Locating an obstacle in a 3D finite depth ocean using the convex scattering support

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Abstract

We consider an inverse scattering problem in a 3D homogeneous shallow ocean. Specifically, we describe a simple and efficient inverse method which can compute an approximation of the vertical projection of an immersed obstacle. This reconstruction is obtained from the far field patterns generated by illuminating the obstacle with a single incident wave at a given fixed frequency. The technique is based on an implementation of the theory of the convex scattering support [1]. A few examples are presented to show the feasibility of the method.

Key words: inverse scattering, shallow ocean, convex scattering support

1 Introduction

In the following article we consider a fixed-frequency inverse scattering problem in a three dimensional waveguide. The waveguide we consider is bounded in one (vertical) direction, while infinite in the other two (horizontal) directions. The geometry, governing equations, and boundary conditions associated with the scattering phenomena are those of linear acoustics in a homogeneous shallow ocean ; described for example in [2]. We develop and present a simple and efficient method to find an approximation of the vertical projection of the convex hull of an immersed obstacle, given the observed far field patterns when this obstacle is illuminated by a single incident monochromatic wave.

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Imaging a scatterer in a shallow ocean is a classical inverse problem which has been considered by many authors. Furthermore, it is well known that the inverse scattering problem associated with a 3D waveguide as such is more difficult than in free space. Actually, because of the physical presence of the top and bottom elements of the waveguide, only a finite number of modes can propagate at long distance. The remainder of the modes are said to be evanescent, which means they decay exponentially as a function of distance. This fact increases the so-called ill-posedness of the inverse problem. Two issues have to be considered: first the identifiability, *i.e.* the uniqueness of the scatterer given the data, secondly the use of a stable inversion technique.

Concerning the first issue, Gilbert *et al* [3] have proved uniqueness when we have measurements of the scattered field on a horizontal plane. However, due to the presence of evanescent modes, uniqueness from the far field patterns cannot be established. In the following article, we present a proof of uniqueness for measurements supported by a cylinder which surrounds the obstacle. Concerning the second issue, several methods have been proposed in both two and three dimensional oceanic models (see for example [4–9] and the bibliography of [6]). The most interesting methods are those for which no *a priori* assumption is made concerning the physical nature of the scatterer. For example the linear sampling method (see an overview in [10]) has been adapted to the 2D ocean [6]. The main drawback of such a method is that many incident waves are required.

Recently, in [11], Kusiak *et al* have developed and implemented a theory based on the so-called convex scattering support. This theory provides the same advantage as the linear sampling method concerning *a priori* knowledge of boundary conditions. Additionally, it allows one to use but one incident wave to deduce information concerning the location, size and shape of the scatterer. Essentially, the reconstruction method consists of producing, by intersecting convex test domains, a minimal convex set which must be contained in the convex hull of the true scatterer. The main goal of our paper is to show how this method can be adapted to approximate the convex hull of the vertical projection of an obstacle in a 3D ocean by using the far field patterns generated by single fixed-frequency illumination of the scatterer.

This paper is organized as follows. In the second section, we briefly recall the various aspects of the direct scattering problem in order to address its inverse counterpart. In particular, we describe our approach in computing the simulated far field patterns using a finite element technique on a bounded domain. In the third section, we prove uniqueness of an acoustically sound soft scatterer having data on a surrounding cylinder. The fourth section is devoted to a brief review of the theory of the convex scattering support. In section five, we describe how that theory may be adapted to our 3D waveguide in order to approximately identify the convex hull of the vertical projection of the obstacle in a shallow ocean. Finally, in the sixth section, we present results which demonstrate the feasibility of our method.

2 The direct scattering problem

Our waveguide is the open domain W included between the two horizontal boundaries $z = 0$ (called 'top' or Γ_0) and $z = h$ (called 'bottom' or Γ_h) in the Cartesian coordinates (x, y, z). The boundary conditions at $z = 0$ and $z = h$ are of the Dirichlet and Neumann types respectively, and the waveguide can therefore be considered as a model of a finite depth ocean in contact with an acoustically-soft medium (such as air) at the top and with an acoustically-hard medium (such as rock) at the bottom.

We suppose that a sound hard or soft obstacle $\mathcal{O}$ is embedded in W. We define Ω to be the open domain of W complementary to $\mathcal{O}$. A monochromatic acoustic wave scatters due to the presence of the obstacle. Let k denote the wavenumber, and let u^i, u^s and u respectively denote the incident, scattered and total fields ($u = u^i + u^s$). The governing equations for u^s in Ω are

$$\begin{cases} (\Delta_3 + k^2)u^s = 0 \;\; in \;\; \Omega \\ u^s|_{\Gamma_0} = 0, \;\; \dfrac{\partial u^s}{\partial z}|_{\Gamma_h} = 0 \qquad \text{(referred by } (BC) \text{ from now on)} \\ \dfrac{\partial u^s}{\partial \nu}|_{\partial\mathcal{O}} = f \;\; \text{or} \;\; u^s|_{\partial\mathcal{O}} = g \\ \qquad (RC). \end{cases} \tag{1}$$

Here, Δ_3 is the three dimensional Laplacian, ν is the outward unit normal on $\partial\mathcal{O}$, $f = -(\partial u^i/\partial\nu)|_{\partial\mathcal{O}}$ and $g = -u^i|_{\partial\mathcal{O}}$. Lastly, (RC) is a radiation condition associated with the behavior of u^s when $r = \sqrt{x^2 + y^2} \to \infty$. Such a condition, which we will come to specify shortly, is necessary to ensure well-posedness of problem (1).

We define the three dimensional cylindrical domain $C(R) = B(R) \times (0, h)$, where $B(R)$ is the open ball of radius R in $\mathbb{R}^2$, and define its two dimensional boundary by $\Sigma(R) = \partial B(R) \times (0, h)$. We assume R to be large enough such that $\mathcal{O}$ is included in $C(R)$, and we define the domains $\Omega' = (B(R)/\overline{\mathcal{O}}) \times (0, h)$ and $\Omega'' = (\Omega/\overline{B(R)}) \times (0, h)$ ($\Omega = \overline{\Omega'} \cup \Omega''$). Finally, let S^1 denote the unit sphere in $\mathbb{R}^2$.

It is well known that any field u^s which satisfies the 3D Helmholtz equation in Ω'' and the boundary conditions (BC) has, in cylindrical coordinates, the following representation in the domain Ω''

$$u^s(r,\theta,z) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}} \Big(a_{mn}H_m^{(1)}(k_n r) + b_{mn}H_m^{(2)}(k_n r)\Big)\psi_m(\theta)w_n(z). \qquad (2)$$

Here, $H_m^{(1,2)}$ are the Hankel functions of the first and second kinds, *c.f.* [12]. The functions ψ_m and w_n are defined by

$$\psi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi}}, \quad w_n(z) = \sqrt{\frac{2}{h}}\sin\Big((n+1/2)\frac{\pi z}{h}\Big). \qquad (3)$$

We note they form respectively orthonormal basis of $L^2(S^1)$ and $L^2((0,h))$. The sequence of complex numbers k_n is defined by

$$k_n = \sqrt{k^2 - \Big((n+1/2)\frac{\pi}{h}\Big)^2}, \quad \Re(k_n) + \Im(k_n) \geq 0. \qquad (4)$$

From now on, we assume that k is chosen such that k_n never vanishes.

We note that, in the domain Ω'', we may also write the scattered field as

$$u^s(r,\theta,z) = \sum_{n\in\mathbb{N}} u_n^s(r,\theta)w_n(z), \qquad (5)$$

where for each n we have

$$u_n^s(r,\theta) = \sum_{m\in\mathbb{Z}} \Big(a_{mn}H_m^{(1)}(k_n r) + b_{mn}H_m^{(2)}(k_n r)\Big)\psi_m(\theta). \qquad (6)$$

Assuming that only the outgoing or evanescent contributing modes in the expansion of u^s are physically acceptable, which will be considered as our radiation condition (RC), it then follows that all of the coefficients b_{mn} in (2) necessarily vanish. The modes u_n^s for $n \in [0, N-1]$ having $\Im(k_n) = 0$ correspond to the so-called propagating waves, while the modes for $n \geq N$ having $\Re(k_n) = 0$ correspond to the evanescent ones. The radiation condition (RC) is equivalent to an infinite number of classical Sommerfeld conditions in two dimensions, that is for each n

$$\lim_{r\to+\infty} \sqrt{r}\Big(\frac{\partial u_n^s}{\partial r} - ik_n u_n^s\Big) = 0, \qquad (7)$$

and each of the above limits hold uniformly in all directions θ.

It can be established, using the same arguments as in [13], that problem (1) in the unbounded domain Ω is equivalent to the following problem in the bounded domain Ω'

$$\begin{cases} (\Delta_3 + k^2)u' = 0 \;\; in \;\; \Omega' \\ \qquad (BC) \\ \dfrac{\partial u'}{\partial \nu}|_{\partial\mathcal{O}} = f \;\; or \;\; u'|_{\partial\mathcal{O}} = g \\ \dfrac{\partial u'}{\partial \nu}|_{\Sigma(R)} = \mathcal{T}(u'|_{\Sigma(R)}). \end{cases} \tag{8}$$

Here, $\mathcal{T} : H^{\frac{1}{2}}(\Sigma(R)) \rightarrow H^{-\frac{1}{2}}(\Sigma(R))$ is the Dirichlet to Neumann operator defined by $\mathcal{T}(\chi) = (\partial\mathcal{S}(\chi)/\partial\nu)|_{\Sigma(R)}$, where $\mathcal{S}(\chi)$ is the solution of the following well-posed problem in Ω''

$$\begin{cases} (\Delta_3 + k^2)u'' = 0 \;\; in \;\; \Omega'' \\ \qquad (BC) \\ u''| [illegible] \chi \\ [illegible] RC). \end{cases} \tag{9}$$

For cl[illegible] we offer an explicit ex[illegible]. Since the functions defined by χ[illegible]$,z) = \frac{1}{\sqrt{R}}\psi_m(\theta)w_n(z)$ fo[illegible]rmal basis of $L^2(\Sigma(R))$, then $\chi \in$ [illegible]$(\Sigma(R))$ may be expanded [illegible]

$$[illegible](\theta, z) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}} d_{mn}\chi_{mn}([illegible] \tag{10}$$

and thus we obtain the ide[illegible]

$$\mathcal{T}(\chi)(\theta, z) = \sum_{n\in\mathbb{N}}\sum_{m\in [illegible]} [illegible] \frac{[illegible]_n R)}{[illegible]_n R)}\chi_{mn}(\theta, z). \tag{11}$$

Following [13], it ma[illegible]oved that problem (1) is equivalent to problem (8) in the [illegible] knowledge, the uniqueness property has been proved only in the case of the acoustically-soft obstacle (Dirichlet data f), when $\partial\mathcal{O}$ and f are sufficiently smooth (say $\partial\mathcal{O}$ of class C^2 and $f \in H^{\frac{3}{2}}(\partial\mathcal{O})$, which implies that $u' \in H^2(\Omega')$), and when the scatterer $\mathcal{O}$ satisfies a convexity condition of the type $\nu.(x, y, 0) = \nu_x x + \nu_y y \geq 0$ for all points of its boundary. The corresponding proof is established in [2] amended by [14] (see also p. 92 in [15] for a simplification in the proof). When this convexity condition is not satisfied, or in the case of the acoustically-hard obstacle (Neumann data g), the well-posedness of (1) or the equivalent problem (8) remains an open problem.

In order to compute the simulated far field patterns which form the data of the inverse problem, a finite element method based on a weak formulation of problem (8) was employed. In particular, the specific basis functions χ_{mn} defined above play a crucial role in the finite element method since they diagonalize the operator $\mathcal{T}$.

3 A uniqueness result for the inverse problem

We begin by assuming that the incident wave u^i is of the form $u^i(x, y, z) = w_n(z)e^{ik_n\tilde{d}.\tilde{x}}$, where $\tilde{d} = (d_x, d_y) \in S^1$ and $\tilde{x} = (x, y)$. We choose $n \in [0, N-1]$ such that u^i is a propagating wave which satisfies $\Delta_3 u^i + k^2 u^i = 0$ in W and the boundary conditions (BC). We now offer our uniqueness theorem.

Theorem 1 *Suppose $\mathcal{O}_{1,2}$ are two soft obstacles whose boundaries are of class C^2 and satisfy the convexity condition $\nu_x x + \nu_y y \geq 0$. If we assume that for an infinite number of incident waves u^i_q with propagation direction $\tilde{d}_q$, the corresponding total fields u_{1q} and u_{2q} coincide on the cylinder $\Sigma(R)$, then $\mathcal{O}_1 = \mathcal{O}_2$.*

The proof uses the same arguments as Schiffer's proof in [16] (p. 173), and is strongly inspired from the one detailed in [17] (p. 107) in the case of three dimensional free space. To complete the proof we need the two following lemmas, the first one being a classical result of spectral theory, and the second one being proved in [17] (p. 21).

Lemma 1 *For a given wavenumber k, the space of solutions of the homogeneous Dirichlet problem for the Helmholtz equation inside a bounded domain D with Lipschitz-continuous boundary has a finite dimension.*

Lemma 2 *Every solution u of the two dimensional Helmholtz equation outside a given ball $B(R)$ which satisfies the radiation condition*

$$\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = \sqrt{x^2 + y^2}, \tag{12}$$

where the limit above holds uniformly in all directions $\hat{x} = \tilde{x}/r$, has the following asymptotic behavior :

$$u(\tilde{x}) = \frac{e^{ikr}}{\sqrt{r}}\Big(u^{\infty}(\hat{x}) + O(\frac{1}{r})\Big), \ \ r \rightarrow +\infty, \tag{13}$$

uniformly in all directions $\hat{x}$.

Here, the function u^{∞} defined on the unit sphere S^1 is known as the far field pattern of u and is given by

$$u^{\infty}(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial B(R)} \Big(u(y)\frac{\partial e^{-ik\hat{x}.y}}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} e^{-ik\hat{x}.y}\Big)\, ds(y).$$

Proof of theorem 1 Let us suppose that $\mathcal{O}_1 \neq \mathcal{O}_2$ and that both satisfy the assumptions of theorem 1. We know from section 2 that the direct problem (1) for $\mathcal{O}_1$ and $\mathcal{O}_2$ is well-posed for all q. The total fields u_{1q} and u_{2q}, and consequently the scattered fields u^s_{1q} and u^s_{2q}, coincide on the cylinder $\Sigma(R)$, and therefore in the exterior domain Ω'' as well because of the expansion formula (2) with $b_{mn} = 0$. From the unique continuation principle, u_{1q} and u_{2q} coincide in the exterior domain $\Omega/(\overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2})$. Since $u_{1q}|_{\partial\mathcal{O}_1} = 0$ and $u_{2q}|_{\partial\mathcal{O}_2} = 0$, in each connected component D of $\mathcal{O}_1 \cup \mathcal{O}_2/(\overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2})$, either u_{1q} or u_{2q} is solution of a homogeneous Dirichlet problem for the Helmholtz equation inside D.

With the aid of lemma 1, we complete the proof by proving that u_{iq} $(i = 1, 2)$ are linearly independent in domain D for all q. This implies that the connected components D are empty, and hence $\mathcal{O}_1 = \mathcal{O}_2$. For simplicity, we set $u_{iq} = u_q$ and assume there exists a $P \in \mathbb{N}$ and some constants c_q, $q \in [1, P]$, such that in D,

$$\sum_{q=1}^{P} c_q u_q = 0. \tag{14}$$

Again, from the unique continuation principle, (14) holds in Ω''. The functions u^s_q have a similar representation as (5) in the domain Ω'', so that using $u_q = u^i_q + u^s_q$ in (14), multiplying the result by $w_n(z)$ and integrating the product in z on $(0, h)$, yields

$$\sum_{q=1}^{P} c_q e^{ik_n\tilde{d}_q.\tilde{x}} + \sum_{q=1}^{P} c_q u^s_{q,n}(\tilde{x}) = 0, \ \ \forall \tilde{x} \in \mathbb{R}^2/B(R). \tag{15}$$

Multiplying (15) by $e^{-ik_n\tilde{d}_p.\tilde{x}}$ for a given $p \in [1, P]$ and integrating the result over

$\tilde{x}$ on $\partial B(R)$ gives

$$\sum_{q=1}^{P} c_q \int\limits_{r=R} e^{ik_n(\tilde{d_q}-\tilde{d_p}).\tilde{x}}\, d\tilde{x} + \sum_{q=1}^{P} c_q \int\limits_{r=R} u^s_{q,n}(\tilde{x}) e^{-ik_n\tilde{d_p}.\tilde{x}}\, d\tilde{x} = 0. \tag{16}$$

The first integral becomes

$$\int\limits_{r=R} e^{ik_n(\tilde{d_q}-\tilde{d_p}).\tilde{x}}\, d\tilde{x} = \int\limits_{0}^{2\pi} e^{ik_n||\tilde{d_q}-\tilde{d_p}||Rcos\theta}\, Rd\theta = 2\pi R\, J_0(k_n||\tilde{d_q}-\tilde{d_p}||R), \tag{17}$$

where J_0 is the classical Bessel function of the first kind, *c.f.* [12].

As for the second integral, since each $u^s_{q,n}$ satisfies the 2D Sommerfeld condition (7), it follows from lemma 2 that they have the asymptotic behavior

$$u^s_{q,n}(\tilde{x}) = \frac{e^{ik_n r}}{\sqrt{r}}\Big(u^\infty_{q,n}(\hat{x}) + O(\frac{1}{r})\Big), \quad r \to +\infty,$$

uniformly in all directions $\hat{x}$, with $u^\infty_{q,n}$ defined on the unit sphere S^1 by

$$u^\infty_{q,n}(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k_n}} \int\limits_{\partial B(R)} \Big(u_{q,n}(y)\frac{\partial e^{-ik_n\hat{x}.y}}{\partial\nu(y)} - \frac{\partial u_{q,n}(y)}{\partial\nu(y)} e^{-ik_n\hat{x}.y}\Big)\, ds(y)$$

for a sufficiently large R. Hence, there exists a constant C_1 for a sufficiently large R such that $\forall q \in [1, P]$,

$$|u^s_{q,n}(\tilde{x})| \le \frac{C_1}{\sqrt{R}}, \quad \forall \tilde{x} \in \partial B(R).$$

Therefore, there exists another constant C_2 such that

$$\Big|\sum_{q=1}^{P} c_q \int\limits_{r=R} u^s_{q,n}(\tilde{x}) e^{-ik_n\tilde{d_p}.\tilde{x}}\, d\tilde{x}\Big| \le \sum_{q=1}^{P} |c_q| \int\limits_{r=R} \frac{C_1}{\sqrt{R}}\, d\tilde{x} = C_2\sqrt{R}.$$

Dividing (16) by $(2\pi R)$ we obtain

$$c_p + \sum_{q\neq p} c_q J_0(k_n||\tilde{d_q}-\tilde{d_p}||R) = O(\frac{1}{\sqrt{R}}), \quad R \to +\infty.$$

Passing to the limit $R \to +\infty$, it follows that $c_p = 0$ for all $p \in [1, P]$. This completes the proof. ■

4 The convex scattering support in two dimensions

The recent theory developed in [1,11] treated an inverse scattering problem posed in free space. In this section, for the sake of self containment, we recall the main definitions and properties which substantiate that theory. This theory leads to a summability test which characterizes the convex scattering support. From that accurate but unpractical criterion we derive a heuristic but practical one, as well as an identification strategy based on that new criterion.

Definition 1 *A test domain D ($D' = \mathbb{R}^2/\overline{D}$) supports the far field pattern $u^\infty \in L^2(S^1)$ iff there exists a field $u^s \in H^1_{loc}(D')$ satisfying $(\Delta_2 + k^2)u^s = 0$ in D', and for which u^∞ is the far field pattern corresponding to u^s.*

Definition 2 *The intersection of all convex domains that support u^∞ is a convex domain that supports u^∞. It is called the convex scattering support of u^∞ and is denoted $cS_k supp(u^\infty)$.*

The properties on which definition 2 relies are proved in [11]. From the two previous definitions we immediately have the following

Proposition 1 *If u^∞ is the far field pattern produced by a scatterer $\mathcal{O}$ with convex hull $ch(\mathcal{O})$, then $cS_k supp(u^\infty) \subset ch(\mathcal{O})$.*

As proposition 1 indicates, the convex scattering support is a minimal set included in the convex hull of the obstacle. Unfortunately, in general, it is not possible to find a maximal set that could contain the obstacle. Other proprieties of $cS_k supp(u^\infty)$ can however be found in [1] and [11].

Definition 3 *Let $S^\infty_D : H^{-1/2}(\partial D) \to L^2(S^1)$ be defined by*

$$(S^\infty_D)\varphi(\hat{x}) = \int_{\partial D} \varphi(y)\, \Phi^\infty(\hat{x}, y)\, dy, \quad \Phi^\infty(\hat{x}, y) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y}.$$

The theory of the scattering support is based in part on the following theorem.

Theorem 2 *Assuming that k is such that the homogeneous Dirichlet problem for the Helmholtz equation inside D admits only the trivial solution, S^∞_D is a compact, injective operator with dense range. Furthermore, D supports $u^\infty \in L^2(S^1)$ iff*

$$\sum_{p\in\mathbb{N}} \frac{|(u^\infty, g_p)|^2}{\sigma_p^2} < +\infty, \tag{18}$$

where $\{\sigma_p, f_p, g_p\}$ ($p \in \mathbb{N}$) is a singular system of S^∞_D.

Sketch of the proof The proof is standard, and we recall the principle steps in the following. For a proof of the fact that S_D^∞ is a compact, injective operator with dense range, see [18]. First, it may be demonstrated, *c.f.* [19], that the operator $S_D : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ defined by $S_D\varphi(x) = \int_{\partial D} \varphi(y)\,\Phi(x,y)\,dy$, where Φ is the two dimensional fundamental solution $\Phi(x,y) = iH_0^{(1)}(k|x-y|)/4$, is an isomorphism when k is such that the homogeneous Dirichlet problem for the Helmholtz equation inside D admits only the trivial solution. Since $S_D^\infty\varphi$ is the far field pattern associated to the scattered field $S_D\varphi$, it follows that D supports $u^\infty \in L^2(S^1)$ iff $u^\infty \in \text{Range}(S_D^\infty)$ (see lemma 3.6. in [11]).

Secondly, since S_D^∞ is a compact injective operator between $H^{-\frac{1}{2}}(\partial D)$ and $L^2(S^1)$, there exists a singular system $\{\sigma_p, f_p, g_p\}$, $p \in \mathbb{N}$, *c.f.* [17]. Precisely, $\{\sigma_p\}$ is a strictly positive sequence of $\mathbb{R}$, and the sequences of functions $\{f_p\}$ and $\{g_p\}$ are respectively orthonormal basis on $H^{-1/2}(\partial D)$ and $L^2(S^1)$ such that $S_D^\infty f_p = \sigma_p\, g_p$ and $S_D^{\infty *} g_p = \sigma_p\, f_p$. Here, $S_D^{\infty *}$ denotes the adjoint operator of S_D^∞.

Finally, since S_D^∞ has dense range in $L^2(S^1)$, then Picard's theorem, *c.f.* [17], states that D supports $u^\infty \in L^2(S^1)$ iff (18) is satisfied. ■

The two following propositions, which result from simple calculations, enable one to establish criterion (18) when D is any ball $B(C,R)$ in $\mathbb{R}^2$ of center C and radius R.

Proposition 2 *In the particular case $D = B(O,R)$, the functions g_p coincide with the ψ_m ($m \in \mathbb{Z}$) defined by (3) while the corresponding σ_m are*

$$\sigma_m = \sqrt{\frac{\pi R}{2k}}|J_m(kR)|, \tag{19}$$

where the J_m are the classical Bessel functions of the first kind, c.f. *[12].*

Proposition 3 *If S_C^∞ and u_C^∞ respectively denote the operator $S_{B(C,R)}^\infty$ and the function $u_C^\infty(\hat{x}) = e^{ik\hat{x}.C}u^\infty(\hat{x}) \in L^2(S^1)$, one has*

$$u^\infty \in \text{Range}(S_C^\infty) \;\text{ iff }\; u_C^\infty \in \text{Range}(S_O^\infty). \tag{20}$$

From propositions 2 and 3, it follows that when $D = B(C,R)$, with $C = (C_x, C_y) \in \mathbb{R}^2$, criterion (18) is simply: $B(C,R)$ supports u^∞ iff

$$\sum_{m\in\mathbb{Z}} \frac{|c_m|^2}{\sigma_m^2} < +\infty, \quad c_m = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-im\theta} e^{ik(C_x\cos\theta + C_y\sin\theta)} u^\infty(\theta)\,d\theta. \tag{21}$$

A practical and useful implementation of criterion (21) is delicate because of the interpretation of $+\infty$. To deal with this issue, we have derived a simplified and heuristic criterion which relies on the fact that, as a function of the index m and for a fixed argument z, the function $|J_m(z)|$ of m is a bounded oscillating function for $|m| \ll z$, and a rapidly decaying one to zero when $|m| \gg z$, exhibiting a region of rapid transition near $m = z$, *c.f.* [12]. Similarly, for a real bounded obstacle $\mathcal{O} \subset W$ and for a given C, we observe that the Fourier coefficients $|c_m|$ also possess a similar behavior. Namely, we witness a rapid accumulation to zero when $m \ll -m_-^C$ and $m \gg m_+^C$, the lower and upper bounds $-m_-^C$ and m_+^C being directly read on the $|c_m|$-curve. Criterion (21) is then 'equivalent' to: $B(C, R)$ supports u^∞ 'iff'

$$kR \geq \max(m_-^C, m_+^C). \tag{22}$$

Criterion (22) provides, for a given $C = (C_x, C_y)$, the smallest ball of center C which supports u^∞. Its radius is simply $\max(m_-^C, m_+^C)/k$.

The identification strategy we have chosen consists of the basic following scheme. First, we assume *a priori* that the obstacle is fully contained within the ball $B(C_0, R_0)$. Next, we select a finite collection of balls $B(C_i, R_i)$ $(i \in I)$, the centers C_i of which are equally distributed on the circle $\partial B(C_0, R_0)$, the radii R_i of which are obtained by computing $R_i = \max(m_-^{C_i}, m_+^{C_i})/k$. Finally, we construct the intersection of the collection $B(C_i, R_i)$ for $i \in I$. This provides an approximation of the convex scattering support $cS_k supp(u^\infty)$, and hence an approximation of the convex hull of $\mathcal{O}$.

Remark 1 *The Picard series given in (21) dictates that our 2D inverse problem of finding the obstacle $\mathcal{O}$ from the far field pattern u^∞ yields a severely ill-posed problem because of the rapid accumulation of the singular values σ_m to zero as $|m|$ tends to infinity. In a sense, our simplified criteria (22) is a form of regularization as it acts as a spectral cut-off.*

Remark 2 *Criteria (22) requires the computation of the Fourier coefficients c_m of the function $e^{ik\hat{x}.C}u^\infty(\hat{x})$ in $L^2(S^1)$. In practice, only a finite number M of such coefficients are calculated. It is necessary that M should be consistent with the value of the frequency $k|C|$. That is $M > 2k|C|$ to satisfy the Nyquist sampling requirement. For a given wavenumber k, this amounts to adjusting the size of the discrete Fourier basis to the size of the searching region.*

5 The three dimensional waveguide inverse problem

In the following section we demonstrate how we may adapt the above simplified criterion and strategy of identification to determine the vertical projection $P_z\mathcal{O}$

u_n^∞ for $n \in [0, N-1]$, which we define in the following. In the case when $u^s \in H^2(\Omega')$, which holds in the situation we described at the end of section 2, that is when the direct scattering problem (8) is well-posed, then the function $u^s(r, \theta, z)$ of z for fixed $\tilde{x} = (r, \theta)$ in $\mathbb{R}^2/\overline{P_z\mathcal{O}}$ belongs to $L^2((0, h))$. Since the functions w_n defined by (3) form an orthonormal basis of $L^2((0, h))$, the expansion (5) holds for all $\tilde{x} = (r, \theta)$ outside $P_z\mathcal{O}$, and not only for all $\tilde{x}$ outside $B(R)$. It is then easy to see that for $n \in [0, N-1]$, the two dimensional scattered fields u_n^s solve the Helmholtz equation with wavenumber k_n outside $P_z\mathcal{O}$. Furthermore, we saw that the u_n^s satisfy the 2D Sommerfeld condition (7), which enables one to define, for all $n \in [0, N-1]$, the corresponding far field patterns u_n^∞ with the help of lemma 2.

We summarize these observations by the following system (note that $k_n \in \mathbb{R}^+$ when $n \in [0, N-1]$) :

$$\forall n \in [0, N-1], \quad \begin{cases} (\Delta_2 + k_n^2) u_n^s = 0 \ \text{ in } \ \mathbb{R}^2/\overline{P_z\mathcal{O}} \\ u_n^s(\tilde{x}) = u_n^\infty(\hat{x}) \dfrac{e^{ik_n r}}{\sqrt{r}} + \mathcal{O}(\dfrac{1}{r^{\frac{3}{2}}}), \ \ r \to +\infty. \end{cases} \tag{23}$$

Using the asymptotic behavior of the Hankel functions $H_m^{(1)}$ in formula (6) (see [12]), we obtain that

$$u_n^\infty(\theta) = \sum_{m \in \mathbb{Z}} c_{mn} \psi_m(\theta), \quad c_{mn} = a_{mn} \sqrt{\frac{2}{\pi k_n}} e^{-i(2m+1)\frac{\pi}{4}}, \tag{24}$$

the a_{mn} being the coefficients in (2).

Using definition 1, system (23) means precisely that $P_z\mathcal{O}$ supports u_n^∞, for all $n \in [0, N-1]$. Hence, proposition 1 leads to

$$\bigcup_{n=0}^{N-1} cS_{k_n} supp(u_n^\infty) \subset ch(P_z\mathcal{O}). \tag{25}$$

We finally conclude that the N convex scattering supports of the far field patterns which are associated with the propagating waves (with wavenumber k_n) enable one to find an approximation of $P_z\mathcal{O}$. In practice, for each $n \in [0, N-1]$, the simplified criterion and the identification strategy described at the end of the previous section are performed in order to approximate the corresponding convex scattering support $cS_{k_n} supp(u_n^\infty)$.

It will turn out in the next section that the numerical results are satisfactory even if one single convex scattering support $cS_{k_n} supp(u_n^\infty)$, *i.e.* for a given $n \in [0, N-1]$, is used.

Remark 3 *The equations (24) suggest a very simple method to compute simulated data once the direct scattering problem (8) is solved. The Fourier coefficients of the far field patterns u_n^∞ (i.e. coefficients c_{mn}) are deduced from the coefficients a_{mn} of (2) via the second equation of (24), which are themselves obtained by projecting $u^s|_{\Sigma(R)}$ onto the basis χ_{mn}.*

Remark 4 *Applying criterion (22) in order to approximate $cS_{k_n} supp(u_n^\infty)$ requires n to satisfy an inequality of the type $k_n R_{\mathcal{O}} > 1$, $R_{\mathcal{O}}$ being the horizontal radius of the obstacle. Since k_n decreases when n goes from 0 to $N-1$, it implies that our inversion technique is valuable only if $k_0 R_{\mathcal{O}} > 1$ (in particular, it implies that at least one propagating mode exists), which is equivalent to*

$$k > \sqrt{\left(\frac{\pi}{2h}\right)^2 + \left(\frac{1}{R_{\mathcal{O}}}\right)^2}.$$

6 Some numerical experiments

In the following experiments, we chose $h = 4$ as the height of the waveguide, and we reconstructed the convex hull of the vertical projection of an acoustically-hard obstacle $\mathcal{O}$ for the following four cases.

- Case 1 : a sphere of center $(0, 0, 2)$ and radius 1,
- Case 2 : a sphere of center $(0, 0, 2)$ and radius 0.5,
- Case 3 : an ellipsoid of center $(0, 0, 2)$ and semi-axes 2, 2, and 1,
- Case 4 : two spheres of centers $(0, 1, 2)$ and $(0, -1, 2)$, both of radius 0.5.

The incident field is $u^i(x, y, z) = \sin[(n + 1/2)\frac{\pi z}{h}]e^{ik_n x}$ for $k = 4$ and $n = 0$, k_0 being given by (4). Hence, the number of propagating modes is $N = 5$. The corresponding scattered fields u^s in the bounded domain Ω' (solution of problem (8)), and the corresponding 2D far field patterns u_n^∞ ($n \in [0, 4]$), are obtained using the Finite Element code Melina, *c.f.* [20]. Figure 1 shows the real part of the total field u in the case 4. Using the computed data u_n^∞ (see remark 3), for each $n \in [0, N-1]$ we employ the strategy described in section 4. The set $P_z(\mathcal{O})$ is approximated by the intersection of 8 balls $B(C_i, R_i)$ ($i \in \{1, 2, \cdots, 8\}$) such that their centers C_i are equally distributed on the circle $\partial B(C_0, R_0)$ with $C_0 = (7, 4)$ and $R_0 = 12$. It amounts to guessing that $P_z(\mathcal{O})$ lies within that circle.

Figures 2 to 7 show the xy-projection of the true obstacle within the waveguide and the result of the reconstruction, both inside the circle $\partial B(C_0, R_0)$, in the following situations: for case 1 with either u_0^∞ or u_1^∞, for case 2 with u_0^∞, for case 3 with u_0^∞, and for case 4 with either u_0^∞ or u_1^∞ (similar results are obtained with the other values of n). No additional noise was added to the error

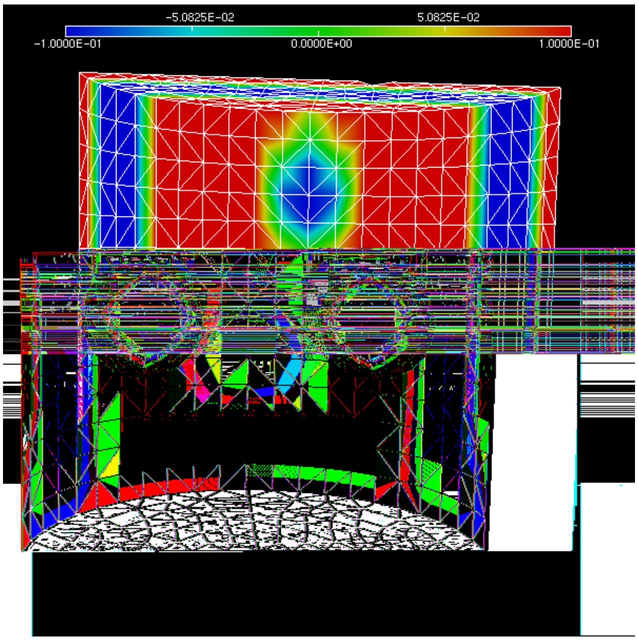

Fig. 1. *FEM computation of the total field u in the case of two spheres (case 4)*

naturally produced by the FEM computation which produced the simulated values of u_n^∞.

These experiments reveal that only one 2D far field u_n^∞ associated to only one incident field u^i enables one to locate the vertical projection of the obstacle and to approximately find its size. The shape of the obstacle is however poorly reconstructed.

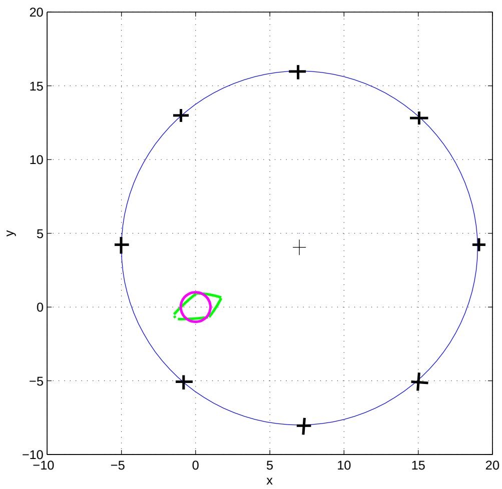

Fig. 2. *Case 1 :* u_0^∞

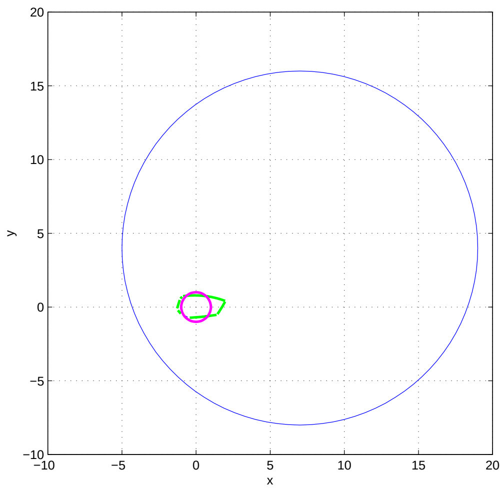

Fig. 3. *Case 1 :* u_1^∞

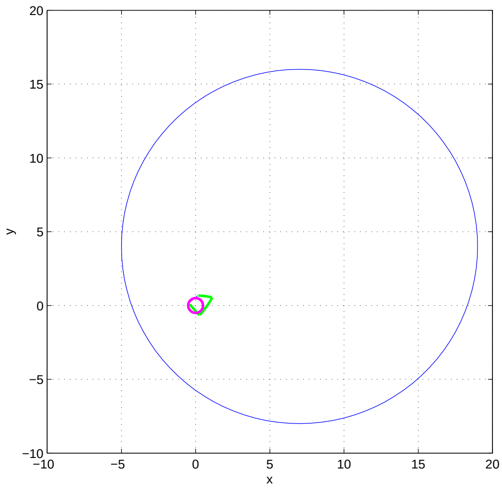

Fig. 4. *Case 2* : u_0^∞

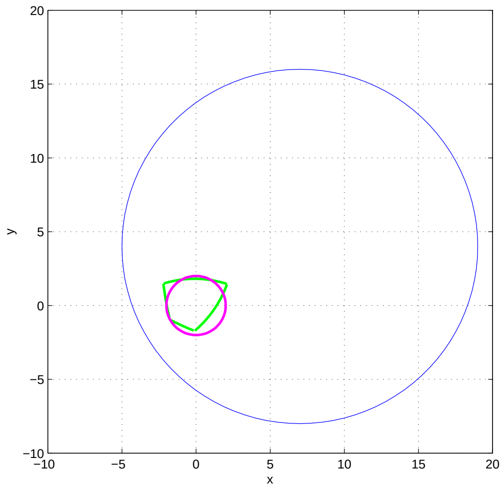

Fig. 5. *Case 3* : u_0^∞

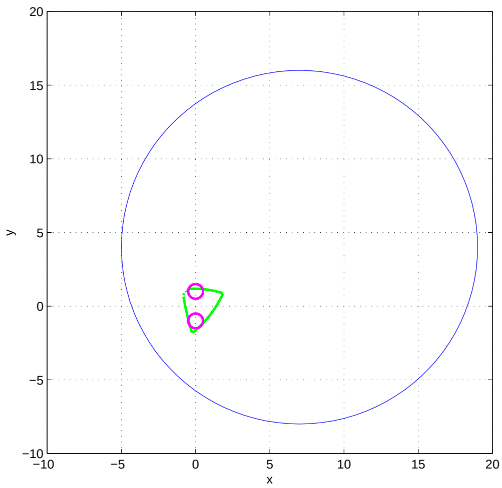

Fig. 6. *Case 4* : u_0^∞

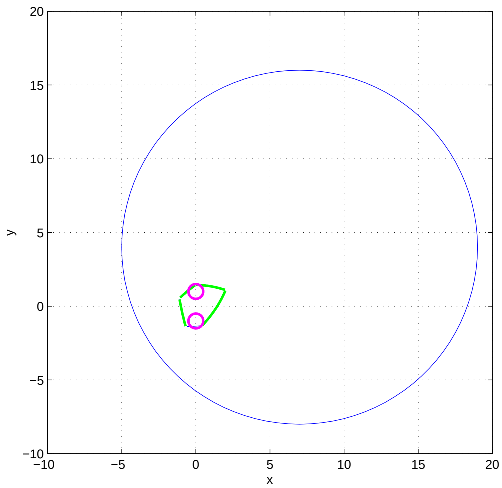

Fig. 7. *Case 4* : u_1^∞

7 Conclusion

The method we have presented provides a reasonable approximation of the vertical projection of an obstacle embedded in a finite depth ocean. One main advantage is that no *a priori* knowledge concerning the boundary conditions on the obstacle is required. Another advantage is that it can be carried out using one single incident wave, and one single frequency. Results are satisfactory even if only one far field pattern is used for identification. As expected, the results are worse than those obtained with techniques using several incident waves, and *a fortiori* many of them, say for instance as with the linear sampling method (see [6]). Furthermore, these results are also worse than those obtained in free space, because of the presence of evanescent modes. Let say that our method enables one to correctly obtain the horizontal position of the obstacle, and to suggest a good idea of the size of its vertical projection.

It would be interesting, in order to complete the information on the shape of the obstacle in the vertical direction, to carry out a method including two steps, in particular for obstacles of revolution. The first step would consist of determining the vertical projection of the obstacle with our method, and the second would consist of identifying the shape of the obstacle in the vertical direction with the help of a method like the one presented in [4].

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