Time harmonic acoustic scattering in presence of a shear flow and a Myers impedance condition

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Talk Abstract

Noise reduction of aircraft engines can be achieved by a well-suited internal coating of the nacelle, which is generally modelized by the Myers impedance boundary condition. We present a numerical method for the time harmonic acoustic radiation in a confined flow in presence of treated boundaries. The case of a uniform flow with the Myers condition in presence of PMLs (Perfectly Matched Layers) leads to a scalar formulation. Well-posedness is proved which ensures the convergence of the finite element discretization. To extend to a shear flow, the vectorial Galbrun's equation is used and we show that the Myers condition is natural and easy to incorporate in Galbrun's framework. We explain why the proof of the well-posedness is not so simple than in the scalar case, even for a uniform flow. Finally the difficulty is solved by replacing the Myers boundary condition by an interface with a solid medium or by enriching the Myers condition.

1 The scalar formulation for a uniform flow

We consider the radiation of an acoustic source f in an infinite 2D duct with treated boundaries (see Figure 1). The wave guide is filled with a compressible fluid in parallel and subsonic uniform flow $U_0(y)$. When the flow is uniform, the scattering problem is scalar. With the time dependence $e^{-i\omega t}$ omitted and in a dimensionless form, the height H of the duct being the reference length, the pressure perturbation p satisfies the convected Helmholtz equation $-\Delta p + \mathrm{D}^2 p = f$ in $\Omega_\infty = \{(x, y); 0 < y < 1\}$ where $\mathrm{D} = M\partial/\partial x - ik$ is the convective derivative

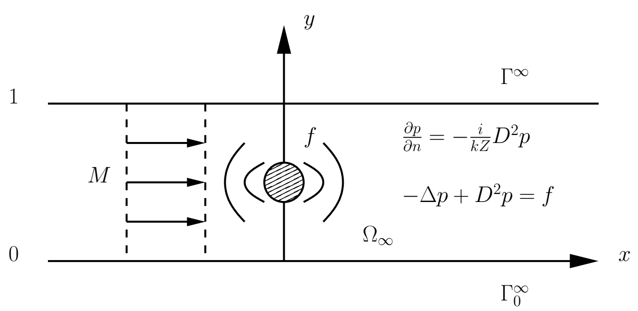

Figure 1: The pressure model

with $M = U_0/c_0$ the Mach number, $k = \omega H/c_0$ the dimensionless wave number and c_0 the speed of sound. The upper wall Γ^∞ is treated and the sound absorption is modelized by the Myers boundary condition [1]: $\partial p/\partial n = -(i/kZ)\mathrm{D}^2 p$ where the constant Z is the dimensionless impedance with $\Re e(Z) > 0$ (to get sound attenuation). The lower wall Γ_0^∞ is rigid $\partial p/\partial n = 0$.

To close the problem some radiation conditions must be imposed. Using the guided modes we have built Dirichlet-to-Neuman operators (DtN) allowing to write exact radiation conditions at finite distance from the source [2]. Another way to select the outgoing solution consists in introducing PMLs: computations are done in a bounded domain Ω composed of the physical domain $\Omega_b = \{(x, y); 0 < x < d, 0 < y < 1\}$ around the source and of surrounding layers $\Omega_\pm^L$ of length L (see Figure 2). The introduction of PMLs amounts to the transformation of the differential operator $\partial/\partial x \longrightarrow \alpha(x)\,\partial/\partial x$ in the governing equations of the problem. The complex function α is assumed to be unity in Ω_b and constant and equal to the complex scalar α^*, satisfying the following hypotheses $\mathrm{Re}(\alpha^*) > 0$, $\mathrm{Im}(\alpha^*) < 0$ (see [3] for a more thorough description and justification).

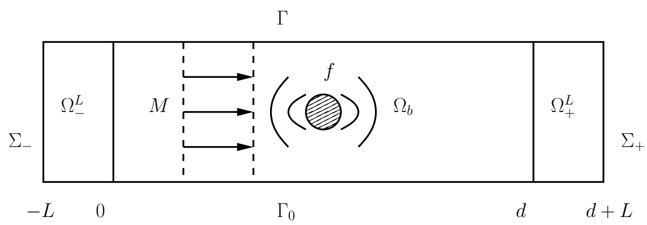

Figure 2: The problem with PMLs

Summing up, for a source $f \in L^2(\Omega)$ and $M < 1$ the radiation problem reads

$$\begin{cases} -\Delta_\alpha p + \mathrm{D}_\alpha^2 p = f \quad (\Omega), \\ \dfrac{\partial p}{\partial n} = -\dfrac{i}{kZ}{\mathrm{D}_\alpha}^2 p \quad (\Gamma) \text{ and } \dfrac{\partial p}{\partial n} = 0 \quad (\Gamma_0), \\ p = 0 \quad (\Sigma_\pm), \end{cases}$$

where $\mathrm{D}_\alpha = M\alpha\partial/\partial x - ik$ and has the equivalent variational form where $H^1_{\Sigma,0}(\Omega) = \{p \in H^1(\Omega),\ p|_{\Sigma_\pm} = 0\}$:

$$\left\{ \begin{array}{l} \text{Find } p \in U = \{p \in H^1_{\Sigma,0}(\Omega) \text{ and } p|_\Gamma \in H^1(\Gamma)\} \\ \text{such that } \ a(p,q) = \int_\Omega f\bar{q} \text{ for all } q \in U, \end{array} \right. \tag{1}$$

where the bilinear form $a(p,q)$ is defined as:

$$\int_\Omega \frac{1}{\alpha}\left(\boldsymbol{\nabla}_\alpha p \cdot \boldsymbol{\nabla}_\alpha \bar{q} - \mathrm{D}_\alpha p\, \overline{\mathrm{D}}_\alpha \bar{q}\right) - \frac{i}{kZ}\int_\Gamma \frac{1}{\alpha}\left(\mathrm{D}_\alpha p\, \overline{\mathrm{D}}_\alpha \bar{q}\right),$$

where $\overline{\mathrm{D}}_\alpha = M\alpha\partial/\partial x + ik$. To prove that the problem (1) is well-posed, we use the Fredholm alternative which ensures the convergence of a finite element discretization. $a(p,q)$ is found to be the sum of a compact part and a coercive part. Coerciveness is obtained by proving that $\exists C > 0$ such that $\forall p \in U$:

$$\left| \int_\Omega (1-M^2)\alpha \left|\frac{\partial p}{\partial x}\right|^2 + \frac{1}{\alpha}\left|\frac{\partial p}{\partial y}\right|^2 - \frac{i}{kZ}\int_\Gamma M^2\alpha\left|\frac{\partial p}{\partial x}\right|^2 \right|$$

$$\geq C\left(\int_\Omega |\boldsymbol{\nabla} p|^2 + \int_\Gamma \left|\frac{\partial p}{\partial x}\right|^2\right),$$

which is true for any value of the impedance as soon as the numerical parameter α^* is chosen such that $2|\arg(\alpha^*)| < \pi/2 - \arg Z$ (remember that $\arg(\alpha^*) < 0$).

On Figure 3 is represented the real part of the pressure radiated by a circular source with an upper treated boundary. The finite element library MELINA is used [4]. The solution computed with the DtN is used as a reference solution to illustrate the efficiency of the use of PMLs. Similar results have been observed in [2] for a potential flow.

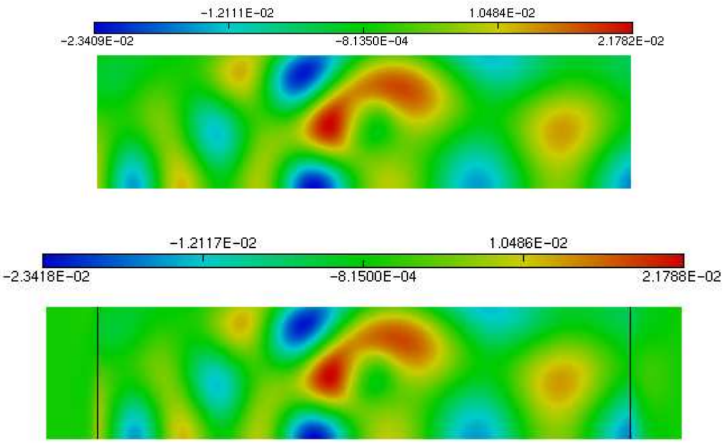

Figure 3: Real part of the acoustic pressure, DtN (top) PMLs (bottom), ($M = 0.3$, $Z = 1 - i$, $k = 7$)

2 The Galbrun's framework for a shear flow

In presence of a shear flow there is no scalar model adapted to a finite element discretization. In the case of rigid boundaries, we have proved that the use of the vectorial Galbrun's equation [5] is a very good alternative: we have developped numerical methods, using finite elements combined with perfectly Matched Layers, to determine the acoustic radiation in presence of a confined shear flow [6], [7] and a 2D complex flow [8], [9].

We will now extend Galbrun's equation to the presence of treated boundaries. We consider for simplicity the case of a uniform flow. Galbrun's equation is well suited to take into account a Myers condition because Galbrun's unknown is the displacement perturbation $\Re e\left(\boldsymbol{u}(x)e^{-i\omega t}\right)$ which is also the natural variable to express the Myers condition (continuity of normal displacement between the fluid and an absorbing solid medium). It reads $\operatorname{div}\boldsymbol{u} = ikZ\boldsymbol{u}\cdot\boldsymbol{n}$ on Γ. Note that in Galbrun's framework the writing of the impedance condition is attractive since it does not depend on the velocity of the flow.

Following [7], for an irrotational source $\operatorname{curl}\boldsymbol{f} = 0$ (then $\operatorname{curl}\boldsymbol{u} = 0$) the augmented Galbrun's equation reads $\mathrm{D}^2\boldsymbol{u} - \boldsymbol{\nabla}(\operatorname{div}\boldsymbol{u}) + \mathbf{curl}(\operatorname{curl}\boldsymbol{u}) = \boldsymbol{f}$. In presence of PMLs, the variational formulation associated to the augmented Galbrun's equation is:

$$\left\{ \begin{array}{l} \text{Find } \ \boldsymbol{u} \in V_0 \ \text{ such that } \ \forall \boldsymbol{v} \in V_0 \\ a_\alpha(\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{v}}, \end{array} \right. \tag{2}$$

where $V_0 := \left\{\boldsymbol{u} \in H^1_{\Sigma,0}(\Omega)^2;\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_0\right\}$ and

$$a_\alpha(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \frac{1}{\alpha}\left(\operatorname{div}_\alpha\boldsymbol{u}\ \operatorname{div}_\alpha\bar{\boldsymbol{v}} + \operatorname{curl}_\alpha\boldsymbol{u}\ \operatorname{curl}_\alpha\bar{\boldsymbol{v}}\right.$$

$$\left. - \mathrm{D}_\alpha\boldsymbol{u}\cdot\overline{\mathrm{D}}_\alpha\bar{\boldsymbol{v}}\right) - ikZ\int_\Gamma \frac{1}{\alpha}\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\left(\bar{\boldsymbol{v}}\cdot\boldsymbol{n}\right).$$

Following the same approach than in the scalar case, for Fredholm alternative to apply a necessary condition would be $|\arg(\alpha^*)| < \pi/2 + \arg Z$. Under this condition, the numerical implementation is straightforward: on Figure 4 is represented the real part of the horizontal component of the displacement radiated by a source $\boldsymbol{f} = \mathbf{1}$ in a circle of radius 0.1 centered in $x = 0$ and $y = 0.8$ in the square $[-1,1]\times[-1,1]$. On Figure 5 a guided mode, analytically determined $(\cos(\gamma y)exp(i\beta_0 x)$ with $\gamma = 1.5486 - 1.7707i$ and $\beta_0 = 5.3847 + 0.3411i)$ is imposed on the left vertical boundary. The error is of only 0.7% in L^2-norm.

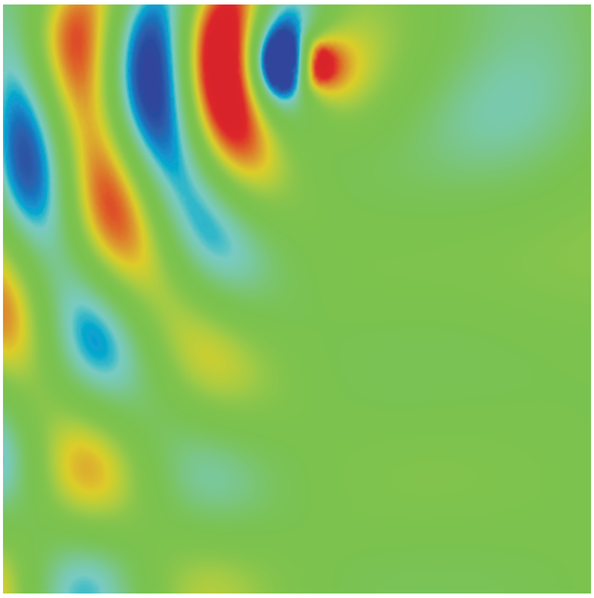

Figure 4: Real part of u_x for a volumic source $M = 0.5$, $Z = 3.5(1 - i)$, $k = 7$

Numerical tests give satisfying results but contrary to the situation with the scalar model, the theory is not satisfying. We will now clarify what are the difficulties to prove well-posedness and two improvements of the Myers condition which solve this difficulty will be proposed.

Figure 5: Real part of u_x for a surfacic source $M = 0.5$, $Z = 1 + i$, $k = 8$

3 Theory for a simplified problem

For the sake of clarity we consider a simpler problem: we take $\alpha = 1$ (no PMLs) everywhere and we only keep the terms with second order derivatives (other terms are compact perturbation and do not modify the conclusion).

3.1 Ill-posedness with the usual Myers condition

We are interested in solving the problem (2) with a_α replaced with $a(\boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{v})$ where

$$b(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \operatorname{div} \boldsymbol{u} \; \operatorname{div} \bar{\boldsymbol{v}} + \operatorname{curl} \boldsymbol{u} \; \operatorname{curl} \bar{\boldsymbol{v}} - M^2 \frac{\partial \boldsymbol{u}}{\partial x} \cdot \frac{\partial \bar{\boldsymbol{v}}}{\partial x}$$

and

$$c(\boldsymbol{u}, \boldsymbol{v}) = -ikZ \int_\Gamma (\boldsymbol{u} \cdot \boldsymbol{n}) \, (\bar{\boldsymbol{v}} \cdot \boldsymbol{n}) \, . \qquad (3)$$

To prove that this problem is well-posed in V we would like to establish the following coerciveness property: $\exists C > 0$ and $\delta \geq 0$ such that $\forall \boldsymbol{u} \in V$

$$|a(\boldsymbol{u}, \boldsymbol{u})| \geq C \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 - \delta \int_\Omega |\boldsymbol{u}|^2 \, . \qquad (4)$$

Let us emphasize that we have to consider an $H^1(\Omega)$ framework because of the convective term $\int_\Omega M^2 \partial \boldsymbol{u}/\partial x \cdot \partial \bar{\boldsymbol{v}}/\partial x$, which is not defined in an $H(\text{curl}) \cap H(\text{div})$ framework, which would be chosen in a similar problem like in electromagnetism [10].

In the rigid case (or $|Z| = \infty$), since $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, coerciveness is easy to prove because we have Costabel's identity [11]: $\forall \boldsymbol{u} \in W = \{\boldsymbol{u} \in V_0; \; \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$,

$$\int_\Omega |\operatorname{div} \boldsymbol{u}|^2 + |\operatorname{curl} \boldsymbol{u}|^2 = \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 \, .$$

Therefore $\forall M < 1$ the inequality (4) holds $\forall \boldsymbol{u} \in W$ with $C = 1 - M^2$ and $\delta = 0$.

With the Myers condition ($|Z| \neq \infty$), Costabel's identity is no longer true and despite the presence of the term $c(\boldsymbol{u}, \boldsymbol{v})$ the coerciveness fails. The contradiction comes from fields $\boldsymbol{u} = \boldsymbol{\nabla} \varphi$ with $\varphi \in H^1(\Omega)$, $\Delta \varphi = 0$ and $\partial \varphi / \partial n$ in $L^2(\Gamma)$ but with $\varphi \notin H^2(\Omega)$.

3.2 Two possible improvements

3.2.1 Well-posedness in the case of a fluid-structure coupling problem

The idea is to exploit the following result $\forall \gamma < 1$ and $\forall \boldsymbol{u} \in V_0$:

$$\int_\Omega |\operatorname{div} \boldsymbol{u}|^2 + |\operatorname{curl} \boldsymbol{u}|^2 + \frac{2}{\gamma} \, ||\boldsymbol{u} \cdot \boldsymbol{n}||^2_{H^{1/2}(\Gamma)} \geq (1 - \gamma) \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 \, . \qquad (5)$$

The inequality (5) means that replacing $c(\boldsymbol{u}, \boldsymbol{v})$ in definition (3) by a term like $(\boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n})_{H^{1/2}(\Gamma)}$ leads to a well-posed problem for M small enough. Such problem is not obtained by simply modifying the Myers boundary condition since such approach would lead to a non-local impedance boundary condition. It corresponds to a situation of practical interest, a 2D elastic layer in slipping contact with the fluid (see figure 6). For this fluid-structure coupling problem the working space becomes $V = \{\boldsymbol{u}|_\Omega \in V_0; \; \boldsymbol{u}|_{\Omega_S} \in H^1(\Omega_S)^2; \; [\boldsymbol{u} \cdot \boldsymbol{n}] = 0 \text{ on } \Gamma$ and $\boldsymbol{u} = \boldsymbol{0}$ on $\partial \Omega_s \backslash \Gamma\}$ where Ω_S denotes the solid layer. Thanks to inequality (5) combined with Korn's inequality in Ω_s and the continuity of the trace application from

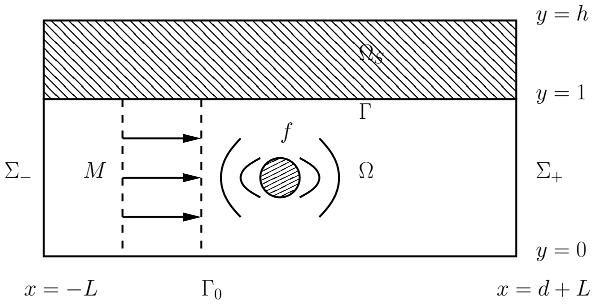

Figure 6: Fluid-solid coupling problem

$H^1(\Omega_S)$ to $H^{1/2}(\Gamma)$ we have proved that $\exists M_0 < 1$ such that $\forall M < M_0$ the fluid-structure problem is well-posed. The bound M_0 is linked to the characteristic parameters of the fluid and of the solid.

3.2.2 *Enriched Myers condition*

An alternative way to get a well-posed problem without considering an extra medium to the fluid is to enrich the Myers condition:

$$\operatorname{div} \boldsymbol{u} = ikZ \left[\boldsymbol{u} \cdot \boldsymbol{n} - \beta \frac{\partial^2}{\partial x^2} (\boldsymbol{u} \cdot \boldsymbol{n}) \right] \quad \text{on} \quad \Gamma,$$

with $\beta > 0$. Note that this new condition involves a second order tangential derivative, which was also the case for the "usual" Myers boundary condition when expressed versus the pressure. The enrichment consists in adding the term

$$\int_\Omega ikZ\beta \frac{\partial}{\partial x} (\boldsymbol{u} \cdot \boldsymbol{n}) \frac{\partial}{\partial x} (\bar{\boldsymbol{v}} \cdot \boldsymbol{n}),$$

to $c(\boldsymbol{u}, \boldsymbol{v})$ of equation (3) and in reducing the functional space to $V = \{\boldsymbol{u} \in V_0; \boldsymbol{u} \cdot \boldsymbol{n} \in H^1(\Gamma)\}$. To get coerciveness we use again the inequality (5). By the compactness of the embedding $H^1(\Gamma) \subset H^{1/2}(\Gamma)$, γ can be chosen as large as necessary and therefore the enriched problem is well-posed $\forall M < 1$ and $\forall \beta > 0$.

Notice that β can be chosen small, the limit $\beta \to 0$ corresponding to approach the usual Myers condition. Finally, including PMLs in the enriched model, it is possible to extend the results of the scalar model to the enriched model and to ensure the convergence of a finite element discretization under the condition $2|\arg(\alpha^*)| < \pi/2 + \arg Z$. Numerical illustrations are in progress.

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