Approximate Models for Wave Propagation Across Thin Periodic Interfaces

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Abstract

This work deals with the scattering of acoustic waves by a thin ring that contains regularly spaced inhomogeneities. We first explicit and study the asymptotic of the solution with respect to the period and thickness of the inhomogeneities using so-called matched asymptotic expansions. We then build simplified models replacing the thin ring with Approximate Transmission Conditions that are accurate up to third order with respect to the layer width. We pay particular attention to the study of these approximate models and the quantification of their accuracy.

Résumé

Cet article est consacré à l'étude de la diffraction d'une onde acoustique par un structure constituée d'un anneau mince de diélectrique contenant un grand nombre d'hétérogénéités disposées périodiquement. Nous commençons par écrire et justifier un développement asymptotique de la solution en fonction de l'épaisseur de l'anneau et de la distance entre les hétérogénéités. Puis, nous construisons des modèles approchés stables et bien posés d'ordres 2 et 3 dans lesquels l'anneau périodique est remplacé par une condition de transmission.

Keywords: effective transmission conditions, periodic thin interfaces, matched asymptotic expansions, homogenization, Helmholtz equation

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1. Introduction

This work is dedicated to the study of asymptotic models associated with acoustic waves scattering from thin rings that contain regularly spaced inhomogeneities. We are interested in situations where the thickness of the ring and the distance between two consecutive inhomogeneities are very small compared to the wavelength of the incident wave and the diameter of the ring. One easily understands that in those cases, numerical computation of the solution would become prohibitive as the small scale (denoted by) goes to 0, since the used mesh needs to accurately follow the geometry of the inhomogeneities. In order to overcome this difficulty, we shall derive so-called *Approximate Transmission Conditions* which are transmission conditions that only involve the traces of the field and of its normal derivative on the boundary of the ring and which yield on approximations of the exact solution that polynomially converge to the exact one (as ! 0). The numerical discretization of approximate problems is expected to be much less expensive than the exact one, since the used mesh has no longer to be constrained by the small scale.

The use of such approximate models is a rather classical topic in the modeling of wave propagation phenomena when a (geometrical) small scale is present (a typical reference is [1] from the engineering literature). As explained above, the main idea would be to replace an exact problem, which is difficult and expensive to numerically solve (basically due to the need of local mesh refinement imposed by the small scale), with an approximate one which is numerically much cheaper.

Without being exhaustive, let us indicate some works from the mathematical literature that share similarities with our problem or employed methods. For instance, first order approximate boundary conditions have been derived for electromagnetic scattering problems from perfect conductors coated with periodic thin structures in [2, 3] and [4] for the Maxwell equations in planar geometries, in [5] and [6] for the Helmholtz equation in circular and smooth geometries. Higher order conditions have been derived in [7] and [8] for the Laplace problem and in [9, 10] for the Helmholtz equation. The case of approximate transmission conditions has been studied in [11, 12] for perforated thin conductors and [13] for dielectric thin periodic layers for the Laplace equation: in [11] the two first terms of the asymptotic expansion of the solution has been obtained. The case of effective transmission conditions modeling highly conductive thin sheets is treated in [14]. The goal of this work is to complement the above mentioned studies in two directions. The first one is to provide a description of the asymptotic of the solution with respect to the small parameter in the case of thin circular periodic interfaces. We shall employ for that purpose a method which mixes the techniques of periodic-homogenization ([15, 16]) and the so-called matched asymptotic expansions. The latter method has been developed in ([17]) to treat singular perturbation problems which arise in fluid mechanics. A standard work on the matched asymptotic expansions applied to the Helmholtz equation can be found in [18, 19] and complex situations are studied in [20]. For recent applications, we refer the reader to [21, 22]. In these first investigations, we shall restrict ourselves to the scalar scattering problem modeled by the Helmholtz equation in two dimensions and only consider the first three terms of the asymptotic expansion. A complete description of the asymptotic expansion is the subject of a forthcoming paper [23]. The second main contribution is the derivation of *variational* and *stable* approximate interface conditions which are accurate up to $O(\varepsilon^2)$ and up to $O(\varepsilon^3)$ errors. We show that natural conditions (those directly provided from the asymptotic of the exact solution) may lead to unstable problems (with respect to the small parameter). Using a shifting of the interfaces we obtain a new family of approximate transmission conditions that become stable for appropriate choice of the shifting parameter. We give a detailed discussion on the stability properties of the obtained conditions in terms of the small parameter and the frequency of the problem. We show for instance that while centered conditions stability properties are frequency dependent, the noncentred ones are uniformly stable with respect to the frequency. Up to our knowledge, this type of instability and the proposed solutions have not been introduced before in the literature related to scattering problems.

The remaining of this article is organized as follows. We describe in Section 2 the setting of the problem and introduce some notation. Section 3 is dedicated to the formal construction of a matched asymptotic expansion of the solution. Section 4 contains the rigorous justification of this expansion up to the third order. The derivation and error analysis of approximate transmission conditions is done in Section 5.

2. Setting of the Problem

In these first investigations, we shall restrict ourselves to the scalar scattering problem modeled by the Helmholtz equation in two dimensions. Let μ_ε and ρ_ε denote the acoustical characteristics of the medium Ω where ε is a small parameter that will be introduced later. Then the acoustic field u_ε satisfies:

$$\nabla \cdot (\mu_\varepsilon \nabla u_\varepsilon) + \omega^2 \rho_\varepsilon u_\varepsilon = f \text{ in } \Omega; \tag{1}$$

where ω denotes the pulsation of time variation and f denotes a given source term. In order to simplify the exposure of the error analysis we shall assume that Ω is a disk:

$$\Omega = \left\{ (x;y) \in \mathbb{R}^2; \sqrt{x^2 + y^2} < R_e \right\}:$$

Moreover, we replace the radiation condition satisfied by u_ε by the following impedance condition on the boundary of Ω denoted by S_{R_e}:

$$\frac{\partial u_\varepsilon}{\partial r} + i\omega u_\varepsilon = 0 \text{ on } S_{R_e}: \tag{2}$$

Readers who are familiar with scattering problems can be easily convinced that the theoretical treatment of the scattering problem in $\mathbb{R}^2$ (where (2) is replaced by the Sömmerfeld radiation condition) can be deduced with minor modifications.

We shall denote by $(r > 0; \theta \in [-\pi; \pi))$ the polar coordinates in the $(x;y)$ plane. We assume that the medium Ω is made of a thin ring of mean radius R and thickness ε, namely $\Omega_R^\varepsilon := \{|r - R| < \varepsilon/2\}$ embedded in a homogeneous medium (μ_∞, ρ_∞). The parameter ε is a small geometrical parameter that can be arbitrarily close to 0 in the sequel. This ring contains many regularly spaced inhomogeneities in the angular direction, which means in particular that and are periodic with respect to (see Figure 1). The size of the inhomogeneities, as well as the spacing between two inhomogeneities will be supposed to be proportional to the same small parameter . More precisely we assume that there exist two functions

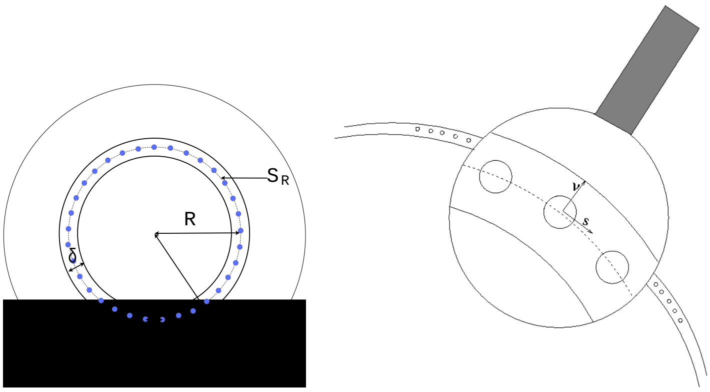


Figure 1: The domain (left) and a zoom on the periodic ring (right)

$: \mathbb{R} \quad \mathbb{R} \mapsto \mathbb{R}^+$ and $: \mathbb{R} \quad \mathbb{R} \mapsto \mathbb{R}^+$ of the scaled variables $(s; \ )$ satisfying

$$\begin{cases} (\ ; s+1) = (\ ; s); \\ (\ ; s+1) = (\ ; s); \end{cases} \quad \text{and} \quad \begin{cases} (\ ; s) = \ _\infty \\ (\ ; s) = \ _\infty \end{cases} \quad \text{if } | \ | > \frac{1}{2}; \tag{3}$$

and such that

$$(r; \ ) = \frac{r \quad R}{\ }; \frac{R}{\ } \quad \text{and} \quad (r; \ ) = \frac{r \quad R}{\ }; \frac{R}{\ } \quad \text{for } r > 0 \text{ and } \ \in [0; 2 \ ]. \tag{4}$$

We further make standard assumptions material properties bounds,

$$\begin{cases} \exists (\ _m; \ _M) \in \mathbb{R}^2; \quad 0 < \ _m < \ < \ _M; \\ \exists (\ _m; \ _M) \in \mathbb{R}^2; \quad 0 < \ _m < \ < \ _M: \end{cases} \tag{5}$$

Finally we shall assume that the support of the source term f does not intersect the thin ring $_R$.

Philosophy behind approximate interface conditions. We first recall that the variational problem associated with (1) and (2) can be written as: find $u \in H^1(\ )$ such that

$$a \ (u \ ; v) = L(v) \quad \forall v \in H^1(\ ); \tag{6}$$

where $a \ (u; v) := \int \ \nabla u \ \nabla \bar{v} \quad \omega^2 \ u\bar{v} \ dx + i\omega \ _\infty \int_{S_{R_e}} u\bar{v} \ ds$ and $L(v) := \int f \ \bar{v} \ dx$:

This problem is well-posed and is stable uniformly with respect to .

Proposition 1. *Problem (6) is well-posed. Moreover, there exists a constant C independent of such that,*

$$\|u \ \|_{H^1(\ )} \quad C \sup_{v \in H^1(\ ); v \neq 0} \frac{|a \ (u \ ; v)|}{\|v\|_{H^1(\ )}} \quad \forall u \in H^1(\ ): \tag{7}$$

The proof is standard and is given in [24].
We remark that a can be split into two parts:

$$\int_{\ \setminus \ _R} \ _\infty \nabla u \ \nabla \bar{v} \quad \omega^2 \ _\infty u\bar{v} \ dx + i\omega \ _\infty \int_{S_{R_e}} u\bar{v} \ ds; \tag{8}$$

$$\int_{\ _R} \ \nabla u \ \nabla \bar{v} \quad \omega^2 \ u\bar{v} \ dx: \tag{9}$$

When approximating problem (6) using finite elements, the main problem comes from the approximation of the term (9), since ε_δ and μ_δ have fast variations. The goal of approximate transmission conditions would be to replace this term by a boundary integral of the form

$$\int_{S_R} \mathcal{B}_j \begin{pmatrix} u(R^+,\theta) \\ u(R^-,\theta) \end{pmatrix} \cdot \begin{pmatrix} \overline{v}(R^+,\theta) \\ \overline{v}(R^-,\theta) \end{pmatrix} ds, \tag{10}$$

where $\mathcal{B}_j$ is a local boundary operator that takes into account the characteristics of the medium inside the small ring. Roughly speaking, we say that $\mathcal{B}_j$ is a transmission operator of order j if (10) approximates (9) up to $O(\delta^{j+1})$ error. The larger is the value of j, the more complicated is the expression of $\mathcal{B}_j$. In principle, it would be possible to build $\mathcal{B}_j$ for any order j. However, in practice, calculations become extremely heavy for $j \geq 3$. We shall restrict ourselves here to $j = 1$ and $j = 2$.
The process of obtaining the $\mathcal{B}_j$ is based on two main steps.

- In the region $|r - R| \gg \delta$ we first prove that the solution has a polynomial asymptotic expansion of the form
$$u_\delta = \sum_{n \in \mathbb{N}} \delta^n u_n.$$
This step is very technical. The rigorous analysis employs the technique of matched asymptotic expansions, and indeed requires introducing the expansion of the field in the region $|r - R| \leq \delta$.

- Then, in order to derive an approximate model of order j, we truncate the asymptotic expansion at $n = j$, and consider $u_j := \sum_{n=0}^{j} \delta^n u_n$. Then we shall observe, from analytical expressions of the near fields, the existence of $\mathcal{B}_j$ such that
$$\int_{r=R} \mathcal{B}_j \begin{pmatrix} u(R^+,\theta) \\ u(R^-,\theta) \end{pmatrix} \cdot \begin{pmatrix} \overline{v}(R^+,\theta) \\ \overline{v}(R^-,\theta) \end{pmatrix} ds = \int_{\Omega_R} \left(\nabla u_j \cdot \nabla \overline{v} - \omega^2 u_j \overline{v} \right) dx + O(\delta^{j+1}).$$

As we shall notice, possible expressions for $\mathcal{B}_j$ are not unique. The main difficulty is to derive ones that have "good" stability properties, namely ones for which the approximate solution satisfies an uniform stability estimate similar to (7). Unfortunately we were not able to come up with a systematic procedure to automatically derive "good" expressions for $\mathcal{B}_j$. The ones provided in the last section have been suggested by the difficulties encountered when studying the well-posedness of the approximate model associated with the "natural" expressions provided by the asymptotic expansion.
Let us end this part by introducing some short notation that will be useful in designing the approximate transmission conditions. The interior domain Ω^-, exterior domain Ω^+ and interface S_R are defined as

$$\Omega^- := \left\{ (x,y) \in \mathbb{R}^2; \sqrt{x^2+y^2} < R \right\};$$
$$\Omega^+ := \left\{ (x,y) \in \mathbb{R}^2; R < \sqrt{x^2+y^2} < R_e \right\};$$
$$S_R := \left\{ (x,y) \in \mathbb{R}^2; \sqrt{x^2+y^2} = R \right\}.$$

Let $u \in H^1(\Omega^+) \cap H^1(\Omega^-)$. We abbreviate the exterior and interior values of u on the interface S_R by u^+ and u^-:

$$u^+(\theta) := u(R^+,\theta); \qquad u^-(\theta) := u(R^-,\theta).$$

The jump and mean values across S_R respectively denoted by $[u]$ and $\langle u \rangle$ are defined by:

$$[u] := u^+ - u^-; \qquad \langle u \rangle := \frac{1}{2}\left(u^+ + u^- \right). \tag{11}$$

3. Formal Asymptotic Expansion of the solution

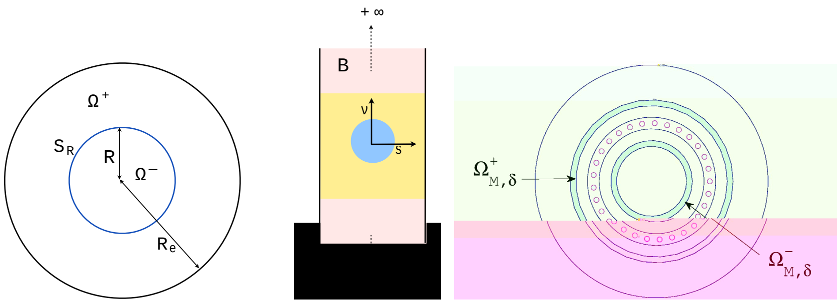
3.1. The formal ansatz

In order to build approximate models, we first seek a complete description of the asymptotic behavior of u_ε as ε goes to zero. The natural form would be an expansion of u_ε in powers of ε. However, due to the fast variations with respect to angular coordinates, it is not possible to write a uniform expansion of the solution in the whole domain Ω. Roughly speaking, the solution u_ε oscillates more rapidly in the vicinity of the periodic ring than far from it. In order to take into account these two distinct behaviors we shall employ the technique of matched asymptotic expansion. It consists in considering two distinct expansions of the solution respectively in the far field zone ($|r - R| \gg \varepsilon$) and in the near-field zone ($|r - R| \ll \varepsilon$), then match the two expansions in an intermediate zone $\varepsilon \ll |r - R| \ll 1$. More precisely we make the following ansatz:

Far from the periodic ring, we assume that a standard power series expansion in ε holds:

$$u_\varepsilon(r,\theta) = \begin{cases} \displaystyle\sum_{n\in\mathbb{N}} \varepsilon^n u_n^+(r,\theta) & r \geq R, \\ \displaystyle\sum_{n\in\mathbb{N}} \varepsilon^n u_n^-(r,\theta) & r \leq R. \end{cases} \tag{12}$$

where the far field terms $u_n^\pm$ are defined in $\Omega^\pm$ (see figure2(a)). We shall denote by $u_n : \Omega \to \mathbb{C}$ whose restriction to $\Omega_\pm$ is $u_n^\pm$.



(a) Ω^+ and Ω^- (b) B (c) Overlapping zones

Figure 2: Domains for far and near field terms

Near the periodic ring, we have to take into account the periodicity of θ and ε. That is why we use a more complicated ansatz inspired by the theory of periodic homogenization [25, 2, 7],

$$u_\varepsilon(r,\theta) = \sum_{n\in\mathbb{N}} \varepsilon^n U_n\left(\frac{r-R}{\varepsilon}, \frac{R\theta}{\varepsilon}; \theta\right), \tag{13}$$

where the near field terms $U_n(s,\nu;\theta) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ are 1-periodic functions with respect to s. Notice that these functions will then be systematically identified to their restrictions to

$$B^* := B \times [0; 2\pi] \quad \text{where} \quad B :=]-1/2; 1/2[\times \mathbb{R} \tag{14}$$

is the infinite periodicity cell of Figure 2(b).

The expansions (13) and (12) are assumed to be also valid in two overlapping zones $\Omega^+_{M,\delta}$ and $\Omega^-_{M,\delta}$ defined by (see Figure 2(c)):

$$\Omega^+_{M,\delta} := \{(r,\theta) \in \mathbb{R}^2,\ M\varepsilon \leq r - R \leq \delta\},$$

$$\Omega^-_{M,\delta} := \{(r,\theta) \in \mathbb{R}^2,\ M\varepsilon \leq -(r - R) \leq \delta\},$$

where, $\eta^\pm$ are chosen such that $0 < \eta^- < \eta^+$ and,

$$\lim_{\delta\to 0} \eta^\pm = 0; \qquad \lim_{\delta\to 0} \frac{\eta^\pm(\delta)}{\delta} = +\infty : \tag{15}$$

For instance, $\eta^-(\delta) = \sqrt{\delta}$ and $\eta^+(\delta) = 2\sqrt{\delta}$ would be convenient.

Let us notice that, using properties (15) of η, for the near field, overlapping areas correspond to ν going to $\pm\infty$. On the contrary, for the far field, the overlapping areas correspond to $r \to R$.

A detailed analysis of the behavior of far and near fields will allow us to find conditions that match far and near field expansions in the overlapping zones.

Validity of expansions (12) and (13) will be fully justified by the error analysis of Section 4.

In the two following sections, we shall formally derive the equations satisfied by far fields terms $u_n^\pm$ and near fields terms U_n.

3.2. Identification of far fields and near fields equations

Far fields equations. The derivation of these equations is immediate. Substituting u^δ by its far field expansion in (12) (1) and (2) and formally separating the different powers of δ, we obtain the equations satisfied by the far fields terms $u_n^\pm$.

$$\begin{cases} \mu_\infty \Delta u_n^\pm + \omega^2 \varepsilon_\infty u_n^\pm = \begin{cases} f & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} & \text{in } \Omega^\pm; \\ \dfrac{\partial u_n^+}{\partial r} + i\omega u_n^+ = 0 & \text{on } S_{R_e}: \end{cases} \tag{16}$$

We emphasize that $u_n^\pm$ are not entirely defined since we did not prescribe yet boundary conditions on S_R. Two scalar functions of the variable θ are needed to fix traces of u_n^+ and u_n^- on S_R.

Near fields equations. As expected, due to the scaling appearing in (13), the derivation of these equations is much more involved that for the near fields. Let us introduce some useful notation. To any function of three variables $U(s;\nu;\theta)$ we shall associate the function of two variables $(U)^\delta(r;\theta)$ such that

$$(U)^\delta(r;\theta) := U\left(\frac{r-R}{\delta}; \frac{R\theta}{\delta}; \theta\right): \tag{17}$$

The calculations of this section are essentially based on the following differentiation rules

$$\frac{\partial (U)^\delta}{\partial r} = \frac{1}{\delta}\left(\frac{\partial U}{\partial \nu}\right)^\delta; \quad \frac{\partial (U)^\delta}{\partial \theta} = \left(\frac{\partial U}{\partial \theta} + \frac{R}{\delta}\frac{\partial U}{\partial s}\right)^\delta \tag{18}$$

that we shall also rewrite for simplicity as (this defines the symbol $\equiv$),

$$\frac{\partial (U)^\delta}{\partial r} \equiv \frac{1}{\delta}\frac{\partial U}{\partial \nu}; \quad \frac{\partial (U)^\delta}{\partial \theta} \equiv \frac{\partial U}{\partial \theta} + \frac{R}{\delta}\frac{\partial U}{\partial s}: \tag{19}$$

Setting $U_n^\delta := (U_n)^\delta$, ansatz (13) can be rewritten as

$$u^\delta(r;\theta) = \sum_{n\in\mathbb{N}} \delta^n U_n^\delta(r;\theta): \tag{20}$$

Setting $r^2 \quad := r^2 \; r \,:(\quad r\,) + \quad \omega^2$, using the expression of the Laplace operator in polar coordinates and replacing r by $R + \quad$, one easily computes that, thanks to (18)

$$
\begin{aligned}
r^2 \quad U_n \quad & {}^{-2} \quad R^2 \frac{\partial}{\partial} \quad \frac{\partial U_n}{\partial} + R^2 \frac{\partial}{\partial s} \quad \frac{\partial U_n}{\partial s} \\
& + \quad {}^{-1} \quad 2R \quad \frac{\partial}{\partial} \quad \frac{\partial U_n}{\partial} + R \frac{\partial U_n}{\partial} + R \frac{\partial}{\partial s} \quad \frac{\partial U_n}{\partial} + R \frac{\partial^2 U_n}{\partial s \partial} \\
& + \quad {}^{0} \quad \frac{\partial U_n}{\partial} + {}^2 \frac{\partial}{\partial} \quad \frac{\partial U_n}{\partial} + \frac{\partial^2 U_n}{\partial^2} + R^2 \; \omega^2 U_n \\
& + \quad 2\, R \; \omega^2 U_n \\
& + \quad {}^2 \quad {}^2 \; \omega^2 U_n :
\end{aligned} \tag{21}
$$

Since u solves the homogeneous Helmholtz equation in the vicinity of the periodic ring, we have formally

$$
\sum_{n \in \mathbb{N}} {}^n \, r^2 \quad U_n = 0:
$$

Introducing (21) in the previous equation, and collecting the terms in $\;^n$, we obtain the equations for the near fields U_n that can be written in the following form (we adopt the convention that $U_n = 0$ if $n < 0$):

$$
\boxed{A_0(\frac{\partial}{\partial};\frac{\partial}{\partial s})U_n = \; A_1 U_{n-1} \;\; A_2 U_{n-2} \;\; A_3 U_{n-3} \;\; A_4 U_{n-4} \quad \text{in } R^2 \quad [0;2\;]} \tag{22}
$$

where $A_0(\frac{\partial}{\partial};\frac{\partial}{\partial s})$ is the differential operator in $(\;;s)$ given by

$$
A_0(\frac{\partial}{\partial};\frac{\partial}{\partial s})U := R^2 \quad \frac{\partial}{\partial}(\quad \frac{\partial U}{\partial}) + \frac{\partial}{\partial s}(\quad \frac{\partial U}{\partial s}) \tag{23}
$$

while the operators $A_j; 1 \quad j \quad 4$ are the differential operators in $(\;;s;\;)$ given by

$$
\begin{aligned}
& A_1 U := A_1^{(1)} \frac{\partial U}{\partial} + A_1^{(0)} U; \quad \text{with} \begin{cases} A_1^{(0)} U := 2R \quad \frac{\partial}{\partial} \quad \frac{\partial U}{\partial} + R \quad \frac{\partial U}{\partial} \\ A_1^{(1)} U := R \frac{\partial}{\partial s} \quad U \quad + \; R \;\frac{\partial U}{\partial s}; \end{cases} ; \\
& A_2 U := A_2^{(0)} U + \quad \frac{\partial^2 U}{\partial^2}; \quad \text{with} \quad A_2^{(0)} U := {}^2 \frac{\partial}{\partial} \quad \frac{\partial U}{\partial} + \quad \frac{\partial U}{\partial} + \omega^2 \, R^2 U; \\
& A_3 U := 2\, R \omega^2 \; U; \\
& A_4 U := \; {}^2 \omega^2 \; U:
\end{aligned} \tag{24}
$$

Equations (22) are complemented with the periodicity conditions

$$
U_n(\;;s+1) = U_n(\;;s): \tag{25}
$$

The fact that the operator $A_0(\frac{\partial}{\partial};\frac{\partial}{\partial s})$ is a differential operator only in the variables $(\;;s)$ will play an essential role in the process of decoupling the "fast" variables $(\;;s)$ from the variable $\;$. Indeed in the resolution of (22, 25), the variable will play the role of a parameter (via the right hand side).

As for the near fields, (22, 25) does not allow to entirely define (inductively) the U_n's. More precisely, we shall need to prescribe their behavior for large . First, we exclude exponentially growing (in) solutions, that can be expressed by the fact that we shall look for the functions $U_n(\ ;\ )$ in the space

$$V(B) = \{ U \in H^1_{loc}(B) \,/\, e^{-|\ |^{\frac{1}{2}}} U \text{ is bounded} \}.$$

In we define

$$\mathrm{Ker}\, A_0 = \Big\{ U \in V(B) \,/\, A_0(\frac{\partial}{\partial}, \frac{\partial}{\partial s}) U = 0; u \text{ satisfies } (25) \Big\}$$

(22, 25) defines U_n up to a function of $\mathrm{Ker}\, A_0$. It remains to describe more precisely $\mathrm{Ker}\, A_0$. In the trivial case where the functions and are constant, one easily sees using separation of variables in $(s;\ )$ that

$$\mathrm{Ker}\, A_0 = \mathrm{span}\{1;\ \}.$$

In the general case, $\mathrm{Ker}\, A_0$ is still of dimension 2. Obviously, it still contains the constant functions, but the function has to be replaced by a function that behaves as when goes to infinity. In the following, it will be useful to split B into three parts, as illustrated in Figure 2(b),

$$B = B_- \cup B_0 \cup B_+; \quad B_\pm = \{(\ ;s) \in B \,/\, \quad 1/2\}; \quad B_0 = \{(\ ;s) \in B \,/\, |\ | < 1/2\};$$

and introduce the two artificial interfaces between B_0 and $B_\pm$:

$$_\pm = \Big\{ (\ \frac{1}{2}; s); \ \frac{1}{2} < s < \frac{1}{2} \Big\}$$

Lemma 2. *There exists a unique function $N \in \mathrm{Ker}\, A_0$ such that*

$$|N(\ ;s) \quad | \text{ is bounded in } B, \tag{26}$$

$$\int_{_+} N\, ds + \int N\, ds = 0 \tag{27}$$

and

$$\mathrm{Ker}\, A_0 = \mathrm{span}\{1; N\}. \tag{28}$$

Remark 3. *In fact, (26) defines N up to an additive constant. The condition (27) allows us to fix this constant. It has been chosen in order to simplify later the writing of the matching conditions. For the following, we shall set:*

$$N_\infty := \int_{_+} N\, ds = \int N\, ds. \tag{29}$$

Proof. One possible approach for the proof of this results consists in reducing the problem of finding N to the bounded domain via non homogeneous DtN conditions as in [22]. We shall prefer here a direct variational approach in weighted Sobolev spaces. Let us simply sketch the proof. We introduce the weighted-periodic functional space $W^1(B)$

$$W^1(B) := \Big\{ U \in H^1_{loc}(B); \frac{U}{(1+\ ^2)^{\frac{1}{2}}} \in L^2(B); U(\ ;\ \frac{1}{2}) = U(\ ;\ \frac{1}{2}) \Big\}; \tag{30}$$

equipped with the inner product

$$(U;V)_{W^1} := \int_B \nabla U \cdot \nabla \overline{V}\, ds\, d \; + \int_B \frac{U \overline{V}}{1+\ ^2}\, ds\, d\ : \tag{31}$$

Note that the constant functions belong to $W^1(B)$. Next we introduce the closed subspace of $W^1(B)$

$$W^1_0(B) := \big\{ U \in W^1(B); U \text{ satisfies } (27) \big\} \tag{32}$$

It is well known that, by Hardy's inequality (see [26]), the semi-norm

$$|U|_{W_0^1} := \left(\int_B |\nabla U|^2 \, ds \, d \right)^{\frac{1}{2}} \tag{33}$$

in a norm on $W_0^1(B)$, equivalent to the norm in $W^1(B)$.

The next step consists in noticing that

$$(26;27) \iff \tilde{N} := N \quad \in W_0^1(B)$$

This property is obtained by looking at the Fourier series expansion (with respect to s) of the function N in the domains $B_\pm$ inside which N is harmonic (see for instance the proof of Proposition 4). Choosing $\tilde{N}$ as a new unknown, finding N amounts to finding $\tilde{N} \in W_1^0(B)$ such that

$$A_0\left(\frac{\partial}{\partial}; \frac{\partial}{\partial s}\right)\tilde{N} = R^2 \frac{\partial}{\partial} \quad \text{in } D'(R^2); \quad N(\,;s+1) = N(\,;s):$$

The existence and uniqueness of $\tilde{N}$, thus N, is then a direct consequence of general result in Proposition 7 (take $f = 0$ and $g = R^2(\ _\infty \quad);0)^t$.

To prove that $\operatorname{Ker} A_0 = \operatorname{span}\{1; \ \}$, we first show that if $U \in \operatorname{Ker} A_0$, then there exists $a \in R$ such that $U - aN \in W^1(B)$ (see below). Indeed, using the Fourier series (in s) expansion of U in the domains $B_\pm$ in which U is harmonic, we can show the existence of $a^\pm$ such that

$$U = a^\pm \quad + o(|\ |^{-\infty});$$

where $o(|\ |^{-\infty})$ is defined in (43). Then, it is sufficient to show that $a^+ = a^-(:= a)$ which is obtained formally by integrating the equality

$$A_0\left(\frac{\partial}{\partial}; \frac{\partial}{\partial s}\right)U = 0$$

over the bounded domain B_0. This can be made rigorous with an appropriate truncation process. The details are left to the reader.

Then it suffices to remark that (using Green's formula and the density of compactly supported function in $W^1(B)$ [27]), $\operatorname{Ker} A_0 \cap W^1(B)$ is nothing but the space of constant functions. □

The above result means that (22) determines inductively U_n up to two additive constants in $(\ ;s)$, that is to say two functions of the variable . We have the same degree of indetermination as for the far fields.

To determine entirely $(u_n^+; u_n^-; U_n)$, at each step n, we thus need a priori $4 = 2 + 2$ additional equations relating functions of $\ \in [0;2\ ]$.

3.3. Matching conditions

As already said, to complete the construction of the terms of our formal expansions, we need additional matching conditions that will be obtained by saying that the expansions (12) and (13) (or (20)) must coincide in the two overlapping zones $^\pm_{M;}$.

To make this identification, we shall use for convenience the coordinates $(\ ;\ )$ instead of $(r;\ )$ and particular series expansions of the near and far fields u_n and U_n.

Series expansions for the far fields. In addition to equations (16), we impose the condition that $u_n^\pm$ is smooth at the vicinity of the interface S_R. This allows us to use simple Taylor expansions, i. e.

$$u_n^\pm(R + \ ;\ ) = \sum_{k \in \mathbb{N}} \frac{^k\ ^k}{k!} \frac{\partial^k u_n^\pm}{\partial r^k}(R;\ ): \tag{34}$$

Series expansions for the near fields. For the near fields, we use the Fourier series expansion in s of

For $\ell \neq 0$; $\quad A_{1;\ell}\,U_{n-1;\ell} + A_{2;\ell}\,U_{n-2;\ell} + A_{3;\ell}\,U_{n-3;\ell} + A_{4;\ell}\,U_{n-4;\ell} \in e^{\mp 2\pi|\ell|\rho}\,P_{2n-1}$

For $\ell = 0$; $\quad A_{1;0}\,U_{n-1;0} + A_{2;0}\,U_{n-2;0} + A_{3;0}\,U_{n-3;0} + A_{4;0}\,U_{n-4;0} \in P_{n-1}$

Then, using (36) and property (40), we deduce that

For $\ell \neq 0$; $\quad U_{n;\ell} \in e^{\mp 2\pi|\ell|\rho}\,P_{2n}$;

For $\ell \neq 0$; $\quad U_{n;0} \in P_{n+1}$;

which is nothing but (41) using (35). □

In order to simplify the notation, we shall rewrite (41) as

$$\forall\, \theta \in [0; 2\pi]; \quad U_n(\rho; s; \theta) = p_n^{\pm}(\rho; \theta) + o(|\rho|^{-\infty}); \quad p_n^{\pm}(\rho; \theta) = \sum_{k=0} p_{n;k}^{\pm}(\theta)\,\rho^k: \tag{42}$$

where θ plays the role of a parameter and $o(|\rho|^{-\infty})$ is a canonical notation for any smooth function $\varphi(\rho; s)$ in $V(B)$ satisfying the following property

$$\begin{cases} \varphi(\rho; s) = o(|\rho|^{-\infty}) \iff \exists\; k \in \mathbb{N} \text{ and } \{\varphi_\ell \in P_k;\ \ell \in \mathbb{Z}^*\} \text{ such that} \\ \varphi(\rho; s) = \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \varphi_\ell(\rho)\, e^{\mp 2\pi|\ell|\rho}\, e^{2i\pi\ell s} \quad \text{in } B_{\pm} \end{cases} \tag{43}$$

where the above series are uniformly convergent in $B_{\pm}$. Thus, such functions are C^{∞} in $B_{\pm}$ and satisfy

$$\begin{cases} \text{(i)} \quad \forall (p;q) \in \mathbb{N}^2;\ \exists\, C_{pq} > 0 \;/\; \left|\dfrac{\partial^{p+q}\varphi}{\partial\rho^p \partial s^q}(\rho; s)\right| \le C_{pq}\, e^{-|\rho|}; \quad \text{in } B_{\pm}; \\ \text{(ii)} \quad \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(\rho; s)\, ds = 0; \quad \forall\, |\rho| > \frac{1}{2}: \end{cases} \tag{44}$$

Notice that

$$\begin{cases} \forall\, m \in \mathbb{N};\ \varphi = o(|\rho|^{-\infty}) \implies \rho^m \varphi = o(|\rho|^{-\infty}); \\ \forall (p;q) \in \mathbb{N}^2;\ \varphi = o(|\rho|^{-\infty}) \implies \dfrac{\partial^{p+q}\varphi}{\partial\rho^p \partial s^q}(\rho; s; \theta) = o(|\rho|^{-\infty}): \end{cases} \tag{45}$$

and that (the proof is similar to the proof of Proposition 4)

$$\begin{cases} U \in V(B) \text{ and } A_0\left(\dfrac{\partial}{\partial\rho}; \dfrac{\partial}{\partial s}\right) U = V \text{ with } V = o(|\rho|^{-\infty}) \\ \implies \exists\; (a;b) \in \mathbb{C}^2 \text{ such that } U = a + bN + o(|\rho|^{-\infty}): \end{cases} \tag{46}$$

As we shall see, in the formula (42), only the polynomials $p_n^{\pm}(\rho; \theta)$, i. e. the coefficients $p_{n;k}^{\pm}(\theta)$, play a role in the matching conditions.

Remark 6. *With the help of the notation* $o(|\rho|^{-\infty})$*, we can describe the behavior at infinity more precisely by (see also Remark 3)*

$$N(\rho; s) = \rho - N_{\infty} + o(|\rho|^{-\infty}): \tag{47}$$

Obtaining the matching conditions. Everything in this section is quite formal but will be justified a posteriori. We substitute (34) and (4) respectively in (12) and (13) to obtain on the one hand:

$$u_\delta(r,\theta) = \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} \delta^n \frac{(r-R)^k}{k!}\frac{\partial^k u_n^{\pm}(R,\theta)}{\partial r^k}, \quad (r,\theta) \in \Omega^{\pm}_{M,\delta}, \tag{48}$$

and on the other hand,

$$u_\delta(r,\theta) = \sum_{n=0}^{+\infty}\sum_{k=0}^{n+1} \delta^{n-k}(r-R)^k p^{\pm}_{n,k}(\theta) + o\left(\delta/|r-R|\right)^{+\infty}, \quad (r,\theta) \in \Omega^{\pm}_{M,\delta}, \tag{49}$$

where $o\left(\delta/|r-R|\right)^{+\infty} = \sum_{n\in\mathbb{N}}\sum_{l\in\mathbb{Z}} \delta^n p^{\pm}_{n,l}(\theta) e^{2i\pi l \frac{R}{\delta}\theta} e^{-2\pi|l||r-R|/\delta}$ decays exponentially with $\frac{|r-R|}{\delta}$.

Exchanging the order of summation in (49) gives

$$u_\delta(r,\theta) = \sum_{n=-1}^{+\infty}\sum_{k=0}^{+\infty} \delta^n (r-R)^k p^{\pm}_{n+k,k}(\theta) + o\left(\delta/|r-R|\right)^{+\infty}, \quad (r,\theta) \in \Omega^{\pm}_{M,\delta}. \tag{50}$$

Neglecting the exponentially decaying remainder in (50) (will be justified by the error estimates), the identification of (48) with (50), δ and $(r-R)$ being considered as independent variables, leads to

$$\boxed{p^{\pm}_{n,k}(\theta) = \begin{cases} 0 & \text{if } k = n+1, \\ \dfrac{1}{k!}\dfrac{\partial^k u^{\pm}_{n-k}(R,\theta)}{\partial r^k} & \text{if } 0 \le k \le n. \end{cases}} \tag{51}$$

We get $2\times(n+2)$ matching conditions at stage $n+1$ while only 2×2 additional equations are needed. In fact, only the matching conditions corresponding to $k = 0, 1$ are sufficient to provide the needed information : indeed, it can be proved (as in [23] for instance) that the matching conditions for $2 \le k \le n+1$ can be deduced form the matching conditions for $k = 0, 1$ and from the near fields and far fields equations. For our purpose, we shall use the fact that the first line of (51) tells us that the polynomials $p_n^{\pm}$ are in fact of degree n instead of $n+1$. Finally, the information that we shall retain for the inductive construction of the (u_n^+, u_n^-, U_n) are:

$$\boxed{p_n^{\pm} \in P_n; \quad p^{\pm}_{n,0}(\theta) = u^{\pm}_n(R,\theta); \quad p^{\pm}_{n,1}(\theta) = \frac{\partial u^{\pm}_{n-1}}{\partial r}(R,\theta)} \tag{52}$$

3.4. *Construction of the first terms of the asymptotic expansion*

A preliminary technical result. In what follows, we shall be faced with problems of the type:

$$\begin{cases} \text{Find } U \in H^1_{loc}(\mathbb{R}^2) \text{ such that } \; U(\sigma, s+1) = U(\sigma, s) \text{ and} \\[2mm] A_0\left(\dfrac{\partial}{\partial \sigma}, \dfrac{\partial}{\partial s}\right) U = F, \qquad \text{in } \mathcal{D}'(\mathbb{R}^2); \end{cases} \tag{53}$$

where $F \in \mathcal{D}'(\mathbb{R}^2)$ is a distribution of the following form

$$F = \nabla\cdot g - f, \quad (f, g) \in L^2_{loc}(\mathbb{R}^2) \times L^2_{loc}(\mathbb{R}^2)^2, \quad (f, g) \text{ are 1-periodic in } s. \tag{54}$$

It is an exercise to check that (53) is equivalent to (U being identified with its restriction to $\mathcal{B}$)

$$\begin{cases} \text{Find } U \in H^{1,loc}_{per}(\mathcal{B}) \text{ such that} \\[2mm] \displaystyle\int_{\mathcal{B}} \nabla U \cdot \nabla \overline{V}\, d\sigma\, ds = \int_{\mathcal{B}} g \cdot \nabla \overline{V}\, d\sigma\, ds + \int_{\mathcal{B}} f\, \overline{V}\, d\sigma\, ds, \quad \forall V \in H^1_{c,per}(\mathcal{B}). \end{cases} \tag{55}$$

where $H^{1,loc}_{per}(B) := \{V \in H^1_{loc}(B) \,/\, V(\cdot\,; 1/2) = V(\cdot\,; -1/2)\}$, $H^1_{per}(B) := H^{1,loc}_{per}(B) \cap H^1(B)$ and $H^1_{c,per}(B)$ is the subspace of $H^1_{per}(B)$ made of functions which are compactly supported in $\overline{B}$.

In this paragraph, we are interested in solutions U that are bounded at infinity, which can be reduced to imposing:

$$U \in W^1(B). \tag{56}$$

To expect the existence of such a solution, we need additional properties of f and g, namely

$$(1+\rho^2)^{\frac{1}{2}} f,\ g \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)^2; \tag{57}$$

which implies that the linear form in $H^1_c(B)$ at the right hand side of (55) extends continuously as a linear form on $W^1(B)$ (see (31)), namely that, formally, $F \in W^1(B)'$.

Finally, to get uniqueness, we need to impose one additional linear condition, as for defining N. Once again, for convenience, we shall choose

$$\int_{\Gamma_+} U\, ds + \int_{\Gamma_-} U\, ds = 0; \qquad (\Leftrightarrow \quad U \in W^1_0(B); \quad \text{cf } (32)) \tag{58}$$

Proposition 7. *Assume that (57) holds. Then, the problem (53, 56, 58) admits a unique solution if and only if the following compatibility condition is satisfied:*

$$\int_B f\, ds\, d\nu = 0. \tag{59}$$

Proof. From (55) and (57), by density of $H^1_c(B)$ in $W^1(B)$, we see that U is solution of (53, 56, 58) if and only if U is solution of the problem

$$R^2 \int_B \nabla U \cdot \nabla \overline{V}\, ds\, d\nu = \int_B g \cdot \nabla \overline{V}\, ds\, d\nu + \int_B f\, \overline{V}\, ds\, d\nu; \quad \forall V \in W^1(B). \tag{60}$$

Taking $V = 1$, we see that the existence of U implies (59).

Reciprocally, since $W^1_0(B) \subset W^1(B)$, we see that, if U exists, it is solution of the variational problem

$$\begin{cases} \text{Find } U \in W^1_0(B) \text{ such that} \\ R^2 \displaystyle\int_B \nabla U \cdot \nabla \overline{V}\, ds\, d\nu = \int_B g \cdot \nabla \overline{V}\, ds\, d\nu + \int_B f\, \overline{V}\, ds\, d\nu; \quad \forall V \in W^1_0(B). \end{cases} \tag{61}$$

which admits a unique solution U thanks to Hardy's inequality and Lax-Milgram's lemma.

To conclude, it remains to check that this solution is also solution of (60) which is a consequence of

$$W^1(B) = W^1_0(B) + \mathrm{span}\{1\}$$

and of the fact that (59) implies that the equality (60) is automatically true for $V = 1$. □

Equations for $n = 0$. For $n = 0$, (16), (22), (42) and (52) reduce to

$$\begin{cases} \mu_\infty \Delta u_0^\pm + \omega^2 \varepsilon_\infty u_0^\pm = f \quad \text{in } \Omega^\pm; & \dfrac{\partial u_0^+}{\partial r} + i\omega u_0^+ = 0 \quad \text{on } S_{R_e}; \\ A_0\!\left(\dfrac{\partial}{\partial \nu}, \dfrac{\partial}{\partial s}\right) U_0 = 0 \quad \text{in } B^*; & U_0(\nu, s, \theta) = p^\pm_{0,0}(\theta) + o(|\nu|^{-\infty}) \text{ in } B^*_\pm; \end{cases} \tag{62}$$

with the matching conditions (obtained from the first equalities in (52) with $n = 0$, note that the second equality gives $0 = 0$ since we already know that $p_0^\pm \in P_0$)

$$u_0^\pm(R,\theta) = p_{00}^\pm(\theta); \tag{63}$$

Next we remark that the second line of (62) means that for each θ, $U_0(\cdot;\theta)$ belongs to $\mathrm{Ker}\, A_0$ (as defined in (28)) and is bounded at infinity. Due to lemma 2, this means that $U_0(\cdot;\theta)$ is constant, i. e., taking into account the second part of the second line of (62),

$$U_0(s,\nu;\theta) \equiv U_0(\theta) \quad \text{and} \quad p_{00}^\pm(\theta) = U_0(\theta). \tag{64}$$

Then the matching conditions (63) give

$$u_0^+(R,\theta) = u_0^-(R,\theta) = U_0(\theta). \tag{65}$$

Note that this implies that u_0 (the function in Ω whose restriction to $\Omega_\pm$ is $u_0^\pm$) is continuous across S_R and we shall set

$$u_0(R,\theta) := u_0^+(R,\theta) = u_0^-(R,\theta).$$

This is not sufficient for entirely defining u_0 from the first line of (62) : a second transmission condition is missing. It will be obtained from the equations for $n = 1$.

Equations for $n = 1$. For $n = 1$, (16), (22), (42) and (52) reduce to

$$\begin{cases} \Delta u_1^\pm + \omega^2 u_1^\pm = 0 \quad \text{in } \Omega_\pm; \quad \dfrac{\partial u_1^+}{\partial r} + i\omega u_1^+ = 0 \quad \text{on } S_{R_e}; \\ A_0\left(\dfrac{\partial}{\partial \nu}, \dfrac{\partial}{\partial s}\right) U_1 = -A_1 U_0 \quad \text{in } B^*; \quad U_1(\nu,s;\theta) = p_{1,0}^\pm(\theta) + p_{1,1}^\pm(\theta)\,\nu + o(|\nu|^{-\infty}) \text{ in } B_\pm^*; \end{cases} \tag{66}$$

with the matching conditions

$$u_1^\pm(R,\theta) = p_{1,0}^\pm(\theta); \quad p_{1,1}^\pm(\theta) = \frac{\partial u_0^\pm}{\partial r}(R,\theta). \tag{67}$$

Since $U_0(\nu,s;\theta) = u_0(R,\theta)$, it follows from the definition of A_1 (see (24)) that

$$A_1 U_0(\nu,s;\theta) = \frac{\partial \mu}{\partial s}(\nu,s)\,\frac{\partial u_0}{\partial \theta}(R,\theta);$$

so that $U_1(\cdot;\theta)$ is solution of

$$A_0\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial s}\right) U_1 = -\frac{\partial \mu}{\partial s}(\nu,s)\,\frac{\partial u_0}{\partial \theta}(R,\theta). \tag{68}$$

The main difference with the equation satisfied by U_0 (see (62)) comes from the right hand side. To get rid of it, we are going to subtract from U_1 a particular solution $U_{1,part}$ of (68). As $\frac{\partial u_0}{\partial \theta}(R,\theta)$ appears as a (constant in (ν,s)) multiplicative factor in the right hand side of (68) it is natural to look for this particular solution of the form:

$$U_{1,part}(\nu,s;\theta) = \frac{\partial u_0}{\partial \theta}(R,\theta)\;\mathcal{U}(\nu,s) \tag{69}$$

where $\mathcal{U}$, that we shall call a profile function, will be solution of

$$A_0\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial s}\right) \mathcal{U} = -\frac{\partial \mu}{\partial s} \quad \text{in } \mathbb{R}^2. \tag{70}$$

We have some liberty in the construction of $\mathcal{U}$. It is natural to look for $\mathcal{U}$ in the space $V(B)$. Since $\mathrm{Ker}\, A_0$ is of dimension 2, we need two additional conditions. We first shall avoid any linear growth in ν by prescribing that:

$$\mathcal{U} \in W^1(B) \tag{71}$$

and, for convenience, we fix the remaining degree of freedom using the same condition as for N, namely:

$$\int_{+} U\, ds + \int U\, ds = 0. \tag{72}$$

The existence and uniqueness of U is guaranteed by Proposition 7. It is sufficient to realize that (70, 71, 72) is nothing but (53) with

$$f = 0 \quad \text{and} \quad g = \quad \infty \quad ; 0 \quad ^t$$

which do satisfy both (57) and (59). Consequently we can state the

Lemma 8. *There exists a unique function* $U \in W^1(B)$ *satisfying (70) and (72).*

By analogy with the definition of N_∞, we define

$$U_\infty := \int_{+} U(; s)\, ds = \int U(\; ; s)\, ds. \tag{73}$$

The reader will notice that in fact

$$U_\infty := \int_{-1/2}^{1/2} U(; s)\, ds = \int_{-1/2}^{1/2} U(\; ; s)\, ds; \quad \forall \quad \frac{1}{2}; \tag{74}$$

and moreover that, according to (43):

$$U(; s) = U_\infty + o(| |^{-\infty}) \qquad \text{in } B_\pm. \tag{75}$$

By construction of $U_{1;part}$, (66) implies that that for each , $U_1(; \;)$ $U_{1;part}(; \;)$ belongs to $\mathrm{Ker}\, A_0$. Thus, by lemma 2, there exist two functions $a_1(\;)$ and $b_1(\;)$ such that

$$U_1(; \;) = a_1(\;) + b_1(\;)\, N + \frac{\partial u_0}{\partial}(R; \;)\, U \tag{76}$$

By identification (for large) between (66) and (76), we deduce, using (47) and (75), that

$$p^{\pm}_{1;0}(\;) = a_1(\;) \quad b_1(\;)\, N_\infty \quad U_\infty \frac{\partial u_0}{\partial}(R; \;) \quad \text{and} \quad p^{+}_{1;1}(\;) = p^{-}_{1;1}(\;) = b_1(\;).$$

Therefore, the matching conditions (67) give

$$u_1^{\pm}(R; \;) = a_1(\;) \quad b_1(\;)\, N_\infty \quad U_\infty \frac{\partial u_0}{\partial}(R; \;); \qquad \frac{\partial u_0^{\pm}}{\partial r}(R; \;) = b_1(\;). \tag{77}$$

We are now in position to determine the first terms of the asymptotics (12) and (13).

Determination of $u_0^+; u_0^-$ and U_0.

The second equation of (77) implies that $\dfrac{\partial u_0}{\partial r}$ is continuous across S_R.

Joined to (62) (first line) and (65), this allows us to conclude that, as intuitively expected, the function u_0 (i. e. the limit of u when tends to 0) is nothing but the solution of the "limit problem" obtained by simply ignoring the presence of the periodic ring:

$$\boxed{\begin{array}{ll} {}_\infty \Delta u_0 + \omega^2 {}_\infty u_0 = f; & \text{in } ; \\[2mm] \dfrac{\partial u_0}{\partial r} + i\omega u_0 = 0; & \text{on } S_R; \end{array}} \tag{78}$$

which entirely defines u_0 (thus u_0^+ and u_0^-). Then, the corresponding near field U_0 is given by (cf (64, 65))

$$U_0(\ ; s;\ ) = u_0(R;\ ): \tag{79}$$

Then, (77) gives

$$u_1^{\pm}(R;\ ) = a_1(\ )\ \ N_{\infty} \frac{\partial u}{\partial r}(R;\ )\ \ U_{\infty} \frac{\partial u}{\partial\ }(R;\ ): \tag{80}$$

Eliminating a_1, we obtain a first transmission condition for u_1, namely:

$$[u_1]\!]\ R;\ ) := u_1^+(R;\ )\quad u_1^-(R;\ ) = 2\,N_{\infty} \frac{\partial u}{\partial r}(R;\ ) + 2\,U_{\infty} \frac{\partial u}{\partial\ }(R;\ ) \tag{81}$$

Again, a second transmission condition is missing. It will be obtained from the equations for $n = 2$.

Taking the sum of the first two equalities of (77) gives $a_1(\ ) = \langle u_1 \rangle(R;\ )$

(87) suggest us to introduce 5 new functions (profiles) $U_{jk} \in V(B)$, $(j,k) \in I$ satisfying

$$A_0\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial s}\right) U_{jk} = F_{jk}, \quad \text{in } B \tag{89}$$

From the regularity of N and U, one easily checks that the F_{jk}'s are of the form (54)

$$F_{jk} = \nabla \cdot g_{jk} - f_{jk}, \quad (f_{jk}, g_{jk}) \in L^2_{loc}(\mathbb{R}^2) \times L^2_{loc}(\mathbb{R}^2)^2, \quad (f_{jk}, g_{jk}) \text{ are 1-periodic in } s.$$

Of course, (f_{jk}, g_{jk}) are not uniquely defined but it is always possible to choose them in such a way that $g_{jk} \in L^2(\mathbb{R}^2)$ with compact support (it suffices to notice that the "non L^2" part of g_{jk} is necessarily compactly supported in B_0, where μ and ρ vary). However, and this is the main difference between (89) and the equation (70) which defined the profile U, f_{jk} does not necessarily decrease at infinity to satisfy (57).

More precisely, one easily checks, using (88) and the expression (24) of the operators $A_1^{(0)}$ and $A_1^{(1)}$ and the properties (45), that

$$f_{jk} = F_{jk}^\infty + o(|\nu|^{-\infty}) \quad \text{in } B_\pm, \tag{90}$$

where the constants F_{jk}^∞ are given by

$$F_{01}^\infty = F_{11}^\infty = 0, \quad F_{00}^\infty = -\omega^2 R^2 \rho_\infty, \quad F_{10}^\infty = -R\,\mu_\infty, \quad F_{02}^\infty = -\mu_\infty. \tag{91}$$

As a consequence, the construction of the functions U_{jk} can not be reduced to the application of Proposition 7. To describe more clearly how we proceed, we distinguish two cases

- For $(j,k) = (0,1)$ or $(1,1)$, $F_{jk}^\infty = 0$ implies that F_{jk} satisfies (57) and thus does belong to the dual space of $W^1(B)$. However, it does not necessarily satisfy the compatibility condition (59) for the problem to be solvable variationally in $W^1(B)$. To overcome this difficulty, the idea is to subtract from U_{jk} an appropriate function so that the new function can be searched in $W^1(B)$, as we did for the construction of N. Let us consider for simplicity the case where $F_{jk} \in L^2(B)$ and has a compact support. In this case, it is easy to see (using Fourier series in s) that if U_{jk} belongs to $V(B)$ and solves (89), then there exists $(a_+, a_-) \in \mathbb{R}^2$ such that,

$$\left|\frac{\partial U_{jk}}{\partial \nu}(s,\nu) - a_\pm\right| \leq C e^{-\delta|\nu|}, \quad \text{for } \pm\nu > 0.$$

  Therefore, integrating (formally) (89) in the domain $|\nu| \leq M$ and letting M go to $+\infty$ that

$$R^2 \mu_\infty (a_+ - a_-) = \int_B F_{jk}\, d\nu\, ds \neq 0 \tag{92}$$

  The fact that a_+ and a_- cannot vanish simultaneously traduces the fact that U_{jk} can not belong to $W^1(B)$. The idea is to subtract from U_{jk} a particular function that has an appropriate linear behavior at infinity, coherent with (92). For symmetry, the most natural choice consists in imposing a linear behavior as

$$C_{jk}|\nu| \quad \text{where} \quad C_{jk} = \frac{1}{2R^2\mu_\infty} \langle F_{jk}, 1 \rangle_B\,,$$

  or equivalently to look for U_{jk} such that

$$U_{jk} - C_{jk}|\nu| \in W^1(B), \quad C_{jk} = \frac{1}{2R^2\mu_\infty} \langle F_{jk}, 1 \rangle_B. \tag{93}$$

  Moreover, to ensure uniqueness, we impose the additional condition (as for U, see (72))

$$\int_{\Sigma_+} \left(U_{jk} - C_{jk}|\nu|\right) ds + \int_{\Sigma_-} \left(U_{jk} - C_{jk}|\nu|\right) ds = 0. \tag{94}$$

Consequently, we look for U_{jk} solution of (89, 93, 94). The uniqueness results from $\mathrm{Ker}\, A_0 \cap W^1_0(B) = \{0\}$. It remains to show the existence of U_{jk}. For technical reasons, it is useful to introduce a cut-off function in order to "avoid" the zone B_0 in which the coefficient is no longer constant. More precisely let $\in C^\infty(\mathbb{R})$ an even function of such that (see Figure 3)

$$(\ ) = 0 \quad \text{if} \quad |\ | \quad \frac{1}{2}; \qquad (\ ) = 1 \quad \text{if} \quad |\ | \quad 1: \tag{95}$$

and we decompose U_{jk} as

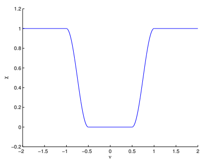


Figure 3: function

$$U_{jk}(s;\ ) = U_{jk}(s;\ ) + C_{jk}\ (\ )\ |\ |; \qquad C_{jk} = \frac{1}{2R^2\ _\infty} \langle F_{jk}; 1\rangle_B : \tag{96}$$

Then, the new unknown functions U_{jk} satisfy

$$A_0(\frac{\partial}{\partial}; \frac{\partial}{\partial s}) U_{jk} = F_{jk} \quad \text{in } B; \tag{97}$$

where the new right hand side

$$F_{jk} := F_{jk} \quad C_{jk}\, A_0(\frac{\partial}{\partial}; \frac{\partial}{\partial s}) \quad |\ | \tag{98}$$

only differs from F_{jk} by a smooth function, compactly supported in the region $1/2 \quad |\ | \quad 1$. Moreover, one sees that:

$$A_0(\frac{\partial}{\partial}; \frac{\partial}{\partial s}) \quad |\ | = R^2\ _\infty \frac{\partial^2}{\partial^2} \quad |\ |$$

and thus

$$\int_B A_0(\frac{\partial}{\partial}; \frac{\partial}{\partial s}) \quad |\ |\ ds\,d = \ _\infty R^2 \int_{-1}^{1} \frac{\partial^2}{\partial^2} \quad |\ | = 2\ _\infty R^2$$

As a consequence

$$\langle F_{jk}; 1\rangle_B = 0$$

and the existence of $U_{jk} \in W^1_0(B)$ solution of is guaranteed by Proposition 7.

For $(j; k) = (0; 0); (1; 0)$ or $(0; 2)$, F_{jk} does no longer belong to the dual space of $W^1(B)$. To remedy this difficulty, the idea consists again in subtracting from U_{jk} its expected behavior at infinity, which can be guessed by integrating explicitly in each domain $B_\pm$ the equation (89) when the right hand side F_{jk} is replaced by its "non-decreasing" part at infinity, which is here nothing but a constant function, explicitly given by (91). To make this rigorous, we use again the cut-off function to decompose U_{jk} as

$$U_{jk} = U_{jk}^{;\infty} + \frac{F^\infty_{jk}}{R^2\ _\infty} \frac{^2}{2} \tag{99}$$

Then, the new unknown functions U_{jk} satisfy

$$A_0(\frac{\partial}{\partial},\frac{\partial}{\partial s})U_{jk}^{;\infty} = F_{jk}^{;\infty}; \quad \text{in } B$$

where the right hand sides

$$F_{jk}^{;\infty} := F_{jk} \quad \frac{F^{\infty}_{jk}}{{}_{\infty}R^2} A_0(\frac{\partial}{\partial},\frac{\partial}{\partial s}) \quad \frac{2}{2} \quad \text{in } B_{\pm}; \tag{100}$$

satisfy, for all $| \ | > \frac{1}{2}$,

$$F_{jk}^{;\infty} = (F_{jk} - F^{\infty}_{jk}) + F^{\infty}_{jk} \quad \frac{\partial^2}{\partial^2} \quad (1 \quad) \frac{2}{2} \quad ; \quad \text{in } B_{\pm}; \tag{101}$$

since the last term has compact support, for all $| \ | > 1$,

$$F_{jk}^{;\infty} = o(| \ |^{-\infty}). \tag{102}$$

We are thus reduced to the same situation encountered in point (i) and we can use the process exposed in (i) to construct $U_{jk}^{;\infty}$. The reader will verify that the conditions (93) and (94), when U_{jk} and F_{jk} are replaced respectively by $U_{jk}^{;\infty}$ and $F_{jk}^{;\infty}$, become, when written in terms of U_{jk}

$$U_{jk} \quad C_{jk} | \ | \quad \frac{F^{\infty}_{jk}}{R^2{}_{\infty}} \frac{2}{2} \in W^1(B); \quad C_{jk} = \frac{1}{2 \ {}_{\infty}R^2} \langle F_{jk}^{;\infty}, 1\rangle_B. \tag{103}$$

$$\int_{+} U_{jk} \quad C_{jk} | \ | \quad \frac{F^{\infty}_{jk}}{{}_{\infty}R^2} \frac{2}{2} \ ds + \int U_{jk} \quad C_{jk} | \ | \quad \frac{F^{\infty}_{jk}}{R^2{}_{\infty}} \frac{2}{2} \ ds = 0. \tag{104}$$

Based on the above construction and using the property (46), we can completely describe the behavior at infinity of the U_{jk}'s, as we already did for N (47) and U (75).
Note that $\langle F_{jk}^{;\infty}, 1\rangle_B = \langle F_{jk} - F^{\infty}_{jk}, 1\rangle_B$ since,

$$\begin{aligned} \langle F_{jk}^{;\infty}, 1\rangle_B &= \langle (F_{jk} - F^{\infty}_{jk}) + F^{\infty}_{jk} \quad \frac{\partial^2}{\partial^2} \quad (1 \quad) \frac{2}{2} \quad , 1\rangle_B \\ &= \langle F_{jk} - F^{\infty}_{jk}, 1\rangle_B. \end{aligned}$$

Let us summarize the results we have obtained:

Lemma 9. *For each $(j,k) \in I$, there exists a unique function U_{jk} solution of (89, 103), 104), where F_{jk} and F^{∞}_{jk} are given by (88) and (91). Moreover, this function satisfies the following behavior at infinity:*

$$U_{jk}(\ ,s) = \frac{F^{\infty}_{jk}}{R^2{}_{\infty}} \frac{2}{2} + \frac{1}{2 \ {}_{\infty}R^2} \langle F_{jk} - F^{\infty}_{jk}, 1\rangle_B | \ | \quad U^{\infty}_{jk} + o(| \ |^{-\infty}) \qquad \textit{in } B_{\pm}. \tag{105}$$

where

$$U^{\infty}_{jk} = \int_{+} \Big[U_{jk} \quad \frac{1}{2R^2{}_{\infty}} \langle F_{jk} - F^{\infty}_{jk}, 1\rangle_B | \ | \quad \frac{F^{\infty}_{jk}}{R^2{}_{\infty}} \frac{2}{2} \Big] ds. \tag{106}$$

Since the U_{jk}'s are properly defined, we can introduce

$$U_{2;part}(\ ,s;\ ) = \langle \frac{\partial u_1}{\partial} \rangle(R;\ ) \quad U + \sum_{(j,k)\in\mathcal{I}} \frac{\partial^{j+k}u_0}{\partial t \partial^k}(R;\ ) \quad U_{jk}. \tag{107}$$

as a particular solution of (87). Therefore, (87) means that $U_2(\ ;\ ) \quad U_{2;part}(\ ;\ )$ belongs to $\operatorname{Ker} A_0$, which implies the existence of two functions $a_2(\ )$ and $b_2(\ )$ such that

$$U_2(\ ;\ ) = a_2(\ ) + b_2(\ ) N + \langle \frac{\partial u_1}{\partial} \rangle(R;\ ) \quad U + \sum_{(j,k)\in\mathcal{I}} \frac{\partial^{j+k}u_0}{\partial t \partial^k}(R;\ ) \quad U_{jk} \tag{108}$$

By identification between (83) and (108), we deduce, using (75) and (105), that

$$\begin{cases} p^{\pm}_{2;0}(\ ) = a_2(\ ) \quad b_2(\ ) N_{\infty} \ \langle \frac{\partial u_1}{\partial} \rangle U_{\infty} \quad \sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u_0}{\partial r^j \partial^k}(R;\ ) \ U^{\infty}_{jk}; \\ p^{\pm}_{2;1}(\ ) = b_2(\ ) \quad \frac{1}{2R^2 \ \infty} \sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u_0}{\partial r^j \partial^k}(R;\ ) \ \langle F_{jk} \ \ F^{\infty}_{jk}; 1 \rangle_B; \\ p^{\pm}_{2;2}(\ ) = \frac{1}{2R^2 \ \infty} \sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u_0}{\partial r^j \partial^k}(R;\ ) \ F^{\infty}_{jk}: \end{cases}$$

As explained in (52), note that the matching conditions (51) automatically hold for $k = 2$. Indeed, using the explicit expression of $F^{\infty}_{j;k}$ (90) and the fact that u_0 satisfies the homogeneous Helmholtz equation in the vicinity of S_R, we have

$$\begin{aligned} p^{\pm}_{2;2}(\ ) &= \frac{1}{2R^2 \ \infty} \left(\ \omega^2 R^2 \ \infty \, u^{\pm}_0(R;\ ) + \ \infty \frac{\partial^2 u^{\pm}_0(R;\ )}{\partial^2} + R \ \infty \frac{\partial u^{\pm}_0(R;\ )}{\partial r} \right); \\ &= \frac{1}{2} \frac{\partial^2 u^{\pm}_0(R;\ )}{\partial r^2}: \end{aligned}$$

Therefore, the matching conditions (84) give

$$\begin{cases} u^{\pm}_2(R;\ ) = a_2(\ ) \quad b_2(\ ) N_{\infty} \ \langle \frac{\partial u_1}{\partial} \rangle U_{\infty} \quad \sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u_0}{\partial r^j \partial^k}(R;\ ) \ U^{\infty}_{jk}; \\ \frac{\partial u^{\pm}_1}{\partial r}(R;\ ) = b_2(\ ) \quad \frac{1}{2R^2 \ \infty} \sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u^{\pm}_0}{\partial r^j \partial^k}(R;\ ) \ \langle F_{jk} \ \ F^{\infty}_{jk}; 1 \rangle_B; \end{cases} \tag{109}$$

Looking more closely to the expression of the F_{jk}, one can compute that

$$\begin{cases} C_{00} := \frac{1}{2R^2 \ \infty} \langle F_{00} \ \ F^{\infty}_{00}; 1 \rangle_B = \frac{\omega^2}{2 \ \infty} \int_B (\ \infty \quad) \, ds \, d \ ; \\ C_{01} := \frac{1}{2R^2 \ \infty} \langle F_{01} \ \ F^{\infty}_{01}; 1 \rangle_B = 0; \\ C_{02} := \frac{1}{2R^2 \ \infty} \langle F_{02} \ \ F^{\infty}_{02}; 1 \rangle_B = \frac{1}{2R \ \infty} \int_B (\ \infty \quad) \frac{\partial U}{\partial s} \, ds \, d \\ \qquad\qquad + \frac{1}{2R^2 \ \infty} \int_B (\ \infty \quad) \, ds \, d \ ; \\ C_{10} := \frac{1}{2R^2 \ \infty} \langle F_{10} \ \ F^{\infty}_{10}; 1 \rangle_B = 0; \\ C_{11} := \frac{1}{2R^2 \ \infty} \langle F_{11} \ \ F^{\infty}_{11}; 1 \rangle_B = \frac{1}{2R \ \infty} \int_B (\ \infty \quad) \frac{\partial N}{\partial s} \, ds \, d \ = U_{\infty}; \end{cases} \tag{110}$$

where we have chosen to make appear the expression of the coefficients C_{jk} as integrals over a bounded domain (B_0) so that they can be easily computed numerically, once N and U have been determined.

The first equality in (110) is obvious. Let us explain how we obtain the second one (respectively the third one), the fourth one (respectively the fifth one) being obtained in the same way (the corresponding details are left to the reader).

For the second equality of (110), according to (88), we write

$$hF_{01} \quad F^{\infty}_{01}; 1i_B = hA_1^{(0)}U; 1i_B$$

Using the equation (70) satisfied by U and the definition of $A_1^{(0)}$ (see (88)), we observe that

$$A_1^{(0)}U = \frac{@}{@s}\Big[2R \quad \frac{@U}{@s} + 2 \; (\quad _{\infty})\Big] + R \quad \frac{@U}{@} \tag{111}$$

namely is of the form (54) with

$$f = R \quad \frac{@U}{@}; \quad g = \Big(0; 2R \quad \frac{@U}{@s} + 2 \; (\quad _{\infty})\Big)^t : \tag{112}$$

Thus

$$hF_{01} \quad F^{\infty}_{01}; 1i_B = \quad R \int_B \frac{@U}{@} \, ds \, d :$$

Let $(\;) \; 2 \; C_0^{\infty}(R)$ be such that $(\;) =$ for $j \; j \quad 1=2$. Using the remark (74) and the fact that $= \; _{\infty}$ outside B_0, the above equality can be rewritten

$$hF_{01} \quad F^{\infty}_{01}; 1i_B = \quad R \int_{B_0} \frac{@U}{@} \quad '(\;) \, ds \, d$$

Since U is solution of (70), we have

$$R^2 \int_B rU \quad r \quad V \; ds \, d = \int_B R \quad \frac{@V}{@s} ds \, d \qquad 8 \; V \; 2 \; W^1(B): \tag{113}$$

Choosing $V = \quad (\;)$, this gives

$$R^2 \int_{B_0} \frac{@U}{@} \quad '(\;) \, ds \, d = 0 \quad =) \quad hF_{01} \quad F^{\infty}_{01}; 1i_B = 0:$$

For the third equality of (110), according to (88), we have

$$hF_{02} \quad F^{\infty}_{02}; 1i_B = h \, A_1^{(1)}U \qquad + \quad _{\infty}; 1i_B \quad :$$

serve that $A_1 \quad 1$

$A \qquad _{01} \quad ^{\infty}_{01} 71i_B =$

Using (110), we can be more explicit in writing the second equation of (109)

$$\frac{\partial u_1^\pm}{\partial r}(R;\theta) = b_2(\theta) \mp \Big(C_{00}\, u_0(R;\theta) + C_{02}\frac{\partial^2 u_0}{\partial \theta^2}(R;\theta) + C_{11}\frac{\partial^2 u_0}{\partial r \partial \theta}(R;\theta)\Big) : \tag{116}$$

We are now in position to determine the second terms of the asymptotics (12) and (13).

Determination of $u_1^+; u_1^-$ and U_1.

Eliminating $b_2(\theta)$ from the second line of (116), we get

$$\Big\langle \frac{\partial u_1}{\partial r}\Big\rangle(R;\theta) = 2\,\omega^2 C_{00}\, u_0(R;\theta) + 2\,C_{02}\frac{\partial^2 u_0}{\partial \theta^2}(R;\theta) + 2\,C_{11}\frac{\partial^2 u_0}{\partial r \partial \theta}(R;\theta): \tag{117}$$

This provides the missing transmission condition for defining u_1 which is thus entirely characterized as the unique function $u_1 \in H^1(\Omega_+ \cup \Omega_-)$ solution of

$$\begin{array}{ll}
\sigma_\infty \Delta u_1 + \omega^2 \varepsilon_\infty u_1 = 0; & \text{in } \Omega^+ \cup \Omega^-;\\[2mm]
[u_1](R;\theta) = 2\,N_\infty \dfrac{\partial u_0}{\partial r}(R;\theta) + 2\,U_\infty \dfrac{\partial u_0}{\partial \theta}(R;\theta); & \text{on } \Gamma\\[2mm]
\Big\langle \dfrac{\partial u_1^\pm}{\partial r}\Big\rangle(R;\theta) = 2\,C_{00}\,\omega^2 u_0(R;\theta) + 2C_{02}\dfrac{\partial^2 u_0}{\partial \theta^2}(R;\theta) + 2U_\infty\dfrac{\partial^2 u_0}{\partial r\partial \theta}(R;\theta); & \text{on } \Gamma\\[2mm]
\dfrac{\partial u_1}{\partial r} + i\omega u_1 = 0; & \text{on } S_{R_e};
\end{array} \tag{118}$$

where N_∞ and U_∞, are given respectively by (29) and (73) and

$$\left\{\begin{array}{l}
C_{00} = \dfrac{1}{2\,\sigma_\infty}\displaystyle\int_B (\varepsilon_\infty - \varepsilon)\,ds\,d\theta;\\[3mm]
C_{02} = \dfrac{1}{2\,\sigma_\infty}\displaystyle\int_B \Big[\frac{\sigma_\infty - \sigma}{R}\frac{\partial U}{\partial s} + \frac{\sigma_\infty - \sigma}{R^2}\Big]\,ds\,d\theta:
\end{array}\right. \tag{119}$$

Moreover, u_1 being known, U_1 is the defined by (82).

From the second line of (109)

$$b_2(\theta) + = \Big\langle\frac{\partial u_1}{\partial r}\Big\rangle(R;\theta) \tag{120}$$

On the other hand, from the first line of (116), we deduce (by sum)

$$a_2(\theta) = \langle u_2\rangle(R;\theta) \tag{121}$$

and (by difference), using (120), we get the first transmission condition for u_2

$$[u_2](R;\theta) = 2\,\Big\langle\frac{\partial u_1}{\partial r}\Big\rangle(R;\theta)\, N_\infty + 2\,\Big\langle\frac{\partial u_1}{\partial \theta}\Big\rangle(R;\theta)\, U_\infty + 2\sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u_0}{\partial r^j \partial \theta^k}(R;\theta)\, U_{jk}^\infty \tag{122}$$

Concerning the near field U_2, note that, using (77), (120) and (121), (108) rewrites

$$U_2(\cdot;\theta) = \langle u_2\rangle(R;\theta) + \Big\langle\frac{\partial u_1}{\partial r}\Big\rangle(R;\theta)\, N + \Big\langle\frac{\partial u_1}{\partial \theta}\Big\rangle(R;\theta)\, U + \sum_{(j;k)\in\mathcal{I}} \frac{\partial^{j+k} u_0}{\partial r^j \partial \theta^k}(R;\theta)\, U_{jk} \tag{123}$$

Determination of $u_2^+; u_2^-$ and U_2.

To entirely determine u_2, it remains to compute $[\partial u_2]$. It can be derived from the problem satisfied by U_3. The method to obtain this condition is similar to the method used to determine $[\partial u_1]$: it is based on the introduction of 7 new additional profile functions. We do not need here to give the definition profile function since the expression one obtains for $[\partial u_2]$ does not depend of this new functions, in the same way that $[\partial u_1]$ did not depend on the U_{jk}'s. We leave to the reader (who will find all the details in [23]) to verify that

$$\frac{\partial u_2}{\partial r}(R;\ ) = 2\omega^2 C_{00}\langle u_1\rangle(R;\ ) + 2C_{02}\left\langle\frac{\partial^2 u_1}{\partial^2}\right\rangle(R;\ ) + 2C_{11}\left\langle\frac{\partial^2 u_1}{\partial r\partial}\right\rangle(R;\ ) + \sum_{j=0}^{1}\sum_{k=0}^{j} 2D_{jk}\,\frac{\partial^{j+k}u_0}{\partial r^j\partial^k} \tag{124}$$

where,

$$D_{00} := \frac{1}{2\nu_\infty R}\int_{B_0}\omega^2\ \ ds\,d\ ;\qquad D_{01} := 0;$$

$$D_{02} := \frac{1}{2\nu_\infty R^2}\int_{B_0} R\ \frac{\partial U_{02}}{\partial} + (\ \ \nu_\infty\ \ )\frac{\partial U_{01}}{\partial s}\ ds\,d \tag{125}$$

$$D_{03} := \frac{1}{2\nu_\infty R^2}\int_{B}(\ \ \nu_\infty\ \ )R\frac{\partial U_{02}}{\partial s}\ \ U\,ds\,d\ ;$$

$$D_{10} := U^{\infty}_{0;0};\qquad D_{11} := U^{\infty}_{0;1};\qquad D_{12} := U^{\infty}_{0;2}. \tag{126}$$

Consequently, we define u_2 as the unique solution in $H^1(\ _+ \cup\ _-)$ of

$$\begin{cases} \nu_\infty\Delta u_2 + \omega^2\ _\infty u_2 = 0; & \text{in } ^+\cup\ ^-;\\ [u_2](R;\ ) = 2\left\langle\frac{\partial u_1}{\partial r}\right\rangle(R;\ )\,N_\infty + 2\left\langle\frac{\partial u_1}{\partial}\right\rangle(R;\ )\,U_\infty \\ \qquad + 2\sum_{(j;k)\in\mathcal{I}}\frac{\partial^{j+k}u_0}{\partial r^j\partial^k}(R;\ )\,U^\infty_{jk} & \text{on } ;\\ \left[\frac{\partial u_2}{\partial r}\right](R;\ ) = 2C_{00}\langle u_1\rangle(R;\ ) + 2C_{02}\left\langle\frac{\partial^2 u_1}{\partial^2}\right\rangle(R;\ ) + 2C_{11}\left\langle\frac{\partial^2 u_1}{\partial r\partial}\right\rangle(R;\ ); \\ \qquad + \sum_{j=0}^{1}\sum_{k=0}^{3-j} 2D_{jk}\,\frac{\partial^{j+k}u_0}{\partial r^j\partial^k} & \text{on } ;\\ \frac{\partial u_2}{\partial r} + i\omega u_2 = 0; & \text{on } S_{R_e}. \end{cases} \tag{127}$$

Note that the near field term U_2 is then given by (123).

Remark 10 (Properties of the profile functions)**.** *By short computations, we can verify that*

$$U^\infty_{11} = 0.$$

Moreover, if f and g are even in s or (which is the case in the applications we are interested in), the profiles functions U, N and $U_{j;k}$ also have symmetry properties. In those cases, the expression of $[\partial u_1]$,

$[u_1]$, $[\partial u_2]$ *and* $[u_2]$ *(see 118-127) can be simplified because some of the constants* $C_{j;k}$, $D_{j;k}$ *and* $U^{\infty}_{j;k}$ *vanish. Let us summarize the parity properties in the following table:*

	and even in s	and even in
U	odd	even
N	even	odd
U_{00}	even	even
U_{01}	odd	odd
U_{02}	even	even
U_{10}	even	even
U_{11}	odd	odd

Table 1: parity of the profile functions

Consequently, if and are even in s

$$U_{\infty} = C_{11} = 0; \qquad \text{and} \qquad U^{\infty}_{01} = D_{11} = 0: \tag{128}$$

In the same way, if and are even in ,

$$U_{\infty} = C_{11} = 0; \qquad \text{and} \qquad U^{\infty}_{00} = U^{\infty}_{02} = U^{\infty}_{10} = D_{00} = D_{02} = D_{10} = D_{12} = 0: \tag{129}$$

4. Mathematical justification of the asymptotic expansion

4.1. Main result

As one is in practice interested in computing the far field, our main error estimate will concern the far field: we estimate the error committed when one replaces the exact solution u by $u_0 + u_1$ in any domain that avoids any (fixed but arbitrarily small) neighborhood of the interface . The precise result is the following:

Proposition 11. *Given* $0 < \; < R$ *, let* $= \{(x;y) \in \mathbb{R}^2; |\sqrt{x^2+y^2} - R| > \}$*. There exists a constant* $C > 0$ *and, for each* $0 < \; < R$*, a constant* > 0 *such that*

$$\forall < \; ; \quad \|u - (u_0 + u_1)\|_{H^1(\;)} \leq C^{\;2}(h\;)^2: \tag{130}$$

It is possible to improve the previous result and to obtain

$$\|u - (u_0 + u_1)\|_{H^1(\;)} \leq C^{\;2}: \tag{131}$$

(this result is for instance proved in [23]). In fact, it is possible to prove the following estimate, that is the equivalent of (130) when three terms of the expansion are considered:

$$\|u - (u_0 + u_1 + {}^2u_2)\|_{H^1(\;)} \leq C^{\;3}(h\;)^3; \tag{132}$$

from which (131) follows immediately. However, in the same way that, as we shall see, the proof of (130) uses u_2 and U_2, the proof makes use of u_3 and U_3 that have not been defined in this paper. That is why we shall restrict ourselves to (130).

4.2. Main steps of the proof

The idea of the proof follows the lines of [19] and consists first in constructing a global approximate solution using the first three terms (and not, curiously, the first two only) of the expansions (13) and (12), that coincides with the truncated far field expansion

$$u^2_{e;}(r;\;) := u_0(r;\;) + \; u_1(r;\;) + {}^2u_2(r;\;); \tag{133}$$

far from the interface S_R, and with the truncated near field expansion

$$U^2_{i;\delta}(r;\theta) := U_0(r;\theta) + \delta U_1(r;\theta) + \delta^2 U_2(r;\theta) \tag{134}$$

in the neighborhood of the interface S_R. This can be done with the help of a smooth cut-off function χ such that

$$\chi(x) = 0 \text{ when } |x| \geq 2; \quad \chi(x) = 1 \text{ when } |x| \leq 1:$$

and a small "distance" parameter $\delta < \eta < R/2$ so that on the one hand the "transition zone" $\{(r;\theta) = |r - R| \leq 2\eta\}$ and the outer zone $\{(r;\theta) = r > R + 2\eta$ or $r < R - 2\eta\}$ are included in the domain where the coefficients ε and μ are constant and on the other hand the transition zone does not intersect the support of f. Denoting $\chi_\eta(r;\theta) = \chi((r - R)/\eta)$, we introduce

$$u^2_{\delta} = (1 - \chi_\eta)\, u^2_{e;\delta} + \chi_\eta U^2_{i;\delta} \tag{135}$$

as a global approximate solution. In the following, we shall estimate the error $u_\delta - u^2_\delta$ in function of δ and η. Later in the proof, η - which is a parameter at our disposal - will be chosen as a function of δ, which tends to 0 with δ according to (15), in order to "optimize" the error estimate. To estimate the error, one starts from the stability estimate (cf. (7)):

$$\left\| u_\delta - u^2_\delta \right\|_{H^1(\Omega)} \leq C \sup_{\|v\|_{H^1(\Omega)}=1} a_\delta(\varepsilon^2_\delta; v): \tag{136}$$

Next, a straightforward calculation gives

$$a_\delta(\varepsilon^2_\delta; v) = \varepsilon^m_\delta(v) + \varepsilon^c_\delta(v); \tag{137}$$

where $\varepsilon^m_\delta(v)$ is the so-called "matching error" (it measures the mismatch between the truncated expansions (133) and (134)) given by

$$\varepsilon^m_\delta(v) = \int (u^2_{e;\delta} - (U^2_{i;\delta})_\delta)\nabla\chi_\eta \cdot \nabla\bar{v} - \int (\nabla(u^2_{e;\delta} - (U^2_{i;\delta})_\delta) \cdot \nabla\chi_\eta)\bar{v}; \tag{138}$$

and where $\varepsilon^m_\delta(v)$ is the so-called "consistency" (it measures how much the truncated expansion (134) fails to satisfy the original Helmholtz equation):

$$\varepsilon^c_\delta(v) = a_\delta\left((U^2_{i;\delta})_\delta;\ \chi_\eta v\right): \tag{139}$$

Estimating $\varepsilon^m_\delta(v)$ and $\varepsilon^c_\delta(v)$ (which is the object of the next two sections) gives a "global" error estimate via (136) and (137), which finally gives (130) via an adequate localization process, since u^2_δ coincides with $u^2_{e;\delta}$ sufficiently "far" from S_R (see Section 4.5).

4.3. Estimating of the matching error

Lemma 12.

$$\left|\varepsilon^m_\delta(v)\right| \leq C \left(\sum_{k=0} \delta^{2-k}\eta^k + \sum_{k=1} \eta^{k-1} \left(\frac{\delta}{\eta}\right)^{2k} e^{-2\eta/\delta} \right) \|v\|_{H^1(\Omega)}: \tag{140}$$

Proof. Let us denote by C_η the support of $\nabla\chi_\eta$,

$$C_\eta := \{(r;\theta) \in \mathbb{R}^+ \times [0;2\pi] \text{ such that } \eta \leq |r - R| \leq 2\eta\}:$$

Using the regularity of far and near fields in C_η (combining the fact that C_η does not intersect the support of f and the fact that $\eta > \delta$, it is clear that the far fields terms u_n and the near fields terms $(U_n)_\delta$ are $C^\infty(C_\eta)$), it is possible to bound $u^2_{e;\delta}$ and $(U^2_{i;\delta})_\delta$ and their derivatives by their L^∞ norm. We then obtain

$$\left|\varepsilon^m_\delta(v)\right| \leq \frac{C}{\eta} \|u^2_{e;\delta} - (U^2_{i;\delta})_\delta\|_{L^\infty(C_\eta)} \|\nabla v\|_{L^1(C_\eta)} + \|\frac{\partial}{\partial r}(u^2_{e;\delta} - (U^2_{i;\delta})_\delta)\|_{L^\infty(C_\eta)} \|v\|_{L^1(C_\eta)} : \tag{141}$$

Next, we exploit the smallness of C_δ ($\mathrm{mes}(C_\delta) \le C\delta$) to bound the terms in v:

$$\|\nabla v\|_{L^1(C_\delta)} \le C\delta^{1/2}\|v\|_{H^1(\Omega)}; \quad \|v\|_{L^1(C_\delta)} \le C\delta\|v\|_{H^1(\Omega)}. \tag{142}$$

The first inequality simply results from Cauchy-Schwartz inequality. The second is more tricky and appears to be a consequence of the following inequality (already used in [28], Lemma 3.10 for instance)

$$\|v\|_{L^2(C_\delta)} \le C\delta^{1/2}\|v\|_{H^1(\Omega)}. \tag{143}$$

whose proof is left to the reader. Note that we need the H^1-norm of v on a larger, δ-independent, domain (here Ω) and that the estimate is sharp (take v constant). The second inequality of (142) then results from Cauchy-Schwartz inequality (to pass from the L^1-norm to the L^2-norm which gives an extra $\delta^{1/2}$) combined with (143).

To estimate $\|u^2_{e;\delta} - (U^2_{i;\delta})\|_{L^1(C_\delta)}$, we first use the Taylor formula with integral rest for the $u^\pm_n$'s to write

$$u^{2;\pm}_{e;\delta}(r,\theta) = \sum_{k=0}^{2}\delta^k\sum_{i=0}^{2-k}\frac{(r-R)^i}{i!}\frac{\partial^i u^\pm_k(R,\theta)}{\partial r^i} + \sum_{k=0}^{2}\delta^k\int_R^r\frac{\partial^{3-k}u^\pm_k(t,\theta)}{\partial r^{3-k}}\frac{(t-R)^{2-k}}{(2-k)!}\,dt, \tag{144}$$

where we can estimate the integral remainders as follows (we use the fact that the far fields and their derivatives are bounded in the vicinity of S_{R_e})

$$\left|\int_R^r\frac{\partial^{3-k}u^\pm_k(t,\theta)}{\partial r^{3-k}}\frac{(t-R)^{2-k}}{(2-k)!}\,dt\right| \le C\delta^{3-k}, \quad \forall (r,\theta)\in C_\delta, \quad k=0,1,2. \tag{145}$$

where, in the rest of this section, C denotes a positive constant whose value may change from one line to the other. Next, we use the modal expansions (41) for the U_n's, that we combine with (42) and the matching conditions (51) to write (in the (r,θ) variables which explains the use of the notation (17))

$$\begin{aligned} U^\pm_{k}(r,\theta) &= \sum_{i=0}^{k}\frac{(r-R)^i}{\delta^i\, i!}\frac{\partial^i u^\pm_{k-i}(R,\theta)}{\partial r^i} + (R^\pm_k)_\delta(r,\theta); \\ (R^\pm_k)_\delta(r,\theta) &= \sum_{l\in\mathbb{Z}} p^\pm_{k,l}\left(\frac{r-R}{\delta},\theta\right)e^{2i\pi l\frac{R\theta}{\delta}}\,e^{-2\pi|l|\frac{|r-R|}{\delta}}. \end{aligned} \tag{146}$$

where $p^\pm_{k,l}\in P^{2k}$. More precisely $R^\pm_0 = 0$ (since U_0 is constant in θ, cf (79)) and one easily proves that (we left the details to the reader),

$$|(R^\pm_k)_\delta(r,\theta)| \le C\left(\frac{|r-R|}{\delta}\right)^{2k}e^{-2\pi|r-R|/\delta}; \quad \forall (r,\theta)\in C_\delta; \quad k=1,2. \tag{147}$$

It follows that we can rewrite $(U^{2;\pm}_{i;\delta})$ as

$$(U^{2;\pm}_{i;\delta}) = \sum_{k=0}^{2}\delta^k\left(\sum_{i=0}^{2-k}\frac{(r-R)^i}{i!}\frac{\partial^i u^\pm_k}{\partial r^i}(R,\theta)\right) + \sum_{k=1}^{2}\delta^k(R^\pm_k)_\delta(r,\theta); \tag{148}$$

Taking the difference between (144) and (148), we are left with remainder terms only, namely:

$$u^{2;\pm}_{e;\delta} - (U^{2;\pm}_{i;\delta}) = \sum_{k=0}^{2}\delta^k\int_R^r\frac{\partial^{3-k}u^\pm_k(t,\theta)}{\partial r^{3-k}}\frac{(t-R)^{2-k}}{(2-k)!}\,dt + \sum_{k=1}^{2}\delta^k(R^\pm_k)_\delta(r,\theta); \tag{149}$$

so that, using (145) and (147), we have

$$\left\|u^2_{e;\delta} - (U^2_{i;\delta})\right\|_{L^1(C_\delta)} \le C\left(\sum_{k=0}^{2}\delta^{3-k}\delta^k + \sum_{k=1}^{2}\delta^k\left\|\left(\frac{|r-R|}{\delta}\right)^{2k}e^{-|r-R|/\delta}\right\|\right). \tag{150}$$

Proceeding similarly (we omit the details), we obtain an analogous estimate for the r derivative, where we lose essentially one power of in the first term and one power of in the second term:

$$\left\| \frac{@}{@r} u^2_{e;} \quad (U^2_{i;}) \right\|_{L^1(C)} \quad C \sum_{k=0} {}^{2-k} \, {}^{k} + \sum_{k=1} {}^{k-1} \, \_ \, {}^{2k} e^{-2} = \quad : \tag{151}$$

Finally, combining (141), (142), (150) and (151), we get (140) since is bounded (the reader will notice that the more penalizing term in (141) is the second one). □

4.4. Near field consistency error

The near field consistency error is due to the fact that the near field $(U_{i;})$ does not solve exactly the Helmholtz equation. More precisely, summing (21) from $n = 0$ to $n = 2$ we get

$$
\begin{aligned}
r^2 \; r \quad r\, U^2_{i;} + !\, {}^2 U^2_{i;} \quad = \; & \frac{1}{^2} \underbrace{A_0(U_0)}_{=0} + \frac{1}{} \underbrace{A_0(U_1) + A_1(U_0)}_{=0} \\
& + \underbrace{A_0(U_2) + A_1(U_1) + A_2(U_0)}_{=0} \\
& + \quad A_1(U_2) + A_2(U_1) + A_3(U_0) \\
& + {}^2 \quad A_2(U_2) + A_3(U_1) + A_4(U_0) \\
& + {}^3 \quad A_3(U_2) + A_4(U_1) \quad + {}^4 \quad A_4(U_2)
\end{aligned}
$$

where the first three terms vanish thanks to (62, 66) and (83).

Formally, by multiplying the above expression by $v = r^2$ and integrating over , we get:

$$"^{c}_{;}(v) = \sum_{j=1} {}^{j} \, "^{c;j}_{;}(v); \tag{152}$$

$$
\begin{aligned}
"^{c;1}_{;}(v) &= \int A_1(U_2) + A_2(U_1) + A_3(U_0) \quad \overline{} v = r^2 \; ; \\
"^{c;2}_{;}(v) &= \int A_2(U_2) + A_3(U_1) + A_4(U_0) \quad \overline{} v = r^2 \; ; \\
"^{c;j}_{;}(v) &= \int A_3(U_2) + A_4(U_1) \quad \overline{} v = r^2 \; ; \\
"^{c;j}_{;}(v) &= \int A_4(U_2) \quad \overline{} v = r^2 \; :
\end{aligned} \tag{153}
$$

where the integrals in the right hand sides might be considered as duality products, which we have chosen to omit to mention explicitly for the simplicity of the notation. Intuitively, the smallness of the consistency is due to two facts: the factors ^{j} in (152) and the fact that the "integrals" in (153) are computed to a "small domain", namely the support of . This leads to the following estimates

Lemma 13.

$$^{j} \, |"^{c;j}_{;}(v)| \quad C \quad ^{j+1} \, \|v\|_{H^1(\;)} \tag{154}$$

Proof. The proof is long and tedious and technically complicated essentially because the functions in factor of v in (153) are not necessarily bounded, even locally. We are going to restrict ourselves to explaining how to get the estimate for $j = 1$ because it gives the more penalizing contribution to the consistency error and however presents all the possible technical difficulties. More precisely, we shall even concentrate ourselves on the first term in the expression of $\varepsilon^{c,1}_{\varepsilon}(v)$, namely

$$\int_\Omega A_1(U_2)\,\nabla\chi/r^2 = \int_\Omega A_1^0(U_2)\,\nabla\chi/r^2 + \int_\Omega A_1^1(\partial U_2)\,\nabla\chi/r^2 : \tag{155}$$

where Ω denotes the support of χ: $\Omega = \left\{(x,y) \in \mathbb{R}^2 \text{ such that } R - 2\varepsilon < \sqrt{x^2+y^2} < R + 2\varepsilon\right\}$. The other terms, as well as the other values of j can be treated similarly.

Step 1 : We prove that $\int_\Omega A_1^0(U_2)\,\nabla\chi/r^2 \leq C\varepsilon^2$.

It appears useful to work in the (s,σ) variables and thus to introduce a preliminary notation which is an equivalent in the variables (σ, s) of the notation (17) in the variables (r,θ). For any function of 3 variables $U(\sigma, s; \theta)$, we define

$$\{U\}_\varepsilon(\sigma, s) := U(\sigma, s; \frac{\varepsilon s}{R}): \tag{156}$$

Similarly, for any function $v(r,\theta)$, we introduce $\{v\}_\varepsilon(\sigma, s) := v(R + \varepsilon\sigma, \frac{\varepsilon s}{R})$.

Using the change of variables $s = R\theta/\varepsilon$, $\sigma = (r - R)/\varepsilon$, we get, after integration by parts

$$\int_\Omega A_1^0(U_2)\,\nabla\chi/r^2 = \varepsilon^2 \int_{\Omega_{\varepsilon,\theta}} \left\{\varepsilon^2 \frac{\partial U_2}{\partial\sigma}\right\}_\varepsilon \frac{\partial}{\partial\sigma}\left\{\frac{\nabla\chi}{r}\right\}_\varepsilon + \left\{\frac{\partial U_2}{\partial\theta}\right\}_\varepsilon \left\{\frac{\nabla\chi}{r}\right\}_\varepsilon \, d\sigma\, ds: \tag{157}$$

where, setting $N_\varepsilon = 2\pi R/\varepsilon$

$$\Omega_{\varepsilon,\theta} := \bigcup_{j=0}^{N_\varepsilon} B_{\varepsilon,\theta} + \begin{pmatrix} j+1/2 \\ 0 \end{pmatrix}; \quad B_{\varepsilon,\theta} =]-\frac{1}{2}, \frac{1}{2}[\times]-\frac{2}{\varepsilon}, \frac{2}{\varepsilon}[: \tag{158}$$

At this stage, it is useful to state a technical lemma whose proof is immediate:

Lemma 14. *Let $a(\theta) \in L^\infty(0, 2\pi)$, $f(\sigma, s) \in L^2_{loc}(\mathbb{R}^2)$, 1 periodic in s and $g \in L^2(\Omega_{\varepsilon,\theta})$, one has the estimate:*

$$\left|\int_{\Omega_{\varepsilon,\theta}} a(\frac{\varepsilon s}{R}) f(\sigma, s) g(\sigma, s)\, ds\, d\sigma\right| \leq \sqrt{\frac{2\pi R}{\varepsilon}}\, \|a\|_{L^\infty} \|f\|_{L^2(B_{\varepsilon,\theta})} \|g\|_{L^2(\Omega_{\varepsilon,\theta})} \tag{159}$$

At this point, we need to use the explicit expression (123) of U_2 from which we deduce that we can write

$$\left\{\varepsilon^2 \frac{\partial U_2}{\partial\sigma}\right\}_\varepsilon (s,\sigma) = \sum_{\ell=1}^{L} a_\ell(\frac{\varepsilon s}{R}) f_\ell(\sigma, s) \tag{160}$$

where the a_ℓ's are defined in function of the far fields $(u_0; u_1; u_2)$ and the f_ℓ's in function of the profiles N, U and U_{jk}. Which is important is not the explicit expression of these functions but the properties:

$$a_\ell(\theta) \in L^\infty(0, 2\pi); \quad f_\ell(\sigma, s) \in L^2_{loc}(\mathbb{R}^2); \quad f_\ell(\sigma, s+1) = f_\ell(\sigma, s);$$

as well as

$$\exists\, C_\ell > 0 \;/\; \forall\, |\sigma| > 1/2, \quad \forall s \in]-\frac{1}{2}, \frac{1}{2}[, \quad |f_\ell(\sigma, s)| \leq C_\ell\, \sigma^{-2}: \tag{161}$$

Then we can apply lemma 14 with $(a; f; g) = \left(a_\ell; f_\ell; \frac{\partial}{\partial\sigma}\left\{\frac{\nabla\chi}{r}\right\}_\varepsilon\right)$ to obtain

$$\left|\int_{\Omega_{\varepsilon,\theta}} \left\{\varepsilon^2 \frac{\partial U_2}{\partial\sigma}\right\}_\varepsilon \frac{\partial}{\partial\sigma}\left\{\frac{\nabla\chi}{r}\right\}_\varepsilon\right| \leq \frac{C}{\sqrt{\varepsilon}} \sum_{\ell=1}^{L} \|a_\ell\|_{L^\infty} \|f_\ell\|_{L^2(B_{\varepsilon,\theta})} \left\|\frac{\partial}{\partial\sigma}\left\{\frac{\nabla\chi}{r}\right\}_\varepsilon\right\|_{L^2(\Omega_{\varepsilon,\theta})}: \tag{162}$$

From the estimate (161), it is straightforward to deduce that

$$\|f_{\cdot}\|_{L^2(B\ ;\ )} \quad C_{\cdot} \quad -\ ^{\frac{5}{2}}: \tag{163}$$

On the other hand, using $\frac{@}{@}\ \frac{\nabla}{r} = \frac{@}{@r}\ \frac{\nabla}{r}$ and the fact that $1=r$ is bounded in $\ ;$, we get

$$\frac{@}{@}^{n}\ \frac{\nabla}{r}^{o}\ _{L^2(\ ;\ )} \quad C \quad k \quad v \quad k_{L^2(\ ;\ )} + k\ ^{n}\frac{@}{@r}(\quad v)^{o}\ k_{L^2(\ ;\ )} : \tag{164}$$

Next we use the following result, proven by using the change of variables $(s;\ )\ !\ \ (r;\ ) = \ R+\ ;\ s=R$

Lemma 15. *There exists a constant $C > 0$ such that for any $f\ 2\ L^2(\ )$,*

$$\|f f g\|_{L^2(\ ;\ )} \quad \frac{C}{\ }\ \|f\|_{L^2(\ )} \tag{165}$$

Finally it suffices to apply lemma 15 and the following inequality (similar to (143))

$$\|v\|_{L^2(\ )} \quad C\ \ ^{\frac{1}{2}}\ \|v\|_{H^1(\ )}; \tag{166}$$

to obtain after elementary manipulations (since   is bounded)

$$\frac{@}{@}^{n}\ \frac{\nabla}{r}^{o}\ _{L^2(\ ;\ )} \quad \frac{C}{\ ^{\frac{1}{2}}}\ \|v\|_{H^1(\ )}: \tag{167}$$

Regrouping (162), (163), and (167), we obtain the desired estimate, namely

$$\int_{\ ;\ }\ ^{n}\ 2\ \frac{@U_2}{@}^{o}\ \frac{@}{@}^{n}\ \frac{\nabla}{r}^{o} \quad C\ \frac{2}{3}\ \|v\|_{H^1(\ )} \tag{168}$$

In the same manner, one can prove that

$$\int_{\ ;\ }\ ^{n}\ \frac{@U_2}{@}^{o}\ ^{n}\ \frac{\nabla}{r}^{o} \quad C\ \frac{2}{3}\ \|v\|_{H^1(\ )}: \tag{169}$$

Without entering the details, let us point out that, to prove (169), we remark that

$$^{n}\ \frac{@U_2}{@}^{o}$$

can also be written in the form (160), where, this time the $f_{\cdot}$'s grow at most linearly in   instead of quadratically as in (161). Finally, the proof of step 1 is achieved by regrouping (157), (168) and (169).

Step 2 : We prove that $\displaystyle\int A_1^1(@U_2)\ \overline{\ \ \nabla}=r^2 \quad C\ (\ ^2 +\ )\ \|v\|_{H^1(\ )}:$

Proceeding as for (157) we first obtain

$$\begin{aligned}\int \frac{1}{r^2}\ A_1^1\ \frac{@U_2}{@}\ \overline{\ \ \nabla} &= \ ^2 \int_{\ ;\ } \frac{@}{@s}\ \frac{@U_2}{@}\ ^{n}\ \frac{\nabla}{r}^{o}\ d\ ds \\ &+ \ ^2 \int_{\ ;\ } \frac{@^2U_2}{@\ @s}\ ^{n}\ \frac{\nabla}{r}^{o}\ d\ ds:\end{aligned} \tag{170}$$

The second term in (170) is treated in the same manner as in step 1 (the details are left to the reader)

$$\int_{\ ;\ } \frac{@^2U_2}{@\ @s}\ ^{n}\ \frac{\nabla}{r}^{o}\ d\ ds \quad C\ \frac{\ }{2}\ \|v\|_{H^1(\ )}:$$

The first term must be treated differently because $f\,@_2(\ @\,U_2)g$ is not in $L^2(\ _;\ )$. That is why we have to transform this term using the following formula

$$\frac{@}{@s}\ \frac{@U_2}{@} = \frac{@}{@s}\ \frac{@U_2}{@}\quad \frac{}{R}\quad \frac{@^2 U_2}{@^2}\ : \tag{171}$$

Using the formula $\frac{@}{@s}\ n\ \frac{\nabla}{r}\ o = \frac{}{R}\ n\ \frac{@}{@}\ \frac{\nabla}{r}\ o$, we obtain, after "integration by parts"

$$\int_{;} \frac{@}{@s}\ \frac{@U_2}{@}\ n\ \frac{\nabla}{r}\ o\ d\,ds = \int_{;} \frac{1}{R}\ \frac{@U_2}{@}\ n\ \frac{@}{@}\ \frac{\nabla}{r}\ o$$
$$\int_{;} \frac{1}{R}\ \frac{@^2 U_2}{@^2}\ n\ \frac{\nabla}{r}\ o\ d\,ds;$$

Using the same method as in Step 1 gives

$$\int_{;} \frac{@}{@s}\ \frac{@U_2}{@}\ n\ \frac{\nabla}{r}\ o\ d\,ds \quad C\ \frac{3}{3}\ kvk_{H^1(\ )} \tag{172}$$

and one concludes by regrouping (169), (171) and (172).

Step 3: From the estimates of steps 1 and 2, we deduce, using (155), that

$$\int A_1(U_2)\quad \nabla = r^2 \quad C\ \ ^2 + (\ ^2 + \ ^2)\ kvk_{H^1(\ )}\quad C\ \ ^2\ kvk_{H^1(\ )}; \tag{173}$$

since $<$ and is bounded, which concludes our proof. ☐

4.5. Proof of the error estimate (130)

Using (140), (152) and (154), we deduce from (136) and (137) that:

$$u \quad u^2_{;}\ \ _{H^1(\ )}\quad C\ \ ^2 + \sum_{k=0} \ ^{2-k}\ ^k + \sum_{k=1} \ ^{k-1}\ \_\ \ ^{2k}\ e^{-2}\ = \tag{174}$$

This is where we are going to choose in function of . If we ignore the third term with the exponentials, we see that choosing proportional to would provide an estimate in ^{2} which is the best that we can expect. Unfortunately, the last term is only in ^{0} because we do not exploit the exponentially decaying (in $=$) terms. That is why we choose a slightly larger , namely

$$= p\ \ j\,h\ \ j; \quad \text{for some } p > 0:$$

The first two terms in the right hand side of (174) are bounded by a constant times $^2 j\,h\ \ j^2$ while the third term is bounded by a constant times $j\,h\ \ j^2\ \ ^{2p}$: Therefore, we optimize our global error estimate by choosing $p = 1 =$. In summary, if we introduce the global approximate solution

$$u^2_{;}\ := u^2_{(\ );} \quad \text{with} \quad (\ ) := \frac{j\,h\ \ j}{}: \tag{175}$$

(we recall that $u^2_{;}$ is defined by (135)), we obtain

$$ku \quad u^2_{;}\ k_{H^1(\ )}\quad C\ \ ^2 j\,h\ \ j^2: \tag{176}$$

The estimate (130) follows easily since, > 0 being fixed (arbitrarily small), and $u^2_{;}\quad u^2_{e;}$ in as soon as $(\ ) < \ = 2$.

5. Construction and analysis of approximate transmission conditions

We are now in a position to build and analyze approximate transmission conditions (ATC) using the explicit expressions of transmission conditions for u_0, u_1, and u_2 respectively given by (78), (118) and (127). The main idea behind the formal construction of these approximate models is the following: In order to obtain an approximate field u_k that approaches the exact one up to $O(\ ^{k+1})$ error terms, it is sufficient that u_k approximates $\sum_{j=0}^{k}\ ^{j} u_j$ up to $O(\ ^{k+1})$ error terms. In order to guarantee the latter it would be sufficient to write a set of equations for u_k that are satisfied by $\sum_{j=0}^{k}\ ^{j} u_j$ up to $O(\ ^{k+1})$ error terms. This is the first step that ensures consistency of the approximate models.

In order to guarantee convergence with predicted rates, one has to make sure that the identified approximate model is well posed and provide stability estimates uniformly with respect to . As we shall see, the latter considerations lead us to propose more than one family of ATC.

We shall denote in this section the unit circle by and a function defined on can be expressed in terms of the curvilinear abscissa . We choose this notation since the analysis of the problems related to (ATC) remains valid if $= \overline{\ ^{-}} \setminus \overline{\ ^{+}}$ is an arbitrary regular non intersecting curve.

5.1. A "natural" family of ATC

A first family of ATC corresponds with the natural expressions provided by (78), (118) and (127). The associated approximate models can be written in the form

$$
\begin{array}{|ll|}
\hline
{}_{\infty}4\, u_k + !^{2}\ {}_{\infty} u_k = f; & \text{in } \ ^{+} [\ \ ^{-}; \\
u_k = S^{\ ;k} \left\langle r \frac{@u_k}{@r} \right\rangle + K^{\ ;k}\ u_k & \text{on } \\
\left[r \frac{@u_k}{@r} \right] = D^{\ ;k}\ u_k \quad ?K^{\ ;k} \left\langle r \frac{@u_k}{@r} \right\rangle & \text{on } \\
\frac{@u_k}{@r} + i!u_k = 0; & \text{on } S_{R_e}; \\
\hline
\end{array}
\qquad (177)
$$

where $S^{\ ;k}$, $K^{\ ;k}$, $D^{\ ;k}$ are given loca5(vide)op.962or73.39 cm75046 9.9626 Tf 13.2]TJ/F46 9.9626 Tf 13.2]TJ/F46 9.96J/F46

Remark 16. *Following Remark 10, the expression of ATC can be simplified when and have some symmetries.*
For instance, if

$$\begin{cases} (s; \ ) = \ (\ s; \ ) \quad \textit{and} \quad (s; \ ) = \ (s; \ \ ); \\ (s; \ ) = \ (\ s; \ ) \quad \textit{and} \quad (s; \ ) = \ (s; \ \ ); \end{cases}$$

then

$$U_{\infty} = 0 \quad \textit{and} \quad U^{\infty}_{i;`} = 0 \quad \textit{for all } i;`:$$

In particular the ATC of order 1 and 2 have the same expressions.

In the case where $= \ _0$ *and* $= \ _0$ *are constant, then we further have*

$$N_{\infty} = \frac{\infty \quad 0}{0}; \qquad C_{00} = \frac{1}{\infty}(\ \infty \quad 0) \quad \textit{and} \quad C_{02} = \frac{\infty \quad 0}{\infty R^2}:$$

5.2. A larger family of ATC

To overcome the problem encountered with the first family of ATC, we propose a larger class of ATC obtained by writing an approximate problem on the domain $= \ ^+ [\ ^-$ where 0 is a given parameter and

$$^+ := f R + \quad r \quad R_e g \quad \text{and} \quad ^- := f 0 \quad r \quad R \quad g:$$

We shall assume that the support of f is strictly contained in . In this case the ATC would be interface conditions that couple boundary terms at $r = R +$ and $r = R$. For u defined on we shall denote

$$u^{\pm} := u j_{r = R \pm} \ ; \quad hu i \ := \frac{1}{2}(u^+ + u^-) \quad \text{and} \quad [u] \ := u^+ \quad u^-:$$

The new family of ATC can be derived from (177) by observing that if $u = \sum_{j=0}^{k} \ ^j u_j$ for $k = 0; 1$ or 2, then

$$[u] = O(\ ) ; \qquad r \frac{@u}{@r} = O(\ ) ;$$

and

$$\frac{@^2 u}{@r^2} = \ \frac{1}{r}\frac{@u}{@r} \quad \frac{1}{r^2}\frac{@^2 u}{@^2} \quad \frac{!^2 \ \infty}{\infty} u \quad \text{at} \quad r = R \quad :$$

Therefore, using Taylor expansions, one can check that

$$[u] = [\ u] \quad \frac{2}{R} \quad r \frac{@u}{@r} \quad + O(\ ^3);$$

$$r \frac{@u}{@r} = \ r \frac{@u}{@r} \ + 2 \ R \quad \frac{1}{R^2}\frac{@^2}{@^2} + \frac{!^2 \ \infty}{\infty} \quad hu i \quad + O(\ ^3);$$

$$hu i = hu i \ + O(\ ^2) \quad \text{and} \quad r \frac{@u}{@r} \ = \ r \frac{@u}{@r} \ + O(\ ^2):$$

Remembering that $u^k = \sum_{j=0}^{k} \ ^j u_j + O(\ ^{k+1})$ and using previous observations, it is easily seen from (177) that the following family of approximate problems have the same consistency properties as (177), i.e. they are verified by $\sum_{j=0}^{k} \ ^j u_j$ up to $O(\ ^{k+1})$ terms. These problems can be written as (in order to av introducing new notation we shall still denote by u^k the solution to an approximate problem of order k)

$$
\begin{array}{|ll|}
\hline
{}_\infty \Delta u_k + \omega^2 \, {}_\infty u_k = f; & \text{in } \Omega^+ \cup \Omega^-; \\
u_k = S^{;k} \left\langle r \frac{\partial u_k}{\partial r} \right\rangle + K^{;k} \langle u_k \rangle; & \\
\left[r \frac{\partial u_k}{\partial r} \right] = D^{;k} \langle u_k \rangle - {}^{?}K^{;k} \left\langle r \frac{\partial u_k}{\partial r} \right\rangle; & \\
\frac{\partial u_k}{\partial r} + i\omega u_k = 0; & \text{on } S_{R_e}; \\
\hline
\end{array}
\qquad (178)
$$

where

$$S^{;0} = K^{;0} = D^{;0} = 0$$

$$
\begin{cases}
S^{;1} = 2 \dfrac{N_\infty +}{R}; \qquad K^{;1} = 2 \, U_\infty \dfrac{\partial}{\partial}; \\
D^{;1} = 2 R \left(\omega^2 (C_{00} - \dfrac{\infty}{\infty}) + (C_{02} - \dfrac{}{R^2}) \dfrac{\partial^2}{\partial^2} \right);
\end{cases}
$$

$$
\begin{cases}
S^{;2} = S^{;1} + 2 \, {}^2 \dfrac{U^\infty_{1;0}}{R}; \qquad K^{;2} = K^{;1} + 2 \, {}^2 (\omega^2 U^\infty_{0;0} + U^\infty_{0;1} \dfrac{\partial}{\partial} + U^\infty_{0;2} \dfrac{\partial^2}{\partial^2}); \\
D^{;2} = D^{;1} + 2 \, {}^2 R (\omega^2 D_{00} + D_{01} \dfrac{\partial}{\partial} + D_{02} \dfrac{\partial^2}{\partial^2}):
\end{cases}
$$

Note that, for $= 0$ one indeed retrieve the same expressions as in (177).

Unfortunately the stability properties of the obtained problems are not completely satisfactory. To illustrate this fact let us consider the symmetric case described in Remark 16. In this case the expressions of the conditions of order 1 and 2 coincide. As shown in appendix Appendix A, if one chose such that

$$C_{02} - \frac{}{R^2} < 0 \quad \text{and} \quad N_\infty + > 0$$

then the approximate problems are well posed and one obtain stability estimates that are uniform with respect to sufficiently small . However, contrary to the case of the exact problem, for a given (sufficiently small) the uniqueness of the solution does not hold for all frequencies ω (see Proposition 25). Even if this fact may not be essential for approximate problems at a given frequency, we found it desirable to see whether approximate problems that are well posed for all frequencies can be derived. This is why we propose and analyze in the following non centered formulations of the ATC. Let us remark that the uniform stability with respect to and the frequency ω has more impact on time-dependent problems since a lack of it may lead to severe instability of any (explicit) time-dicretization scheme associated with the approximate problem (see for instance [29, 30]).

5.3. A family of non centered ATC

We shall propose here another family of ATC for which the stability properties in terms of the small parameter and the frequency ω are similar to the exact one. However this will be done at the expense of breaking the symmetry of the roles played the boundary terms at each side of . The idea of proposing such condition has been inspired by the encountered difficulties in the uniqueness proof: namely, for the centered conditions, if the solution vanishes in one domain, this does not automatically implies that it is also zero in the other one (for all values of).

Since the treatment of the second order condition requires an extra work as compared to the first order one, we separate the two cases.

5.3.1. Non centered first order approximate condition

The derivation of the non centered expressions relies on the observation that for a given function u,

$$\langle u\rangle = (u)^{+} - \frac{1}{2}[u] \quad \text{and} \quad \langle u\rangle = (u)^{-} + \frac{1}{2}[u]\ . \tag{179}$$

Therefore, from (178) with $k = 1$ one easily deduces that the approximate problem defined by

$$\begin{array}{|ll|}
\hline
{}_{\infty}\Delta u_1 + \omega^2\, {}_{\infty} u_1 = f; & \text{in } \Omega^{+} \cup \Omega^{-};\\
\left[u_1 \right] = S^{;1} \left(\frac{\partial u_1}{\partial r} \right)^{+} + K^{;1} \left(u_1 \right)^{-}; & \\
\left[\frac{\partial u_1}{\partial r} \right] = D^{;1} \left(u_1 \right)^{-} - K^{\star;1} \left(\frac{\partial u_1}{\partial r} \right)^{+}; & \\
\frac{\partial u_1}{\partial r} + i\omega u_1 = 0; & \text{on } S_{R_e};\\
\hline
\end{array} \tag{180}$$

also constitutes an approximation of the exact problem up to $O(\,^2)$ terms. Indeed, another condition (that admits similar properties) can be obtained by interchanging the plus and minus signs in the previous expression.

5.3.2. Non centered second order approximate condition

The derivation the second order condition is more technical and needs two steps. The first step is similar to the previous case and relies on the use of the identities (179) in order to transform the transmission conditions in (178). After straightforward calculations one obtain that, if $u_2 = u_0 + u_1 + {}^2u_2 + O(\,^3)$ and satisfies (178), then

$$\left[u_2 \right] = S^{;2} \left(\frac{\partial u_2}{\partial r} \right)^{+} + (K^{;2} + {}^2M) \left(u_2 \right)^{-} + O(\,^3)$$

$$\left[\frac{\partial u_2}{\partial r} \right] = D^{;2} \left(u_2 \right)^{-} - (K^{\star;2} + {}^2M) \left(\frac{\partial u_2}{\partial r} \right)^{+} + O(\,^3);$$

where

$$M := 2\omega^2 N_\infty C_{00} + 2(U_\infty)^2 - 2N_\infty C_{02} \frac{\partial^2}{\partial^2} \tag{181}$$

and

$$C_{00} := C_{00} \frac{\infty}{\infty}; \quad C_{02} := C_{02} \frac{}{R^2}; \quad N_\infty = N^{\infty} + \ . \tag{182}$$

Neglecting the $O(\,^3)$ term would provide an ATC that has similar structure as in (180). However, variational analysis of the associated problem (see Section 5.4) show that one needs the boundary operator on to be elliptic with respect to $H^2(\ )$ norm. This is required by the second order derivatives present in $K^{;2} + {}^2M$, but cannot be guaranteed from the variational formulation of the problem. This is why we shall modify the expression of this condition in a consistent way (i.e. preserving the $O(\,^3)$ accuracy) by replacing the operator $K^{;2} + {}^2M$ with a pseudo-differential operator of order 1. The adopted (classical [10, 30]) technique is based on the use of so called Padé approximations and indeed this can be done in a number of different manners. One possibility would be to write

$$K^{;2} + {}^2M = 1 + 2(U_\infty + {}^2U^{\infty}_{0;1}) \frac{\partial}{\partial} + 2\,{}^2\omega^2\, U^{\infty}_{0;0}\, N_\infty C_{00}$$

$$+ \ 1 + 2\,{}^2\, U^2_\infty + U^{\infty}_{0;2}\, N_\infty C_{02} \frac{\partial^2}{\partial^2}$$

We then remark that (at least formally)

$$1 + 2\,{}^2\, U_\infty^2 + U_{0;2}^\infty\; N\; {}_\infty C_{02}\, \frac{\partial^2}{\partial^2} \;=\; 1 \;\; 2\,{}^2\, U_\infty^2 + U_{0;2}^\infty\; N\; {}_\infty C_{02}\, \frac{\partial^2}{\partial^2} {}^{-1} + O({}^4): \tag{183}$$

Let us set

$$\begin{aligned} \overleftarrow{K}^{\,;2} := \;& 1 + 2(\,U_\infty + {}^2U_{0;1}^\infty)\frac{\partial}{\partial} + 2\,{}^2\omega\,{}^2\;\overline{U}_{0;0}^\infty\; N\; {}_\infty \overline{C}_{00} \\ & + \;1\;\; 2\,{}^2\, U_\infty^2 + U_{0;2}^\infty\; N\; {}_\infty C_{02}\, \frac{\partial^2}{\partial^2} {}^{-1} : \end{aligned} \tag{184}$$

Observing that

$$K^{\,;2} + {}^2M = \overleftarrow{K}^{\,;2} + O({}^4) \quad \text{and} \quad {}^?K^{\,;2} + {}^2M = {}^?\overleftarrow{K}^{\,;2} + O({}^4)$$

we conclude that the system of equations

$$\boxed{\begin{array}{ll} {}_\infty 4\, u_2 + \omega^2\,{}_\infty u_2 = f; & \text{in } {}^+ \,[\, {}^- ; \\ u_2 = S^{\,;2}\; r\frac{\partial u_2}{\partial r}{}^+ + \overleftarrow{K}^{\,;2}\; u_2{}^- ; & \\ \left[r\frac{\partial u_2}{\partial r} \right] = D^{\,;2}\; u_2{}^- \;\; {}^?\overleftarrow{K}^{\,;2}\; r\frac{\partial u_2}{\partial r}{}^+ ; & \\ \frac{\partial u_2}{\partial r} + i\omega u_2 = 0; & \text{on } S_{R_e}; \end{array}} \tag{185}$$

would be consistent up to $O({}^3)$ error terms. This is the family of non centered second order conditions that we shall consider in the sequel.

5.4. Analysis of a model problem related to non centered

This section is devoted to proving well posedness of the problems associated ATC and prove uniform stability with respect to and ω. In fact we shall provide here a general framework that encompass both first and second order conditions. We consider problems of the form

$$\boxed{\begin{array}{ll} {}_\infty 4\, w + \omega^2\,{}_\infty w = f; & \text{in } {}^+ \,[\, {}^- ; \\ w = S\; r\frac{\partial w}{\partial r}{}^+ + K\; w^- + h; & \\ \left[r\frac{\partial w}{\partial r} \right] = D\; w^- \;\; {}^?K\; r\frac{\partial w}{\partial r}{}^+ + g; & \\ \frac{\partial w}{\partial r} + i\omega w = 0; & \text{on } S_{R_e}; \end{array}} \tag{186}$$

where f, g and h are given data and where the boundary operators S, K and D have the following properties.

We shall denote by $\langle ; \rangle$ the duality pairing between spaces defined on that extends the $L^2(\;)$ scalar product.

Hypothesis 17.

1. $S : L^2(\Gamma) \to L^2(\Gamma)$ *is a positive definite and selfadjoint isomorphism.*
2. $D : H^1(\Gamma) \to H^{-1}(\Gamma)$ *is a continuous and selfadjoint operator.*
3. $I + K : H^1(\Gamma) \to L^2(\Gamma)$ *is a continuous and injective operator.*

We first derive the variational formulation associated with (186). Multiplying the first equation with a test function v and integrating over $\tilde{\Omega} := \Omega^+ \cup \Omega^-$ shows that (using the boundary conditions on S_{R_e})

$$\int_{\tilde{\Omega}} \left(\nabla w \cdot \nabla v - \omega^2 \varepsilon_\infty \mu_\infty w\, v \right) dx + i\omega \int_{S_{R_e}} w\, v\, ds + \left\langle \frac{\partial w}{\partial r}^+ ; (v)^+ \right\rangle - \left\langle \frac{\partial w}{\partial r}^- ; (v)^- \right\rangle = \int_{\tilde{\Omega}} (f/\varepsilon_\infty) v\, dx$$

Algebraic transformations show that

$$\left\langle \frac{\partial w}{\partial r}^+ ; (v)^+ \right\rangle - \left\langle \frac{\partial w}{\partial r}^- ; (v)^- \right\rangle = \left\langle (I + K^*) \frac{\partial w}{\partial r}^+ - \frac{\partial w}{\partial r}^- ; (v)^- \right\rangle + \left\langle \frac{\partial w}{\partial r}^+ ; (v)^+ - (I + K)(v)^- \right\rangle$$

Let

$$V := \{ v \in H^1(\Omega^+ \cup \Omega^-) \text{ such that } (v)^- \in H^1(\Gamma) \}.$$

Using the boundary conditions we end up with a variational formulation of the form: $w \in V$ and

$$a(w; v) + b\left(\left(w^+ ; w^- \right); \left((v)^+ ; (v)^- \right) \right) = \ell(v) \tag{187}$$

for all $v \in V$, where

$$a(w; v) := \int_{\tilde{\Omega}} \left(\nabla w \nabla v - \omega^2 \varepsilon_\infty \mu_\infty w v \right) dx + i\omega \int_{S_{R_e}} w v\, ds$$

for all w and v in $H^1(\Omega^+ \cup \Omega^-)$,

$$b((\phi^+; \phi^-); (\psi^+; \psi^-)) := \langle D \phi^-; \psi^- \rangle + \langle (S)^{-1}(\phi^+ - (I + K)\phi^-); \psi^+ - (I + K)\psi^- \rangle$$

for all $(\phi^+; \phi^-)$ and $(\psi^+; \psi^-)$ in $L^2(\Gamma) \times H^1(\Gamma)$ and

$$\ell(v) := \int_{\tilde{\Omega}} (f/\varepsilon_\infty) v\, dx - \left\langle g; (v)^- \right\rangle - \left\langle (S)^{-1} h; (v)^+ - (I + K)(v)^- \right\rangle$$

for all $v \in V$.

Theorem 18. *Let $f \in L^2(\tilde{\Omega})$, $g \in H^{-1}(\Gamma)$ and $h \in L^2(\Gamma)$ and assume that Hypothesis 17 holds. Then:*

> *$w \in V$ satisfies (187) if and only if $w \in V$ satisfies (186) in the distributional sense for the equations in $\tilde{\Omega}$ and in the sense of traces for the boundary or interface conditions. The first interface conditions is an equality between functions in $L^2(\Gamma)$. The second interface conditions is an equality between functions in $H^{-1}(\Gamma)$.*
>
> *Problem (186) has at most one solution $w \in V$.*

Proof. Given the derivation of the variational formulation explained above, the first point became a simple exercise on the interpretation of the variational formulations in the distributional sense.
For the uniqueness, assume that $\ell = 0$, then taking $v = w$ in (187) and considering the imaginary part, on deduces that $w = 0$ on S_{R_e}. The boundary condition on S_{R_e} then implies that the normal derivative vanishes on S_{R_e}. This implies that $w = 0$ in Ω^+. The first interface condition then implies that $(I + K) w^- = 0$ and therefore $w^- = 0$. Then the second interface condition implies that $\partial w / \partial r^- = 0$. This proves that we also have $w = 0$ in Ω^-. □

We now consider the issue of existence and stability. One needs further assumptions on the operators S, K and D.

Hypothesis 19.

1. *The operators* $S : L^2( ) \to L^2( )$, $D : H^1( ) \to H^{-1}( )$, $K : H^1( ) \to L^2( )$ *depend continuously on* > 0.
2. *There exist two constants* $c_1 > 0$ *and* $c_2 \geq 0$ *independent of such that*

$$c_1 \| \|^2_{L^2( )} \quad S \; ; \qquad \text{and} \quad \|S \|_{L^2( )} \quad c_2 \| \|_{L^2( )} \quad \text{for all} \quad \in L^2( ):$$

3. *There exist three constants* $c_1 > 0$, $c_2 \geq 0$ *and* $c_3 \geq 0$ *independent of such that*

$$D \; ; \qquad (c_1 \| \|^2_{H^1( )} \quad c_2 \| \|^2_{L^2( )}) \quad \text{and} \quad \|D \|_{H^{-1}( )} \quad c_3 \| \|_{H^1( )}$$

   for all $\in H^1( )$:
4. *There exists a constant* $c > 0$ *independent of such that*

$$\|K \|_{L^2( )} \quad c \| \|_{H^1( )} \quad \text{for all} \quad \in H^1( ):$$

Theorem 20. *Under the hypothesis of Theorem 18, and assuming that Hypothesis 19 holds, problem (187) has a unique solution* $w \in V$. *Moreover, for all* $_0 >$ *there exists a constant* c *independent of such that*

$$\|w \|_{H^1(\tilde{\ })} \quad c \; \|f \|_{L^2(\tilde{\ })} + {}^{-1=2}\|g\|_{H^{-1}( )} + {}^{-1=2}\|h\|_{L^2( )} \; : \tag{188}$$

for all $\in [0; _0]$.

Proof. The space V equipped with the norm $\|w \|^2_V := \|w \|^2_{H^1(\tilde{\ })} + \| w^- \|^2_{H^1( )}$ is a Hilbert space. Using Hypothesis 19 one easily observes that for fixed > 0, the variational problem (187) is of Fredholm type. Then, existence of solutions immediately follows from the uniqueness result of Theorem 18.

For the stability estimate we shall consider only the case of $= 0$. The proof in the case $\neq 0$ follows the same lines as in the case $= 0$ after operating the change of variables (A.5) in $^+$ and $^-$ (transforming them into $^+_0$ and $^-_0$ respectively). To shorten notation we shall denote $\tilde{\ } := {}^+_0 \cup {}^-_0$.

Let $_0 > 0$ be given and let $T : V^0 \to V^0$ be the operator associated with the sesquilinear form in (187). The first point in Hypothesis 19 shows that T depends continuously on $\in [_1; _0]$ for all $_1 > 0$. The invertibility of this operator for all values of $\in [_1; _0]$ then implies that T^{-1} is uniformly bounded with respect to $\in [_1; _0]$. Inequality (188) is then verified for all $\in [_1; _0]$ for a constant c that may depend on $_1$. We shall know prove that this constant remains bounded as $_1 \to 0$. The proof of such result can be done by a contradiction argument. The main ingredient is the following a priori estimate that can be easily deduced from (187) by taking $v = w$ and by using Hypothesis 19: setting

$$N (w) := \|w\|^2_{H^1(\tilde{\ })} + \| (w)^- \|^2_{H^1( )} + {}^{-1}\| (w)^+ \quad (I + K)(w)^- \|^2_{L^2( )}$$

and

$$` := \|f \|_{L^2(\tilde{\ })} + {}^{-1=2}\|g\|_{H^{-1}( )} + {}^{-1=2}\|h\|_{L^2( )}$$

there exists two constants C_1 and C_2 independent of such that,

$$N (w) \quad C_1 \; \|w \|^2_{L^2(\tilde{\ })} + \| w^- \|^2_{L^2( )} \; + C_2 ` \sqrt{N (w)}: \tag{189}$$

We first assume that there exists a sequence $_n \to 0$ as $n \to \infty$ such that

$$\|w^n\|_{L^2(\tilde{\ })} + {}_n^{1=2}\| w^{n -}\|_{L^2( )} = `^n \to \infty \; :$$

Setting

$$w_n = w^n = \; \|w^n\|_{L^2(\tilde{\ })} + {}_n^{1=2}\| w^{n -}\|_{L^2( )}$$

We deduce from (189) that (w_n) is bounded in $H^1(\tilde\Omega)$. Up to an extracted sub-sequence one can assume that (w_n) weakly converges to some $w \in H^1(\tilde\Omega)$ and that the convergence is strong in $L^2(\tilde\Omega)$. We can also assume that $(w_n)^-$ and $(w_n)^+$ strongly converge to $(w)^-$ and $(w)^+$ in $L^2(\Gamma)$. Moreover, w_n satisfies (186) with data $f = f_n$, $h = h_n$ and $g = g_n$ going to 0 as $n \to \infty$ respectively in $L^2(\tilde\Omega)$, $L^2(\Gamma)$ and $H^{-1}(\Gamma)$. The first equation in (186) implies in particular that $(\nabla\cdot A_n\nabla w_n)$ is bounded in $L^2(\tilde\Omega)$. This shows that $(\partial w_n/\partial\nu)^+$ and $(\partial w_n/\partial\nu)^-$ are bounded in $H^{-1/2}(\Gamma)$ and therefore can be assumed to be weakly convergent to $(\partial w/\partial\nu)^+$ and $(\partial w/\partial\nu)^-$ in $H^{-1/2}(\Gamma)$. Taking the limit in the first and the last equation in (186) we observe that

$$\begin{cases} \nabla\cdot A_\infty\nabla w + \omega^2 n_\infty w = 0; & \text{in } \tilde\Omega; \\ \dfrac{\partial w}{\partial r} + i\omega w = 0; & \text{on } S_{R_e}. \end{cases} \tag{190}$$

Estimate (189) also implies that $((w_n)^+ - (I + K^{n})(w_n)^-) \to 0$ in $L^2(\Gamma)$ as $n \to \infty$. Therefore, from the last property in Hypothesis 19 one concludes that

$; \quad \cup$

Theorem 22.

Assume that the first part of Theorem 21 holds. Then problem (180) has a unique solution u_1 for all $\delta > 0$. There exists a constant $C > 0$ and, for each $0 < \delta < R$, a constant $\varepsilon > 0$ such that

$$\forall \delta < \varepsilon; \quad \|u - u_1\|_{H^1(\Omega)} \le C\,\delta^2. \tag{193}$$

Assume that the second part of Theorem 21 holds. Then problem (180) has a unique solution u_2 for all $0 < \delta < \delta_0$. There exists a constant $C > 0$ and, for each $0 < \delta < R$, a constant $\varepsilon > 0$ and $\eta < \eta_0$ such that

$$\forall \delta < \varepsilon; \quad \|u - u_2\|_{H^1(\Omega)} \le C\,\delta^3 (h\,\eta)^2. \tag{194}$$

Proof. Existence and uniqueness results are immediate consequences of Theorems 20 and 21. Concerning error estimates, we shall not give the entire details but only explain the main steps (which are classical). Thanks to the stability result of Theorem 20 one can prove (for $k = 1$ and $k = 2$) that

$$u_k = \sum_{j=0}^{\infty} \delta^j \tilde{u}_k^j \quad \text{in } \Omega^{\pm}$$

for some regular functions $\tilde{u}_k^j$ defined on $\Omega^{\pm}$. The consistency of ATC would show that $\tilde{u}_k^j = u^j$ for $j = 0, \dots, k$: practically one formally obtain the same set of equations for $\tilde{u}_k^j$ as for u^j (see (78), (118) and (127)). The error estimate is then obtained as consequence of Proposition 11. □

Appendix A. Analysis of a model problem related to centered ATC

A model problem corresponding to centered ATC of order 1 would be

$$\begin{cases} \Delta w + \omega^2 \frac{1}{\gamma} w = \frac{f}{\gamma} & \text{in } \Omega^+ \cup \Omega^-; \\ \langle w \rangle_\Gamma = A_0\, \delta \left\langle \frac{\partial w}{\partial r} \right\rangle_\Gamma; \\ \left[\frac{\partial w}{\partial r} \right]_\Gamma = \delta \left(B_0 \omega^2 \langle w \rangle_\Gamma + B_2 \frac{\partial^2}{\partial \theta^2} \langle w \rangle_\Gamma \right); \\ \frac{\partial w}{\partial r} + i\omega w = 0 & \text{on } S_{R_e}; \end{cases} \tag{A.1}$$

where $A_0 > 0$, $B_2 < 0$ and B_0 are given constants and where $f \in L^2(\Omega^+ \cup \Omega^-)$ is a given function.

Remark 23. *Following the analysis of Section 5.4 in the derivation of variational formulation, the reader can easily verify that Theorem 24 also applies to a more general framework for centered ATC, namely transmission conditions of the form*

$$\begin{aligned} \langle w \rangle_\Gamma &= S_\delta \left\langle \frac{\partial w}{\partial r} \right\rangle_\Gamma + K_\delta [w]_\Gamma; \\ \left[\frac{\partial w}{\partial r} \right]_\Gamma &= D_\delta \langle w \rangle_\Gamma - {}^{t}K_\delta \left\langle \frac{\partial w}{\partial r} \right\rangle_\Gamma \end{aligned} \tag{A.2}$$

where the operators S_δ, K_δ and D_δ satisfy Hypothesis 17 and 19. A similar technique as in section 5.3.2 can be used to propose a second order centered ATC technique that enter into this framework providing that $U^{\infty}_{0,2}$ is non negative.

We consider the Hilbert space,

$$V_\delta = \left\{ u \in H^1(\Omega^+_\delta \cup \Omega^-_\delta); \text{ such that } \langle u \rangle_\delta \in H^1(\Gamma) \right\};$$

equipped with the norm

$$\|v\|_{V_\delta} = \|v\|^2_{H^1(\Omega^+_\delta \cup \Omega^-_\delta)} + \frac{1}{A_0}\int_0^{2\pi} |[v]_\delta|^2 d\theta + |B_2|\delta\infty \int_0^{2\pi} \left(|\langle v\rangle_\delta|^2 + \left|\frac{\partial \langle v\rangle_\delta}{\partial\theta}\right|^2 \right) d\theta:$$

The variational formulation associated with this problem can be written in the form: find $w_\delta \in V_\delta$ such that

$$a_\delta(w_\delta; v) = \int_{\Omega^+_\delta \cup \Omega^-_\delta} \frac{f}{\infty} v \quad \forall v \in V_\delta;$$

where,

$$a_\delta(u;v) = \int_{\Omega^+_\delta \cup \Omega^-_\delta} \nabla u \cdot \nabla v - \frac{\omega^2 \infty}{\infty} uv + i\omega \int_{S_{R_e}} uv$$
$$+ \frac{1}{A_0}\int_0^{2\pi} [u]_\delta [v]_\delta + B_0 \omega^2 \delta \int_0^{2\pi} \langle u\rangle_\delta \langle v\rangle_\delta \, d\theta - B_2 \delta \int_0^{2\pi} \frac{\partial}{\partial\theta}\langle u\rangle_\delta \frac{\partial}{\partial\theta}\langle v\rangle_\delta \, d\theta:$$

It is easily seen that a_δ can be split into a compact part and and a coercive one and therefore the problem is of Fredholm type.
We can now prove that problem (A.1) is well-posed and is stable uniformly with respect to δ.

Theorem 24. *There exist $C_\omega > 0$ and $\delta_0 > 0$ such that, for any $\delta < \delta_0$, for any $u \in V_\delta$.*

$$\|u\|_{V_\delta} \le C_\omega \sup_{v \in V_\delta; v \neq 0} \frac{a_\delta(u;v)}{\|v\|_{V_\delta}}: \tag{A.3}$$

Proof. The proof is done by contradiction: let δ_n be a sequence which tends to 0 when n tends to $+\infty$. We assume that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that:

$$\|u_n\|_{V_{\delta_n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{v \in V_{\delta_n}; v\neq 0} \frac{a_{\delta_n}(u_n;v)}{\|v\|_{V_{\delta_n}}} = 0: \tag{A.4}$$

In order to work in a fix domain (independent of δ), we shall consider F^-_δ and F^+_δ:

$$F^-_\delta := \begin{cases} [0;R] \to [0;R-\delta]; \\ \hat{x} \mapsto \frac{R-\delta}{R}\hat{x}; \end{cases} \qquad F^+_\delta := \begin{cases} [R;R_e] \to [R+\delta;R_e]; \\ \hat{x} \mapsto \frac{R_e-R-\delta}{R_e-R}|\hat{x}| + \frac{R_e\delta}{R_e-R}\frac{\hat{x}}{|\hat{x}|}: \end{cases} \tag{A.5}$$

Since $DF^+_\delta(\hat{x}) = \dfrac{R_e - R - \delta}{R_e - R} I$ and $DF^-_\delta(\hat{x}) = \dfrac{R-\delta}{R} I$, $DF^\pm_\delta$ uniformly tends to the identity matrix when δ tends to 0. The same is true of $\det(DF^\pm_\delta)$ which tends to 1. Moreover there are $C_1^\pm$ and $C_2^\pm$ independent of δ such that

$$C_1^\pm |\xi| < |DF^\pm_\delta \xi| < C_2^\pm |\xi| \quad \forall \xi \in \mathbb{R}^2;$$

$$C_1^\pm < \det(DF^\pm_\delta) < C_2^\pm:$$

We also introduce three open sets independent of δ,

$$\Omega^+ = B(0;R_e) \setminus B(0;R); \qquad \Omega^- = B(0;R); \qquad \Omega = \Omega^+ \cup \Omega^-;$$

and the Hilbert space $\tilde V^0$

$$\tilde V^0 = \left\{ v \in H^1(\Omega^+) \cap H^1(\Omega^-);\ \langle v\rangle \in H^1(\Gamma) \right\};$$

equipped with the norm

$$\|v\|_{\tilde V^0} = \left(\|v\|^2_{H^1(\Omega^+\cup\Omega^-)} + \frac{1}{A_0}\int_0^{2\pi} |[v]|^2\, d\theta + |B_2|\,\delta \int_0^{2\pi} |\langle v\rangle|^2 + \left|\frac{\partial\langle v\rangle}{\partial\theta}\right|^2 d\theta \right): \tag{A.6}$$

Note that $F_\delta^\pm$ transforms $\Omega^\pm$ into $\Omega_\delta^\pm$. Then we define $\hat u_n$

$$\hat u_n(\hat x) := \begin{cases} u_n \circ F_\delta^+(\hat x) & \text{if } |\hat x| < R; \\ u_n \circ F_\delta^-(\hat x) & \text{if } |\hat x| > R; \end{cases}$$

and the bilinear form $\hat a^{\delta_n}$

$$\begin{aligned}
\hat a^{\delta_n}(\hat u_n; \hat v) &:= a^{\delta_n}(u_n; v); \\
&:= \int_{\Omega^+} (DF^+(\hat x)^{-1})(DF^+(\hat x)^{-1*})\,\varepsilon \hat\nabla \hat u_n \cdot \overline{\hat\nabla \hat v}\, |\det DF^+| + \int_{S_{R_e}} i\omega \sqrt{\varepsilon}\, \hat u_n v\, d\sigma \\
&\quad + \int_{\Omega^-} (DF^-(\hat x)^{-1})(DF^-(\hat x)^{-1*})\,\varepsilon \hat\nabla \hat u_n \cdot \overline{\hat\nabla \hat v}\, |\det DF^-| \\
&\quad - \int_{\Omega^+} \varepsilon\omega^2 \hat u_n \hat v\, |\det DF^+| - \int_{\Omega^-} \varepsilon\omega^2 \hat u_n \hat v\, |\det DF^-| \\
&\quad - B_2\,\delta \int_0^{2\pi} \frac{\partial\langle \hat u_n\rangle}{\partial\theta}\frac{\partial \hat v}{\partial\theta}\, d\theta + \omega^2 B_0\,\delta \int_0^{2\pi} \langle \hat u_n\rangle\langle \hat v\rangle\, d\theta \\
&\quad + \frac{\delta}{A_0}\int_0^{2\pi} [\hat u_n][\hat v]\, d\theta \quad \forall \hat v \in \tilde V^0:
\end{aligned} \tag{A.7}$$

Using the properties of $F_\delta^\pm$ and (A.4), we can assert that there exist two constants A and B independent of δ such that

$$0 < A \le \|\hat u_n\|_{\tilde V^0} \le B; \tag{A.8}$$

$$\sup_{\hat v \in \tilde V^0; \hat v \neq 0} \frac{\hat a^{\delta_n}(\hat u_n; \hat v)}{\|\hat v\|_{\tilde V^0}} = 0: \tag{A.9}$$

Therefore, there is a sub-sequence (still denoted by $(\hat u_n)$) and a function $\hat u_0 \in H^1(\Omega^+) \cup H^1(\Omega^-)$ such that

$$\hat u_n \rightharpoonup \hat u_0^+ \text{ weakly in } H^1(\Omega^+);$$

$$\hat u_n \rightharpoonup \hat u_0^- \text{ weakly in } H^1(\Omega^-);$$

$$\hat u_n \rightharpoonup \hat u_0^\pm \text{ weakly in } H^{1/2}(S_R):$$

In addition, it follows from (A.8) that $\frac{1}{A_0}\int_0^{2\pi} |[\hat u_n]|^2\, d\theta \le B$. Consequently, by uniqueness of the weak limit,

$$\lim_{n\to\infty} \int_0^{2\pi} |[\hat u_n]|^2\, d\theta = \int_0^{2\pi} |[\hat u_0]|^2\, d\theta = 0:$$

Therefore $[\hat{u}_0] = 0$ and $\hat{u}_0$ is in $H^1(\Omega)$.
Moreover, since $\|\sqrt{\delta}\langle \hat{u}_n\rangle\|_{H^1(\Gamma)}$ is bounded, we also have

$$\lim_{n\to\infty} \left(B_{2\,\infty}\int_0^{2\pi} \frac{\partial \langle\hat{u}_n\rangle}{\partial\theta}\frac{\partial v}{\partial\theta}\,d\theta + \omega^2 B_{0\,\infty}\int_0^{2\pi}\langle\hat{u}_n\rangle v\,d\theta\right) = 0 \quad \forall v \in V^0.$$

Therefore, letting n tends to $+\infty$ in the bilinear form (A.7) and using test functions in $H^1(\Omega)\cap V^0$, yields

$$0 = \int_\Omega \rho_\infty \nabla \hat{u}_0 \cdot \nabla v - \omega^2 \kappa_\infty \hat{u}_0 v + \int_{S_{R_e}} i \rho_\infty \omega\, \hat{u}_0 v \quad \forall v \in H^1(\Omega)\cap V^0.$$

By density of $H^1(\Omega)\cap V^0$ in $H^1(\Omega)$, the previous equality also holds for any $v \in H^1(\Omega)$:

$$0 = \int_\Omega \rho_\infty \nabla \hat{u}_0 \cdot \nabla v - \omega^2 \kappa_\infty \hat{u}_0 v + \int_{S_{R_e}} i \rho_\infty \omega\, \hat{u}_0 v \quad \forall v \in H^1(\Omega).$$

The previous formulation is the classical variational formulation associated to the problem without a thin layer: hence $\hat{u}_0 = 0$. So $\hat{u}_n$ strongly tends to 0 in $L^2(\Omega^\pm)$ and $\sqrt{\delta}\langle\hat{u}_n\rangle$ strongly tends to 0 in $L^2(\Gamma)$. To obtain a contradiction we only need to check that $\|\hat{u}_n\|^2_{V^\delta}$ tends to 0. But,this verified since

$$\|\hat{u}_n\|^2_{V^\delta} \leq C\left(\hat{a}(\hat{u}_n;\hat{u}_n) + \|\hat{u}_n\|^2_{L^2(\Omega^+)} + \|\hat{u}_n\|^2_{L^2(\Omega^-)} + \|\sqrt{\delta}\langle\hat{u}_n\rangle\|^2_{L^2(\Gamma)}\right);$$

and the right hand side tends to 0 as $n \to \infty$. □

A natural question now is to ask if δ_0 depends on ω. The answer is unfortunately positive:

Proposition 25. *Let* $\bar{B}_0 = \dfrac{B_0}{\omega^2}$.

The set of frequencies such that (A.1) *is ill-posed is included in*
$$\Lambda = \left\{ n^2\frac{B_2}{\bar{B}_0} + \frac{4}{\delta^2 A_0 \bar{B}_0};\quad n \in \mathbb{N}\right\}$$

Moreover, for δ *small enough* (A.1) *has non trivial solutions.*

Proof. Let us suppose that $u = w$ verifies (A.1) with $f = 0$. Taking the imaginary part of the variational formulation implies that $u = 0$ on S_{R_e}. The boundary condition of S_{R_e} implies that $\partial u/\partial r = 0$ on S_{R_e} and therefore $u = 0$ on Ω^+.

$$u = 0 \text{ in } \Omega^+. \tag{A.10}$$

Moreover, we also know that u can be written as a Fourier Series in Ω^-,

$$u := \sum_{n\in\mathbb{Z}} u_n(r) e^{in\theta}. \tag{A.11}$$

Combining (A.10), (A.11) and the transmission conditions ((A.1)(a)-(b)) give

$$\forall n \in \mathbb{Z};\quad \begin{bmatrix} 1 & \dfrac{A_0}{\delta^2} \\ \dfrac{\delta}{2}\left(n^2 B_2 + \bar{B}_0\omega^2\right) & 1 \end{bmatrix} \begin{bmatrix} (u_n)^- \\ \left(r\dfrac{\partial u_n}{\partial r}\right)^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consequently, $(u_n)^-$ and $(r\dfrac{\partial u_n}{\partial r})^-$ can be different from zero if the previous system is degenerated, which means that

$$\omega^2 = n^2\frac{B_2}{\bar{B}_0} + \frac{4}{\delta^2 A_0 \bar{B}_0};$$

and the first part of the proposition is established.

We now present a simple particular case where (A.1) has non-trivial solutions. Consider the case of a thin layer of thickness with constant coefficients $_0$, $_0$ and take $_\infty = \; _\infty = 1$ and $= 1=2$. In this case,

$$A_0 = \frac{1}{_0}; \qquad B_0 = \; _0 \quad \text{and} \quad B_2 = \; _0:$$

Since u is solution of an homogeneous Helmholtz equation in $^-$, it is clear that

$$u(r; \;) = \sum_{n \in Z} c_n J_n(!r \;) e^{in} :$$

Moreover, from the first part of the proof (A.10) we also know that u is zero in $^+$. Let $N \; 2 \; N$ and $N = \frac{2}{}$. Consequently the transmission conditions $(A:1((a) \quad (b)))$ give

$$8n \; 2 \; Z; \quad \begin{cases} ! \; ^2 = \frac{_0}{_0} \left(n^2 \quad \frac{N^2}{2} \right); \\ (1 \quad \frac{}{2}) \frac{}{_0} J_n'(! \; (1 \quad \frac{}{2}) + J_n(! \; (1 \quad \frac{}{2}) = 0: \end{cases}$$

Let us define

$$g_n(\;_0) := (1 \quad \frac{}{2}) \frac{}{_0} J_n' \quad \frac{_0}{_0}(n^2 \quad \frac{N^2}{2})(1 \quad \frac{}{2}) \quad + J_n \quad \frac{_0}{_0}(n^2 \quad \frac{N^2}{2})(1 \quad \frac{}{2}) \quad :$$

On Figure Appendix A we have plotted g_n with respect to for three values of N ($N = 30$, $N = 60$ and $N = 90$) for $_0 = 100$. For each N, we define n as the unique integer such that $N= \quad n \quad \frac{N}{} + 1$. In theses three cases, we verify that $= < \quad 0:1$ by the use of the following formula :

$$\frac{}{} = \frac{2}{N} \frac{!}{2} \frac{p \overline{\;_0}}{p \overline{\;_0}} = \frac{1}{N} \sqrt{n^2 \quad \frac{N^2}{2}}:$$

We can see that g_n has zeros in the three studied cases. That shows that problem (A.1) can be ill-posed.

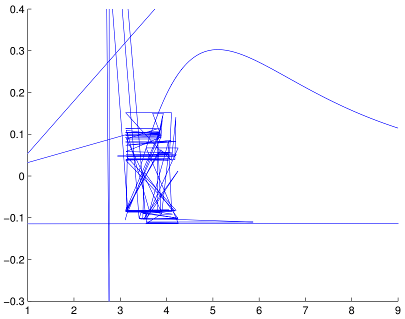


Figure A.4: Graph of g with respect to $_0$

□

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