Effective attenuation in multiscale composite media

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- Wave equation in random media with long- or short-range correlations.
- Explain the apparent attenuation observed in seismic wave propagation.

Geophysics

- Experimental observations for body waves in the earth's crust:

the seismic attenuation factor Q^{-1} defined through the transmission coefficient by

$$|T_L(\omega)|^2 \simeq \exp\Big(-\frac{|\omega|}{c_o} Q^{-1}(\omega) L\Big)$$

has a frequency dependence of the form

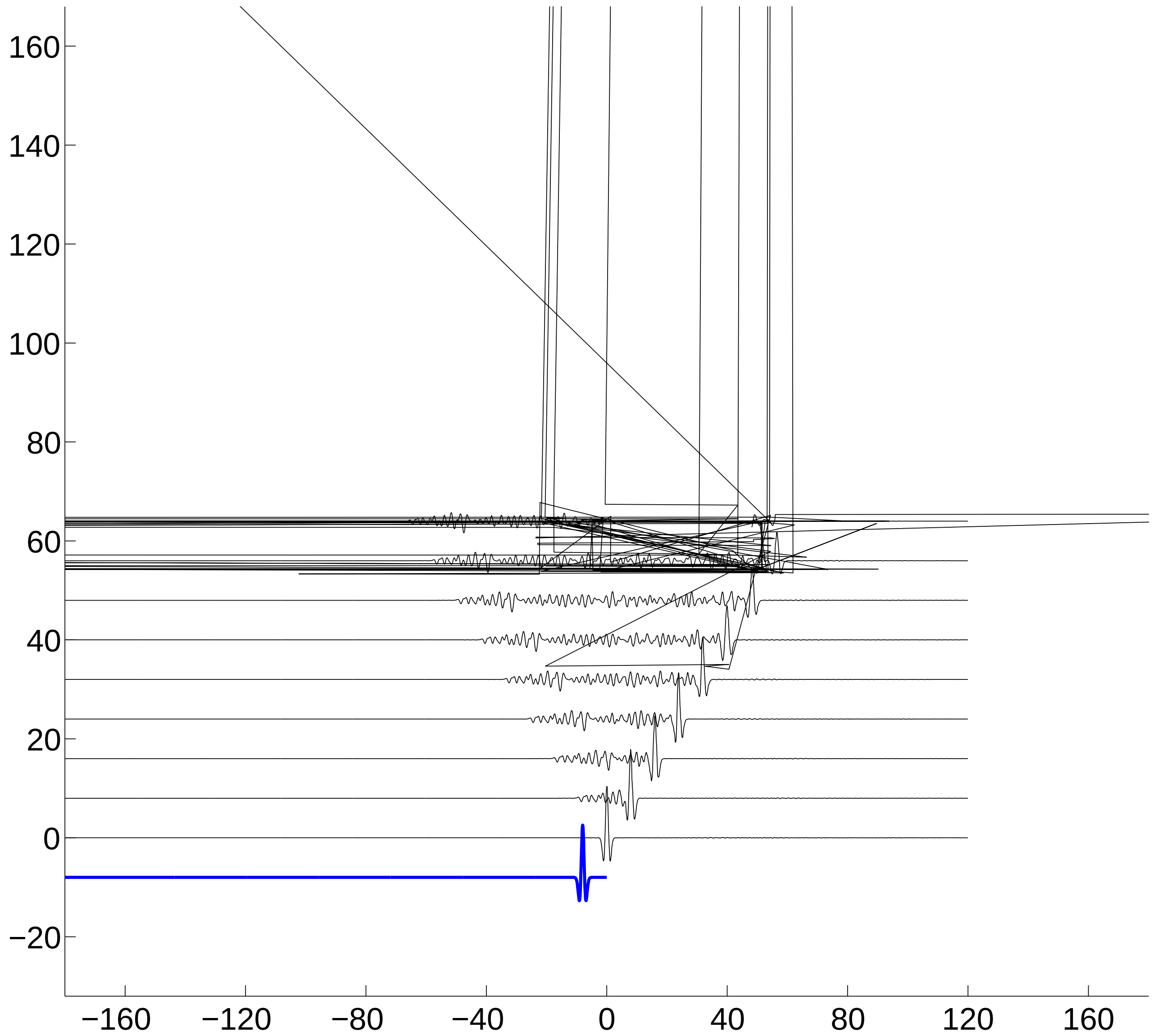
$$Q^{-1}(\omega) \approx |\omega|^\alpha$$

α is obtained through a fitting of measured data:

- $\alpha \simeq 0.[illegible]$ for low frequencies $< 10^{-3}$Hz [Lekic 2009]
- $\alpha \in (-0.[illegible], 0)$ for mid frequencies $(10^{-3}, [illegible])$Hz [Choy 1986, Shito 200[illegible]],
- $\alpha \simeq -[illegible]$ for high frequencies $> [illegible]$ Hz [Choy 1986, Cormier 20[illegible]].

- Two possible mechanisms: scattering and intrinsic attenuation (due to friction, viscosity, ...) [Knopoff 196[illegible], Jackson 1970, ..., Ricard 20[illegible], Is there seismic attenuation in the mantle? ...].

Numerical experiment in one-dimensional random medium



The mixing case

- $\phi(z) = \mathbb{E}[\nu(y)\nu(y+z)]$ decays fast enough at infinity (integrable) and is regular at zero (Lipschitz).

↪ The correlation length ℓ_c can be defined by

$$\ell_c = \frac{1}{\phi(0)} \int_0^\infty \phi(z)\, dz$$

↪ The covariance function can be expanded as

$$\phi(z) = \phi(0)\Big(1 - d_c \frac{|z|}{\ell_c} + o\Big(\frac{|z|}{\ell_c}\Big)\Big), \qquad |z| \ll \ell_c$$

- Example: Binary medium. The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.
$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.
The values n_j are independent and identically distributed (i.i.d.) with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

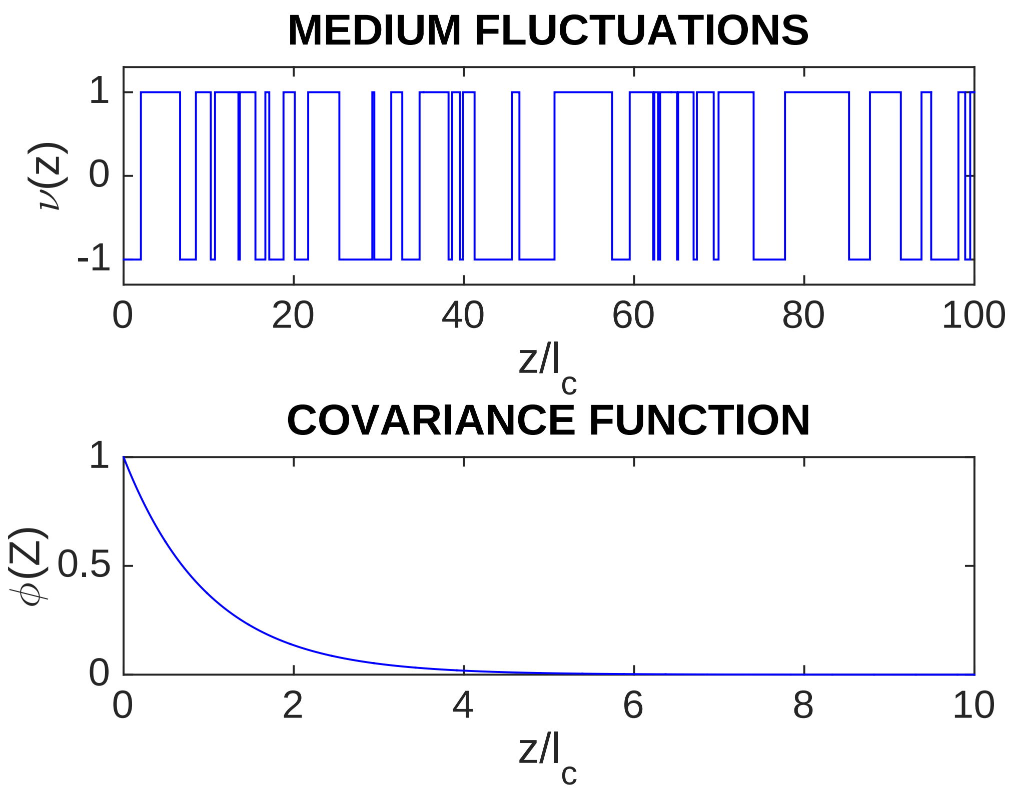
The lengths l_j are i.i.d. with the probability density function (pdf):

$$p_{l_1}(z) = \frac{1}{\ell_c} \exp\Big(-\frac{z}{\ell_c}\Big) \mathbf{1}_{[0,\infty)}(z)$$

↪ The covariance function is

$$\phi(z) = \sigma^2 \exp\Big(-\frac{|z|}{\ell_c}\Big)$$

Binary medium - mixing case



Realizations of a binary medium with exponentially distributed intervals.

Effective pulse propagation - mixing case (1/5)

Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_o} + \varepsilon^2 \tau_c^\varepsilon(L), \qquad \tau_c^\varepsilon(L) = \frac{1}{2c_o\varepsilon}\int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

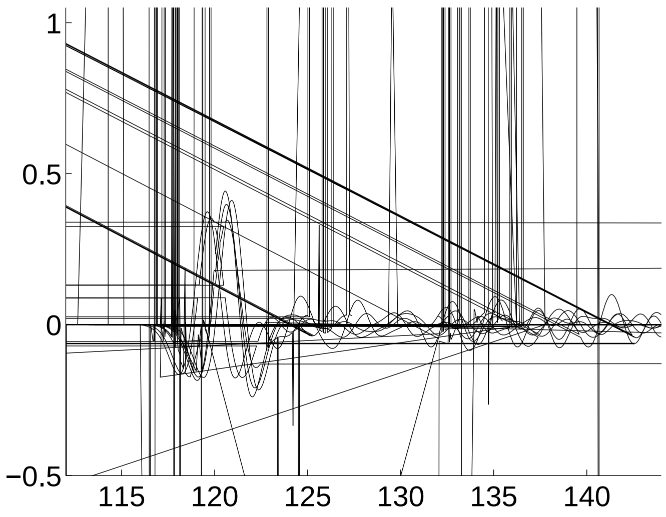
In the limit $\varepsilon \to 0$ at $z = L$:

1) the *random time shift* $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one

2) the pulse profile converges in probability to a *deterministic profile*:

$$p^\varepsilon\left(z = L, t = \tau^\varepsilon(L) + \varepsilon^2\tau\right) \xrightarrow{\varepsilon\to 0} p_L(\tau)$$

Effective pulse propagation - mixing case (2/5)



Effective pulse propagation - mixing case (3/5)

More precisely :

- the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L) \sim$

Effective pulse propagation - mixing case (4/5)

- If $\frac{\omega \ell_c}{c_o} \ll$ (the wavelength $\gg \ell_c$ probes the tail of $\phi(z)$) then:

$$T_L(\omega) \qquad \exp\Big(-\frac{\phi(0)}{8}\frac{\omega^2\ell_c^2}{c_o^2}\frac{L}{\ell_c}\Big)$$

↪ Effective second-order diffusion (attenuation), no effective dispersion.

Here α .

- Remark: What happens if $\int_0^\infty \phi(z)dz \quad \infty$?

- If $\frac{\omega \ell_c}{c_o} \gg$ (the wavelength $\ll \ell_c$ probes the small-z behavior of $\phi(z)$) then:

$$T_L(\omega) \qquad \exp\Big(-\frac{\phi(0)d_c}{6}\frac{L}{\ell_c}-i\frac{\phi(0)}{8}\frac{\omega\ell_c}{c_o}\frac{L}{\ell_c}\Big)$$

↪ Effective constant attenuation, no effective dispersion, additional time shift.

Here α $-$.

- Remark: What happens if $\phi(z)$ is not smooth at zero?

Effective pulse propagation - mixing case (5/5)

- The mean field approach is quantitatively wrong:

$$p_L^{\rm ME}(\tau) = \lim_{\varepsilon \to 0} \mathbb{E}\Big[p^\varepsilon\Big(z = L, t = \frac{L}{c_o} + \varepsilon^2 \tau\Big)\Big] = \frac{1}{2\pi}\int \hat{p}_0(\omega) T_L^{\rm ME}(\omega) e^{-i\omega\tau} d\omega,$$

$$T_L^{\rm ME}(\omega) = T_L(\omega) \exp\Big(-\frac{\gamma(0)}{8}\frac{\omega^2 \ell_{\rm c}^2}{c_o^2}\frac{L}{\ell_{\rm c}}\Big)$$

↪ additional frequency-dependent attenuation that comes from the averaging with respect to the random time shift.

Long-range and short-range correlations

- Long-range correlations: ϕ is not integrable and has a power decay at infinity:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}$$

where $r_{H'} > 0$ and $H' \in (1/2, 1)$ (i.e. $2H' - 2 \in (-1, 0)$).
ℓ_c is the critical length scale beyond which the power law behavior is valid.

- Short-range correlations: ϕ is not smooth at zero:

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H \left| \frac{z}{\ell_c} \right|^{2H} + O\Big(\left| \frac{z}{\ell_c} \right|\Big)\Big)$$

where $d_H > 0$ and $H \in (0, 1/2)$ (i.e. $2H \in (0, 1)$).
ℓ_c is the critical length scale below which the expansion is valid.

Binary medium - long-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf ($H' \in (1/2, 1)$):

$$p_{l_1}(z) = \frac{3-2H'}{\ell_c}\frac{\ell_c^{4-2H'}}{z^{4-2H'}}\mathbf{1}_{[\ell_c,\infty)}(z)$$

Note: The average length of the intervals is $\frac{3-2H'}{2-2H'}\ell_c$ while the variance is infinite.

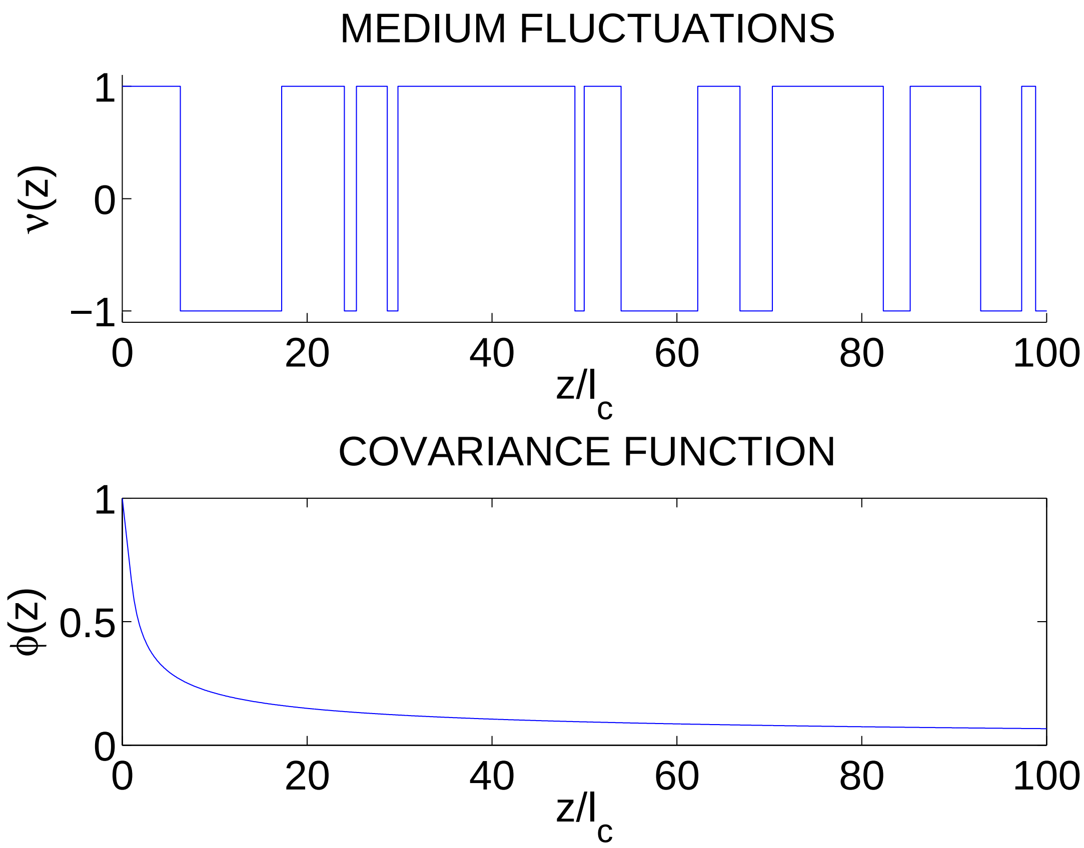
The covariance function is

$$\phi(z) = \sigma^2\left[\frac{1}{3-2H'}\frac{\ell_c^{2-2H'}}{|z|^{2-2H'}}\mathbf{1}_{[\ell_c,\infty)}(|z|) + \left(1 - \frac{2-2H'}{3-2H'}\frac{|z|}{\ell_c}\right)\mathbf{1}_{[0,\ell_c)}(|z|)\right]$$

We have

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_{H'}\left|\frac{z}{\ell_c}\right|^{2H'-2}, \text{ with } r_{H'} = \frac{\sigma^2}{3-2H'}$$

Binary medium - long-range correlations



Realizations of a binary medium with the index H' 0.75.

Generation of intervals longer than the average responsible for the long-range correlation property.

Binary medium - short-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf ($H \in (0, 1/2)$, $\ell_{\rm i} \ll \ell_{\rm c}$):

$$p_{l_1}(z) = \frac{1 - 2H}{\ell_{\rm c}[(\ell_{\rm i}/\ell_{\rm c})^{2H-1} - 1]} \frac{\ell_{\rm c}^{2-2H}}{z^{2-2H}} \mathbf{1}_{[\ell_{\rm i},\ell_{\rm c}]}(z)$$

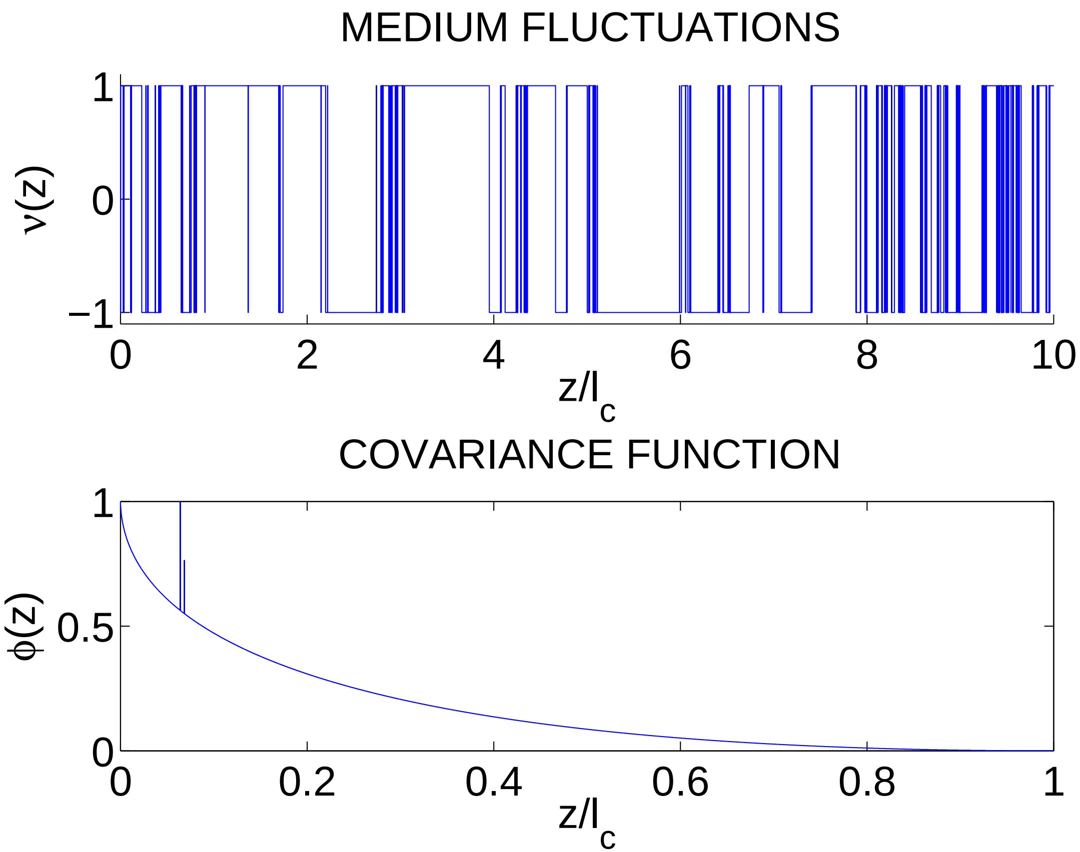
The covariance function is

$$\phi(z) = \frac{\sigma^2}{1-\delta^{2H}}\Big(1 - \frac{1}{1-2H}\frac{|z|^{2H}}{\ell_{\rm c}^{2H}} + \frac{2H}{1-2H}\frac{|z|}{\ell_{\rm c}}\Big)\mathbf{1}_{(\ell_{\rm i},\ell_{\rm c}]}(|z|)$$
$$+ \sigma^2\Big(1 - \frac{2H}{1-2H}\frac{\delta^{2H}-\delta}{1-\delta^{2H}}\frac{|z|}{\ell_{\rm i}}\Big)\mathbf{1}_{(0,\ell_{\rm i}]}(|z|)$$

with $\delta = \ell_{\rm i}/\ell_{\rm c}$. We have

$$\phi(z) \overset{\ell_{\rm i}<|z|\ll\ell_{\rm c}}{\simeq} \sigma^2\Big(1 - d_H\Big|\frac{z}{\ell_{\rm c}}\Big|^{2H} + o\Big(\Big|\frac{z}{\ell_{\rm c}}\Big|\Big)\Big), \text{ with } d_H = \frac{1}{1-2H}$$

Binary medium - short-range correlations



Realizations of a binary medium with the index H 0.25.

Accumulation of very small intervals responsible for the short-range correlation property.

Binary medium - short- and long-range correlations

The process ν is stepwise constant.

$(l_j)_{j\geq 0}$: lengths of the elementary intervals.

$(n_j)_{j\geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf:

$$p_{l_1}(z) = \left(1 - a\delta^{1-2H}\right)\frac{1-2H}{\delta^{2H-1} - 1}\frac{\ell_{\rm c}^{1-2H}}{z^{2-2H}}\mathbf{1}_{[\ell_{\rm i},\ell_{\rm c}]}(z) + a\delta^{1-2H}(3-2H')\frac{\ell_{\rm c}^{3-2H'}}{z^{4-2H'}}\mathbf{1}_{[\ell_{\rm c},\infty)}(z),$$

where $H \in (0, 1/2)$, $H' \in (1/2, 1)$, $a \in (0,1)$, $\delta = \ell_{\rm i}/\ell_{\rm c}$ and $0 < \ell_{\rm i} \ll \ell_{\rm c}$.

The pdf is continuous at $\ell_{\rm c}$ if we choose a as

$$\frac{1}{a} = \frac{3-2H'}{1-2H} - 2\delta^{1-2H}\frac{1+H-H'}{1-2H},$$

but it is not required.

Binary medium - short- and long-range correlations

The covariance function has both H-short-range and H'-long-range properties :

- if $|z| \geq \ell_{\rm c}$ then

$$\phi(z) \simeq \frac{a\sigma^2}{(2-2H)\left(\frac{1-2H}{2H} + a\frac{3-2H'}{2-2H'}\right)} \frac{\ell_{\rm c}^{2-2H'}}{|z|^{2-2H'}}$$

which shows it has the H'-long-range property in the range $|z| \in [\ell_{\rm c}, \infty)$.

- if $\ell_{\rm i} \ll |z| \ll \ell_{\rm c}$ then

$$\phi(z) \simeq \sigma^2 \Big\{ 1 - \frac{1}{1-2H + a2H\frac{3-2H'}{2-2H'}} \frac{|z|^{2H}}{\ell_{\rm c}^{2H}} \Big\}$$

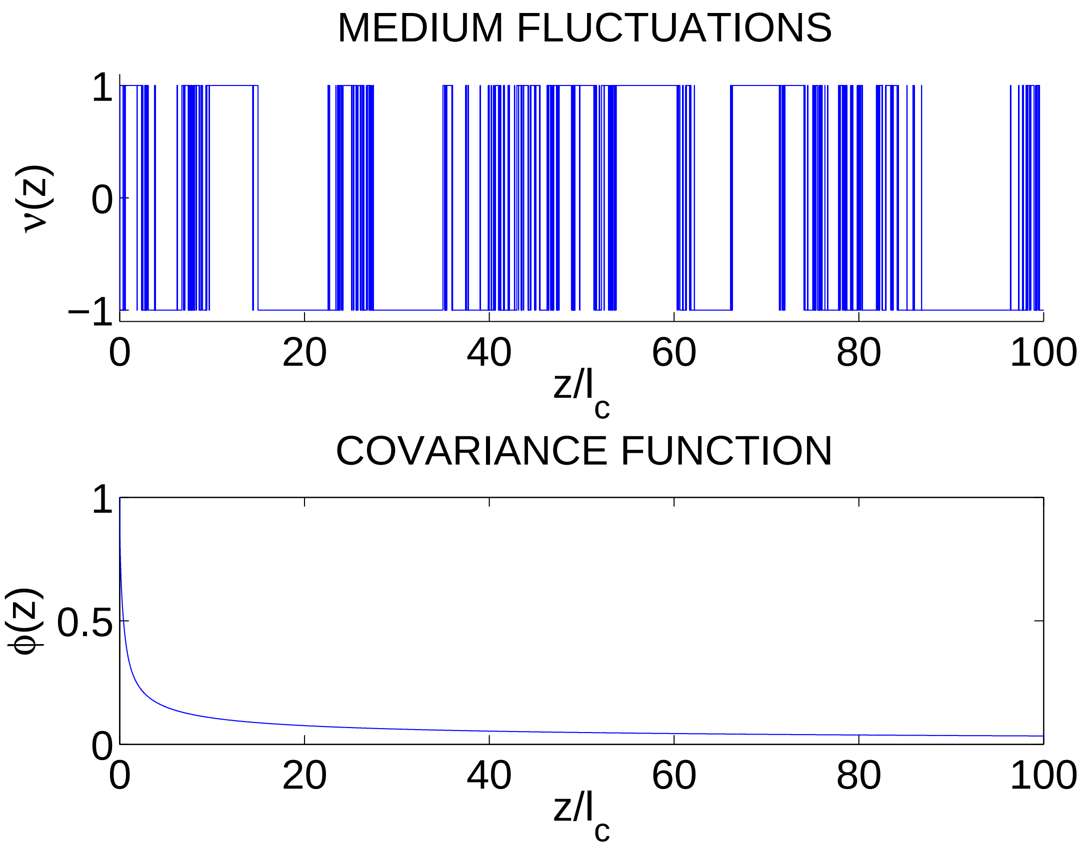
which shows it has the H-short-range property in the range $|z| \in (\ell_{\rm i}, \ell_{\rm c})$.

- if $|z| \leq \ell_{\rm i}$ then

$$\phi(z) \simeq \sigma^2 \Big\{ 1 - \frac{1}{\left(\frac{1-2H}{2H} + a\frac{3-2H'}{2-2H'}\right)\delta^{1-2H}} \frac{|z|}{\ell_{\rm c}} \Big\}$$

which shows it has the regular property in the range $|z| \in [0, \ell_{\rm i}]$.

Binary medium - short- and long-range correlations



Realizations of a binary medium long- and short-range properties with $H = 0.25$ and $H' = 0.75$.

Accumulation of very small intervals responsible for the short-range correlation property and generation of intervals longer than the average responsible for the long-range correlation property

Effective pulse propagation - long-range correlations (1/5)

- Covariance function:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}, \quad H' \in (1/2, 1)$$

- Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_o} + \varepsilon^{3-2H'} \tau_c^\varepsilon(L), \qquad \tau_c^\varepsilon(L) = \frac{1}{2 c_o \varepsilon^{2-2H'}} \int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

- In the limit $\varepsilon \to 0$ at $z = L$:

(1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one

(2) the pulse profile converges in probability to a deterministic profile:

$$p^\varepsilon\left(z = L, t = \tau^\varepsilon(L) + \varepsilon^2 \tau\right) \xrightarrow{\varepsilon \to 0} p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega$$

Note that the time shift is of order $\varepsilon^{3-2H'} \gg \varepsilon^2$ (here $H' > 1/2$ i.e. $3 - 2H' < 2$):
the random time shift is larger than the deterministic deformation.

Effective pulse propagation - long-range correlations (3/5)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ with mean zero and variance ($\sim$ fractional Brownian motion):

$$\mathbb{E}[\tau_c(L)^2] = \frac{\ell_c^{2-2H'}}{c_o^2} \frac{r_{H'}}{H'(2H'-1)} L^{2H'}$$

2) If $\frac{\omega \ell_c}{c_o} \ll 1$ then

$$T_L(\omega) = \exp\Big(-\frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \cos\big((H'-\frac{1}{2})\pi\big) \Big(\frac{|\omega|\ell_c}{c_o}\Big)^{3-2H'} \frac{L}{\ell_c} - i\frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \sin\big((H'-\frac{1}{2})\pi\big) \Big(\frac{|\omega|\ell_c}{c_o}\Big)^{2-2H'} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c}\Big)$$

$\hookrightarrow$ Effective fractional diffusion (attenuation) $\sim |\omega|^{3-2H'}$, $3-2H' \in (1,2)$.

$\hookrightarrow$ Effective fractional dispersion $\sim |\omega|^{2-2H'}\omega$.

Here $\alpha = 2-2H' \in (0,1)$.

Effective pulse propagation - long-range correlations (4/5)

Effective fractional wave equation in the "original" frame (up to random time correction):

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} = \frac{r_{H'}\ell_c^{2-2H'}}{2^{2H'}c_o^{4-2H'}}\int_0^\infty \frac{1}{s^{2-2H'}}\frac{\partial^3 p}{\partial t^3}(t-s)\,ds$$

→ same form as wave equations used in sound propagation in lossy media

→ slightly different from the standard models [M. Caputo, Geophys. J. R. Astron. Soc. 13, 529 (1967); T. L. Szabo, J. Acoust. Soc. Am. 96, 491 (1994)]

→ equivalent to the model proposed in [W. Chen and S. Holm, J. Acoust. Soc. Am. 114, 2570 (2003)]

→ respects causality (dispersion relation satisfies Kramers-Kronig relation).

Effective pulse propagation - long-range correlations (5/5)

- The mean field approach is qualitatively wrong.

$$p^{\mathrm{MF},\varepsilon}(L,\tau) := \mathbb{E}\left[p^{\varepsilon}\left(L, \frac{L}{c_o}+\varepsilon^2\tau\right)\right]$$

↪ additional frequency-dependent decay that comes from the averaging with respect to the random time shift. This term is strong: it becomes of order one for a small propagation distance, of order $\varepsilon^{2-\frac{1}{H'}}$:

$$p^{\mathrm{MF},\varepsilon}\left(\varepsilon^{2-\frac{1}{H'}}L,\tau\right) \xrightarrow{\varepsilon\to 0} p_L^{\mathrm{MF}}(\tau) = \frac{1}{2\pi}\int \hat{p}_0(\omega)\, T_L^{\mathrm{MF}}(\omega)\, e^{-i\omega\tau}\, d\omega$$

where

$$T_L^{\mathrm{MF}}(\tau) = \exp\left(-\frac{r_{H'}}{8H'(2H'-1)}\frac{\omega^2\ell_c^2}{c_o^2}\frac{L^{2H'}}{\ell_c^{2H'}}\right)$$

Thus, the mean field is described by an anomalous diffusion equation.

Effective pulse propagation - short-range correlations (1/2)

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H \Big|\frac{z}{\ell_c}\Big|^{2H} + O\Big(\Big|\frac{z}{\ell_c}\Big|\Big)\Big), \quad H \in (0, 1/2)$$

In the limit $\varepsilon \to 0$ at $z = L$ the pulse profile converges in probability to a deterministic profile:

$$p^\varepsilon\Big(z = L, t = \frac{L}{c_o} + \varepsilon^2\tau\Big) \overset{\varepsilon\to 0}{\longrightarrow} p_L(\tau) = \frac{1}{2\pi}\int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega$$

Note that the random time shift is vanishing.

If $\frac{\omega\ell_c}{c_o} \gg 1$ then

$$T_L(\omega) = \exp\Big(-\frac{\phi(0)d_H}{8}\frac{\Gamma(1+2H)}{2^{2H}}\sin(H\pi)\Big(\frac{|\omega|\ell_c}{c_o}\Big)^{1-2H}\frac{L}{\ell_c} - i\frac{\phi(0)}{8}\frac{\omega\ell_c}{c_o}\frac{L}{\ell_c} + i\frac{\phi(0)d_H}{8}\frac{\Gamma(1+2H)}{2^{2H}}\cos(H\pi)\Big(\frac{|\omega|\ell_c}{c_o}\Big)^{-2H}\frac{\omega\ell_c}{c_o}\frac{L}{\ell_c}\Big)$$

$\hookrightarrow$ Deterministic time shift (proportional to L).

$\hookrightarrow$ Effective fractional diffusion (attenuation) $\sim |\omega|^{1-2H}$, $1 - 2H \in (0, 1)$.

$\hookrightarrow$ Effective fractional dispersion $\sim |\omega|^{-2H}\omega$.

Here $\alpha = -2H \in (-1, 0)$.

Effective pulse propagation - short-range correlations (2/2)

- Remark 1: Effective fractional wave equation in the "original" frame:

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} = \frac{H d_H}{2^{1+2H} c_o^{2-2H} \ell_c^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial^2 p}{\partial t^2}(t-s)\, ds$$

- Remark 2: The mean field approach is quantitatively correct.

Effective pulse propagation - short- and long-range correlations

$$\frac{1}{L}\ln|T_L(\omega)| \approx \begin{cases} -\left(\frac{\omega\ell_c}{c_o}\right)^{3-2H'}\frac{\Gamma(2H'-1)\cos(\pi(H'-1/2))}{2^{2H'}(3-2H')(1+\mathcal{C})}, & \frac{\omega}{c_o} \ll \frac{1}{\ell_c} \\ -\left(\frac{\omega\ell_c}{c_o}\right)^{1-2H}\frac{\Gamma(1+2H)\sin(H\pi)}{2^{2+2H}(1-2H)(1+\mathcal{C}^{-1})}, & \frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i} \\ -\frac{H\delta^{2H-1}}{2(1-2H)(1+\mathcal{C}^{-1})}, & \frac{\omega}{c_o} \gg \frac{1}{\ell_i} \end{cases}$$

with $\mathcal{C} = \sigma^2\frac{(1-2H)(2-2H')(1-a\delta^{1-2H})(1-\delta^{2H})}{a2H(3-2H')(1-\delta^{1-2H})}$.

- Summary :

- For low frequencies $\frac{\omega}{c_o} \ll \frac{1}{\ell_c}$ the H'-long-range correlation property gives an exponent $\alpha = 2-2H' \in (0,1)$.

- For mid frequencies $\frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i}$ the H-short-range correlation property gives an exponent $\alpha = -2H \in (-1,0)$.

- For high frequencies $\frac{\omega}{c_o} \gg \frac{1}{\ell_i}$ the regular property gives an exponent $\alpha = -1$.

- Remember the experimental findings :

- $\alpha \simeq 0.2$ for low frequencies $< 10^{-3}$Hz [Lekic 2009]

- $\alpha \in (-0.5, 0)$ for mid frequencies $(10^{-3}, 1)$Hz [Choy 1986 Shito 2004],

- $\alpha \simeq -1$ for high frequencies > 1 Hz [Choy 1986 Cormier 2011].

Conclusions

• In a random medium with a covariance function that decays at infinity as $|z|^{2H'-2}$, $H' \in (1/2, 1)$, the attenuation factor at low frequency is $Q^{-1}(\omega) \sim |\omega|^{\alpha}$ with $\alpha = 2 - 2H' \in (0, 1)$.

• In a random medium with a covariance function that behaves at zero like $1 - d_H |z|^{2H}$, $H \in (0, 1/2)$, the attenuation factor at high frequency is $Q^{-1}(\omega) \sim |\omega|^{\alpha}$ with $\alpha = -2H \in (-1, 0)$.

• A special frequency-dependent phase is associated to the frequency-dependent attenuation and it ensures that causality and Kramers-Kronig relations are respected.

• Effective fractional wave equations can be written that have the form of equations studied in the literature in the context of wave propagation in lossy media.

• A simple binary medium with short- and long-range correlation properties can explain the frequency dependence of the apparent attenuation observed for seismic body waves.