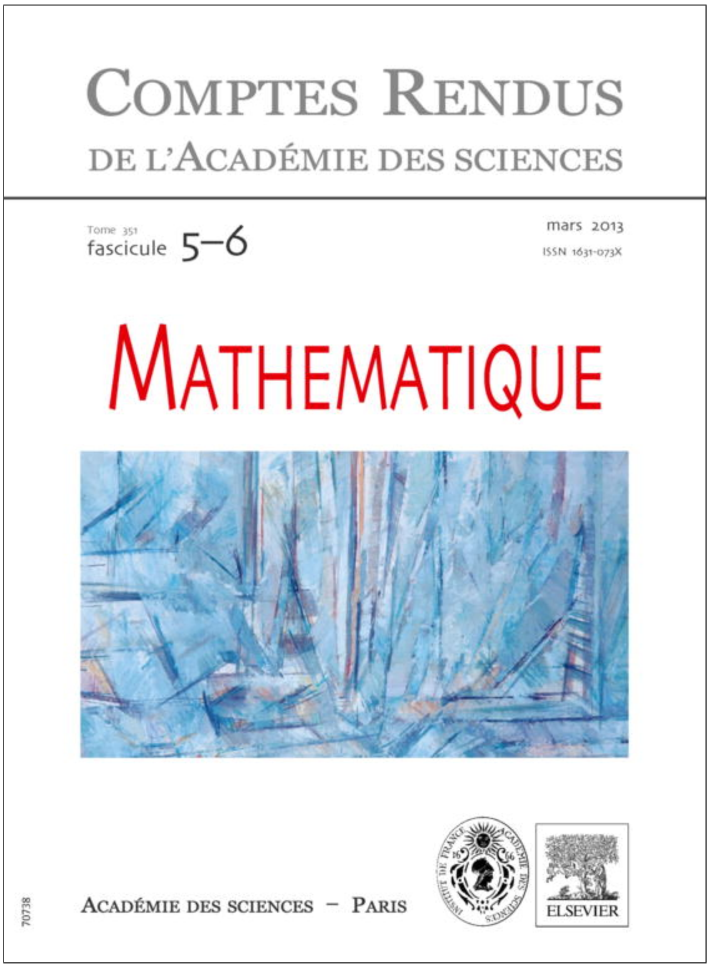
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Partial Differential Equations

A remark on Lipschitz stability for inverse problems

Une remarque sur la stabilité lipschitzienne pour les problèmes inverses

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ABSTRACT

An abstract Lipschitz stability estimate is proved for a class of inverse problems. It is then applied to the inverse medium problem for the Helmholtz equation.

RÉSUMÉ

Une estimation abstraite de stabilité lipschitzienne est prouvée pour une certaine classe de problèmes inverses. Elle est ensuite appliquée à un problème inverse de reconstruction d'indice de réfraction pour l'équation de Helmholtz.

1. Introduction

The stability issue for inverse problems consists in estimating the impact of some variation of the data on the parameter we want to identify. Such analysis is important because the inverse problems are ill-posed in general, and having a theoretical stability estimate enables us to quantify such ill-posedness. The stability estimates answer the following question: if the distance between two data is $\delta > 0$, what is the distance between the corresponding parameters as a function ϕ of δ, with $\phi(\delta) \to 0$ when $\delta \to 0$? The quantification of ill-posedness is given by the convergence rate of ϕ when δ tends toward 0. The stability results that we can collect in the literature are of different types, but we can point out that some assumptions on the parameter are necessary to obtain the function ϕ, for example the boundedness of the parameter with respect to an adapted norm. As a result, these stability estimates are in fact conditional stability estimates. In addition, the stronger are these assumptions, the better is the function ϕ we obtain. If, for example, we think of the well-known Calderon's inverse conductivity problem, where the parameter is the conductivity and the data is the Dirichlet-to-Neumann map, the function ϕ is a logarithm when the conductivity lies in an infinite dimensional space with some *a priori* bounds on the conductivity (see [1]), but ϕ becomes a linear function when the conductivity lies in a finite dimensional space of dimension N and again with some *a priori* bounds on the conductivity (see [2]). Besides, as expected, the constant of linearity grows exponentially when $N \to +\infty$ (see [8]).

The objective of the present paper is to prove an abstract theorem that provides the same kind of Lipschitz stability estimate as in [2] in a general case where the mapping from parameter to data is nonlinear with respect to the appropriate Banach spaces. Basically, such mapping shall be C^1, injective as well as its Fréchet derivative, and the set of parameters shall be a compact and convex subset of a finite dimensional subspace. To illustrate the interest of our theorem, we apply it on the inverse medium problem for the Helmholtz equation. Another application to the inverse Robin problem for the Laplace equation is presented in [3]. The original idea of the proof of our abstract theorem is introduced in [5] in the particular

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context of detection from boundary measurements of an obstacle characterized by two degrees of freedom moving in a fluid. Here, we simply adapt the proof of [5] to a general framework that covers a number of interesting situations. Our proof is elementary and avoids the sophisticated arguments that are used in [2,9] to achieve such result, in particular the arguments related to the quantification of unique continuation. The author is conscious that his stability result is of qualitative rather than quantitative nature, in particular that the Lipschitz constant cannot be expressed in terms of the data, since the proof is based on compactness arguments. However, in [2,9], such Lipschitz constant is not given as an explicit function of the data either (see Theorem 2.7 in [2] and Theorem 2.4 in [9]), even if the intermediate results of quantification of unique continuation in these papers have their own interest. In particular, the exponential growth of the Lipschitz constant with respect to the dimension N of the space can be proved independently of the way the Lipschitz constant is obtained (see for example [9]).

Our paper is organized as follows. The second section concerns the statement and proof of the abstract theorem. The third one is dedicated to the inverse medium problem.

2. The abstract theorem

The aim of this section is to prove the following abstract theorem.

Theorem 2.1. *Let V and H be two Banach spaces, their norms being denoted $\|.\|_V$ and $\|.\|_H$. Let U be an open subset of V, and V_N a finite dimensional subspace of V (of dimension N). Let K_N be a compact and convex subset of $V_N \cap U$.*

Let $T \in C^1(U, H)$ be a mapping such that $T|_{V_N \cap U}$ and $T'(x)|_{V_N}$, $x \in V_N \cap U$, are injective, where $T'(x) \in \mathcal{L}(V, H)$ is the derivative of T in the sense of Fréchet at point x.

Then there exists a constant $C > 0$ such that:

$$\forall x, y \in K_N, \quad \|x - y\|_V \leqslant C \big\| T(x) - T(y) \big\|_H.$$

Proof. Let us consider the mapping $\mathcal{T} : (x, h) \in U \times V \mapsto T'(x)(h) \in H$. Since $x \mapsto T'(x)$ is continuous from U to $\mathcal{L}(V, H)$, it is readily shown that such mapping $\mathcal{T}$ is continuous. Hence, by the injectivity of $T'(x)$ on V_N and the compactness of the set $K_N \times S_N$, where S_N is the unit sphere of V_N, there exists a constant $c > 0$ such that:

$$\big\| T'(x)(h) \big\|_H \geqslant c, \quad \forall x \in K_N,\ \forall h \in S_N,$$

that is:

$$\big\| T'(x)(h) \big\|_H \geqslant c\|h\|_V, \quad \forall x \in K_N,\ \forall h \in V_N. \tag{1}$$

Since the mapping $\mathcal{T}$ is uniformly continuous on the compact set $K_N \times S_N$ there exists $\delta > 0$ such that if $x, y \in K_N$ satisfy $\|x - y\|_V < \delta$ then

$$\big\| \big(T'(x) - T'(y)\big)(h) \big\|_H \leqslant \frac{c}{2},$$

that is

$$\big\| \big(T'(x) - T'(y)\big)(h) \big\|_H \leqslant \frac{c}{2} \|h\|_V, \quad \forall h \in V_N. \tag{2}$$

Let us take $x, y \in K_N$ that satisfy $\|x - y\|_V < \delta$. By denoting $h = y - x$, using the convexity of K_N and the fact that by the chain rule the function $s \in [0, 1] \mapsto T(x + sh) \in H$ is continuously differentiable:

$$\begin{aligned} T(y) - T(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} T(x + sh)\,\mathrm{d}s = \int_0^1 T'(x + sh)(h)\,\mathrm{d}s \\ &= T'(x)(h) + \int_0^1 \big(T'(x + sh) - T'(x)\big)(h)\,\mathrm{d}s. \end{aligned}$$

From (1) and (2), we obtain that if $x, y \in K_N$ satisfy $\|x - y\|_V < \delta$, then:

$$\big\| T(y) - T(x) \big\|_H \geqslant (c/2)\|h\|_V = (c/2)\|y - x\|_V.$$

Consider now the other case $\|x - y\| \geqslant \delta$. If we denote m the minimum of the continuous map $(x, y) \mapsto \|T(x) - T(y)\|_H$ on the compact set $\{(x, y) \in K_N^2,\ \|x - y\|_V \geqslant \delta\}$, we have $m > 0$ because of the injectivity of T on $V_N \cap U$ and:

$$\big\| T(x) - T(y) \big\|_H \geqslant m \geqslant \frac{m}{d} \|x - y\|_V,$$

where d is the diameter of that compact set. We just have to take $C = \max(2/c, d/m)$ in the statement of the theorem to complete the proof. $\square$

3. Application to the inverse medium problem

The scattering of an acoustic wave in an inhomogeneous medium in $\mathbb{R}^3$ is governed by the following system (see [4]):

$$\begin{cases} \Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \\ u = u^i + u^s, \\ \lim\limits_{R\to+\infty} \displaystyle\int_{\partial B_R} \left|\partial u^s/\partial r - \mathrm{i}k u^s\right|^2 \mathrm{d}s = 0, \end{cases} \tag{3}$$

where $k > 0$ is the wave number, $n \in L^\infty(\mathbb{R}^3)$ is a (complex) refractive index such that $n(x) = 1$ in $\mathbb{R}^3 \setminus \overline{B}$ for some open ball B. The data u^i is a smooth function that solves the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$ and is called the incident field, while u^s and u are the scattered field and the total field, respectively. The last equation of the system is the Sommerfeld radiation condition.

Classically, the problem (3) is equivalent to the following one in a bounded domain with an artificial boundary condition: find $u^s \in H^1(B_R)$ such that:

$$\begin{cases} \Delta u^s + k^2 n u^s = k^2\big(1 - n(x)\big)u^i & \text{in } B_R, \\ \partial u^s/\partial r = S_R\big(u^s|_{\partial B_R}\big) & \text{on } \partial B_R, \end{cases} \tag{4}$$

where B_R is an open ball of radius R such that $\overline{B} \subset B_R$, $S_R : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet-to-Neumann map, defined for $g \in H^{1/2}(\partial B_R)$ by: $S_R g = (\partial u_g/\partial r)|_{\partial B_R}$, where u_g is the solution in $\mathbb{R}^3 \setminus \overline{B_R}$ of the Helmholtz equation satisfying the Sommerfeld radiation condition and the Dirichlet condition $u_g = g$ on ∂B_R. It is well known that problem (4) is well-posed as soon as $\mathrm{Im}(n(x)) \geqslant 0$. In addition, it is shown in [4] that u^s has the asymptotic expression:

$$u^s(x) = \frac{e^{\mathrm{i}kr}}{r} u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r^2}\right), \quad r \to +\infty,$$

uniformly for all directions $\hat{x} = x/r \in S^2$, where $r = |x|$ and S^2 is the unit sphere in $\mathbb{R}^3$, and the far field u^∞ is given by

$$u^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial B_R} \left(u^s(y) \frac{\partial e^{-\mathrm{i}k\hat{x}\cdot y}}{\partial r(y)} - \frac{\partial u^s}{\partial r}(y) e^{-\mathrm{i}k\hat{x}\cdot y} \right) \mathrm{d}s(y), \quad \hat{x} \in S^2. \tag{5}$$

The inverse medium problem consists in finding the unknown refractive index n in B from the measurements on S^2 of the far fields $u^\infty(\cdot, d)$ corresponding via (5) to the scattered fields $u^s(\cdot, d)$ that are associated via (4) with plane waves $u^i(x) = e^{\mathrm{i}kx\cdot d}$ in all directions, with $d \in S^2$.

The stability issue for that problem has been addressed first in [10] with the help of ideas from [1]. Such result was improved in [7], where a logarithmic stability estimate is obtained assuming that $1 - n$ is bounded in some Sobolev space $H^s(\mathbb{R}^3)$ with $s > 3/2$, the exponent of the logarithm being specified as a function of s. It should be noted that the inverse medium problem is very close to Calderon's inverse conductivity problem, for which a number of papers concerning the stability issue has been published (see for example a review of them in [11]).

We now establish a Lipschitz stability estimate for our inverse medium problem with the help of Theorem 2.1 and for stronger assumptions on n. More precisely, we apply the theorem with $V = L^\infty(B)$, $H = L^2(S^2 \times S^2)$, $U = L^\infty_+(B)$, where:

$$L^\infty_+(B) := \big\{ n \in L^\infty(B),\ \exists m > 0,\ \mathrm{Im}\big(n(x)\big) \geqslant m \text{ a.e. on } B \big\},$$

and the (nonlinear) operator $T : \{n \in L^\infty(B),\ \mathrm{Im}(n(x)) \geqslant 0 \text{ a.e. on } B\} \to L^2(S^2 \times S^2)$ maps n to the set of far fields $u^\infty(\cdot, d)$ on S^2 for all directions $d \in S^2$, where $u^\infty(\cdot, d)$ corresponds to the scattered field $u^s(\cdot, d)$ that solves problem (4) with $u^i(x) = e^{\mathrm{i}kx\cdot d}$. We then choose any finite dimensional subspace V_N of $L^\infty(B)$, and lastly any compact and convex subset K_N of $V_N \cap U$.

We have the following result.

Theorem 3.1. *There exists a constant $C > 0$ such that*

$$\forall n_1, n_2 \in K_N, \quad \|n_1 - n_2\|_{L^\infty(B)} \leqslant C \left\| u_1^\infty - u_2^\infty \right\|_{L^2(S^2 \times S^2)},$$

where $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, which refer to the refractive indices n_1 and n_2, respectively, are the far fields in the direction $\hat{x}$ of the solutions u_1^s and u_2^s of problem (4) with $u^i = e^{\mathrm{i}kx\cdot d}$.

Remark 1. Here we give a simple example for V_N and K_N. For $i = 1, \ldots, N$, we consider some non-empty open subsets B^i of B such that $B^i \cap B^j = \emptyset$ for $i \neq j$ and $\overline{B} = \bigcup_{i=1}^N \overline{B^i}$. Let us define the subspace V_N of $L^\infty(B)$ as the space of piecewise constant functions n, namely for all $i = 1, \ldots, N$, $n|_{B^i}$ is a constant complex number n^i. Then K_N is defined as the set of $n \in V_N \cap U$ such that, for given real numbers l, L, m, M with $m > 0$,

$$n = \sum_{i=1}^{N} n^i \chi^i, \qquad l \leqslant \operatorname{Re}\bigl(n^i\bigr) \leqslant L, \qquad m \leqslant \operatorname{Im}\bigl(n^i\bigr) \leqslant M,$$

where χ^i is the characteristic function of B^i.

Let us verify the assumptions of Theorem 2.1 concerning T in the three following lemmas, the proof of which may be found in [3]. We simply recall that the proofs of Lemmas 3.2 and 3.4 both use the construction in [6] of complex geometrical optics solutions.

Lemma 3.2. *The mapping* $T : \{n \in L^\infty(B),\ \operatorname{Im}(n(x)) \geqslant 0 \text{ a.e. on } B\} \to L^2(S^2 \times S^2)$ *is injective.*

Lemma 3.3. *The mapping* $T : L^\infty_+(B) \to L^2(S^2 \times S^2)$ *is differentiable and its Fréchet derivative at point* $n \in L^\infty_+(B)$ *is the operator* $T'(n) : L^\infty(B) \to L^2(S^2 \times S^2)$ *which maps* h *to the far fields* $v^\infty_h(\cdot, d)$ *corresponding to the scattered fields* $v^s_h(\cdot, d)$ *for all incidence directions* $d \in S^2$, *where* $v^s_h(\cdot, d)$ *is the solution in* $H^1(B_R)$ *of problem*:

$$\begin{cases} \Delta v^s_h + k^2 n v^s_h = -k^2 h u & \text{in } B_R, \\ \partial v^s_h / \partial r = S_R\bigl(v^s_h|_{\partial B_R}\bigr) & \text{on } \partial B_R, \end{cases} \tag{6}$$

in which $u = u^s + e^{\mathrm{i}kx\cdot d}$ *and* u^s *is the solution of problem* (4)*. In problem* (6)*, the function* $h \in L^\infty(B)$ *has been extended by* 0 *outside* B*, without change of notations. Moreover, the mapping* $n \in L^\infty_+(B) \mapsto T'(n) \in \mathcal{L}(L^\infty(B), L^2(S^2 \times S^2))$ *is continuous.*

Lemma 3.4. *For each* $n \in L^\infty_+(B)$*, the operator* $T'(n) : L^\infty(B) \to L^2(S^2 \times S^2)$ *is injective.*

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