Homogenization of our structure reduces to the homogenization of a periodic truss.
- Homogenization of trusses reduces to simple algebraic formulas.
- Homogenization may lead to second gradient effects. This happens when
  - the structure has some global degrees of mobility,
  - this needs some weak zones simulating hinges,
  - the geometry is designed in such a way that the deformation of a cell is transmitted to its neighbors.
- Results must be extended to welded grids (taking into account the flexion energy of the bars).
- First experimental result : the walls of the sample we have printed are too thick for our asymptotic result to apply. Second gradient terms are hidden by the other terms . . .
- Future : optimize the structure to get an experimental evidence ?