Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations.

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Abstract. We investigate existence and uniqueness for a new class of Backward Stochastic Differential Equations (BSDEs) with no driving martingale. When the randomness of the driver depends on a general Markov process X those BSDEs are denominated forward BSDEs and can be associated to a deterministic problem, called Pseudo-PDE which constitute the natural generalization of a parabolic semilinear PDE which naturally appears when the underlying filtration is Brownian. We consider two types of solutions for the Pseudo-PDEs: *classical* and *of martingale type*.

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1 Introduction

This paper focuses on a new concept of Backward Stochastic Differential Equation (in short BSDE) with no driving martingale of the form

$$Y_t = \xi + \int_t^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right) dV_r - (M_T - M_t), \tag{1.1}$$

defined on a fixed stochastic basis fulfilling the usual conditions. V is a given non-decreasing continuous adapted process, ξ (resp. $\hat{f}$) is a prescribed terminal

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condition (resp. driver). The unknown will be a couple of cadlag adapted processes (Y, M) where M is a martingale. A particular case of such BSDEs are the forward BSDEs (in short FBSDEs) of the form

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}(r)}\right) dV_r - (M_T^{s,x} - M_t^{s,x}), \quad (1.2)$$

defined in a canonical space $\left(\Omega, \mathcal{F}^{s,x}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ where $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ corresponds to the laws (for different starting times s and starting points x) of an underlying forward Markov process with time index $[0, T]$, taking values in a Polish state space E, and which is characterized as the solution of a martingale problem related to a certain operator a. (**??**) will be naturally associated with a deterministic problem involving a, which will be called Pseudo-PDE. The forward BSDE (**??**) seems to be appropriated in the case when the forward underlying process X is a general Markov process which does not rely to a fixed reference process or random field as a Brownian motion or a Poisson measure.

The classical notion of Brownian BSDE was introduced in 1990 by E. Pardoux and S. Peng in [**?**], after an early work of J.M. Bismut in 1973 in [**?**]. It is a stochastic differential equation with prescribed terminal condition ξ and driver $\hat{f}$; the unknown is a couple (Y, Z) of adapted processes. Of particular interest is the case when the randomness of the driver is expressed through a forward diffusion process X and the terminal condition only depends on X_T. The solution, when it exists, is usually indexed by the starting time s and starting point x of the forward diffusion $X = X^{s,x}$, and it is expressed by

$$\left\{\begin{array}{lcl} X_t^{s,x} &=& x + \int_s^t \mu(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dB_r \\ Y_t^{s,x} &=& g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right)dr - \int_t^T Z_r^{s,x}dB_r, \end{array}\right. \quad (1.3)$$

where B is a Brownian motion. Existence and uniqueness of (**??**) (that we still indicate with FBSDE) above was established first supposing essentially Lipschitz conditions on f with respect to the third and fourth variable. μ and σ were also supposed to be Lipschitz (with respect to x). In the sequel those conditions were considerably relaxed, see [**?**] and references therein.
In [**?**] and in [**?**] previous FBSDE was linked to the semilinear PDE

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}\sum_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_ix_j}u + \sum_{i\leq d}\mu_i\partial_{x_i}u + f((\cdot,\cdot), u, \sigma\nabla u) = 0 \quad \text{on } [0,T[\times\mathbb{R}^d \\ u(T,\cdot) = g. \end{array}\right. \quad (1.4)$$

In particular, if (**??**) has a classical smooth solution u then $(Y^{s,x}, Z^{s,x}) := (u(\cdot, X_\cdot^{s,x}), \sigma\nabla u(\cdot, X_\cdot^{s,x}))$ solves the second line of (**??**). Conversely, only under the Lipschitz type conditions mentioned after (**??**), the solution of the FBSDE can be expressed as a function of the forward process $(Y^{s,x}, Z^{s,x}) = (u(\cdot, X_\cdot^{s,x}), v(\cdot, X_\cdot^{s,x}))$, see [**?**]. When f and g are continuous, u is a viscosity solution of (??). Excepted in the case when u has some minimal differentiability properties, see e.g. [?], it is difficult to say something more on v.

Since the pioneering work of [?], in the Brownian case, the relations between more general BSDEs and associated deterministic problems have been studied extensively, and innovations have been made in several directions.
In [?] the authors introduced a new kind of FBSDE including a term with jumps generated by a Poisson measure, where an underlying forward process X solves a jump diffusion equation with Lipschitz type conditions. They associated with it an Integral-Partial Differential Equation (in short IPDE) in which some non-local operators are added to the classical partial differential maps, and proved that, under some continuity conditions on the coefficients, the BSDE provides a viscosity solution of the IPDE. In chapter 13 of [?], under some specific conditions on the coefficients of a Brownian BSDE, one produces a solution in the sense of distributions of the parabolic PDE. Later, the notion of mild solution of the PDE was used in [?] where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory of Fukushima Dirichlet forms, see e.g. [?]. Infinite dimensional setups were considered for example in [?] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space. Concerning the study of BSDEs driven by more general martingales than Brownian motion, we have already mentioned BSDEs driven by Poisson measures. In this respect, more recently, BSDEs driven by marked point processes were introduced in [?], see also [?]; in that case the underlying process does not contain any diffusion term. Brownian BSDEs involving a supplementary orthogonal term were studied in [?]. We can also mention the study of BSDEs driven by a general martingale in [?]. BSDEs of the same type, but with partial information have been investigated in [?]. A first approach to face deterministic problems for those equations appears in [?]; that paper also contains an application to financial hedging in incomplete market.

The main motivation of this paper was to generalize in some aspects the links between (??) and (??). Our BSDEs (??) are associated to a completely general Markov process supposed to solve a *martingale problem* with respect to a given *deterministic* operator a. This Markov process will only be defined by its laws for every starting time and starting point $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$. Our associated deterministic problem, comparable to (??) will be the Pseudo-PDE

$$\left\{\begin{array}{rcll} a(u)(t,x)+f\left(t,x,u(t,x),\sqrt{\Gamma(u,u)}(t,x)\right) &=& 0 & \text{on } [0,T]\times E \\ u(T,\cdot) &=& g, & \end{array}\right. \tag{1.5}$$

where $\Gamma(u,u)=a(u^2)-2ua(u)$ is a potential theory operator called the *square field operator*. When X is a diffusion as in the first line of (??) that operator is of the form $\Gamma(u,u)=\sum\limits_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial_{x_i}u\partial_{x_j}u$. For the Pseudo-PDE, we study solutions of two natures: *classical* solutions, which generalize the $C^{1,2}$ solutions of (**??**) and solutions in the martingale sense.

The main contributions of the paper are essentially the following. In Section **??** we introduce the notion of BSDE with no driving martingale (**??**). Theorem **??** states existence and uniqueness of a solution for that BSDE, when the final condition ξ is square integrable and the driver $\hat{f}$ verifies some integrability and Lipschitz conditions. In Section **??**, we consider an operator and its domain $(a, \mathcal{D}(a))$; V will be a continuous non-decreasing function. That section is devoted to the formulation of the martingale problem concerning our underlying process X. For each initial time s and initial point x the solution will be a probability $\mathbb{P}^{s,x}$ under which for any $\phi \in \mathcal{D}(a)$,

$$\phi(\cdot, X_\cdot) - \phi(s, x) - \int_s^\cdot a(\phi)(r, X_r)dV_r$$

is a local martingale starting in zero at time s. We will then assume that this martingale problem is well-posed and that its solution $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defines a Markov process. In Proposition **??**, we prove that, under one of these probabilities, the angular bracket of every square integrable martingale is absolutely continuous with respect to dV. In Section **??** we suitably define some extended domains for the operators a and Γ, using some locally convex topology. In Section **??** we introduce the Pseudo-PDE (**??**) to which we associate the FBSDE (**??**), considered under every $\mathbb{P}^{s,x}$. We also introduce the notions of classical solution in Definition **??**, and of solution in the martingale sense in Definition **??**, which is fully probabilistic. Classical solutions typically belong to the domain $\mathcal{D}(a)$. In Theorem **??**, we show that, without any assumptions of regularity, there exist Borel functions u and v such that for any $(s, x) \in [0, T] \times E$, the solution of (**??**) verifies

$$\begin{cases} \forall t \geq s : Y_t^{s,x} = u(t, X_t) \quad \mathbb{P}^{s,x} \text{ a.s.} \\ \frac{d\langle M^{s,x}\rangle}{dV}(t) = v^2(t, X_t) \quad dV \otimes d\mathbb{P}^{s,x} \text{ a.e.} \end{cases}$$

Theorems **??** and **??** state that the function u is the unique solution in the martingale sense of (**??**). Corollary **??** asserts that, given a classical solution $u \in \mathcal{D}(a)$, then for any (s, x) the processes $Y^{s,x} = u(\cdot, X_\cdot)$ and $M^{s,x} = u(\cdot, X_\cdot) - u(s, x) - \int_s^\cdot f((\cdot, \cdot, u, \sqrt{\Gamma(u, u)})(r, X_r)dV_r$ solve (**??**) under the probability $\mathbb{P}^{s,x}$. Conversely if the function u defined via the BSDEs belongs to $\mathcal{D}(a)$, then, in Theorem **??**, u is shown to be a classical solution of (**??**), up to a so called zero potential set, see Definition **??**. In the companion paper [**?**], we will also discuss other types of (analytical) solutions, i.e. *mild* and *viscosity* ones. The couple (u, v) will always be a mild solution of (**??**), whereas several assumptions will have to be strengthened in order for u to be a viscosity solution of (**??**). For that reason, the notion of mild solution will appear to be the most natural one at the analytical level. In Section **??** we list some examples which will be developed in [**?**]. These include Markov processes defined as weak solutions of Stochastic Differential Equations (in short SDEs) including possible jump terms, α-stable Lévy processes associated to fractional Laplace operators, and solutions of SDEs with distributional drift.

2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary.

A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$ and if (F, d_F) is a metric space, $\mathcal{C}(E,F)$ (respectively $\mathcal{C}_b(E,F)$, $\mathcal{B}(E,F)$, $\mathcal{B}_b(E,F)$) will denote the set of functions from E to F which are continuous (respectively bounded continuous, Borel, bounded Borel).

Let $(\Omega, \mathcal{F})$, $(E, \mathcal{E})$ be two measurable spaces. A measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ shall often be called a **random variable** (with values in E), or in short r.v. If $\mathbb{T}$ is some set, an indexed set of r.v. with values in E, $(X_t)_{t\in\mathbb{T}}$ will be called a **random field** (indexed by $\mathbb{T}$ with values in E). In particular, if $\mathbb{T}$ is an interval included in $\mathbb{R}_+$, $(X_t)_{t\in\mathbb{T}}$ will be called a **stochastic process** (indexed by $\mathbb{T}$ with values in E). Given a stochastic process, if the mapping

$$\begin{array}{rcl} (t,\omega) & \longmapsto & X_t(\omega) \\ (\mathbb{T}\times\Omega, \mathcal{B}(\mathbb{T})\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{E}) \end{array}$$

is measurable, then the process $(X_t)_{t\in\mathbb{T}}$ will be called a **measurable process** (indexed by $\mathbb{T}$ with values in E).

On a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $p \in \mathbb{N}^*$, L^p will denote the set of random variables with finite p-th moment. Two random fields (or stochastic processes) $(X_t)_{t\in\mathbb{T}}$, $(Y_t)_{t\in\mathbb{T}}$ indexed by the same set and with values in the same space will be said to be **modifications (or versions) of each other** if for every $t \in \mathbb{T}$, $\mathbb{P}(X_t = Y_t) = 1$.

A measurable space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}})$ (where $\mathbb{T}$ is equal to $\mathbb{R}_+$ or to $[0,T]$ for some $T \in \mathbb{R}_+^*$) will be called a **filtered space**.
A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ will be called called a **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets.

Concerning spaces of stochastic processes, in a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$, we will use the following notations and vocabulary, most of them being taken or adapted from [**?**] or [**?**]. We will denote $\mathcal{V}$ (resp $\mathcal{V}^+$) the set of adapted, bounded variation (resp non-decreasing) processes starting at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous; $\mathcal{M}$ will be the space of cadlag martingales.
For any $p \in [1,\infty]$ $\mathcal{H}^p$ will denote the subset of $\mathcal{M}$ of elements M such that $\sup_{t\in\mathbb{T}} |M_t| \in L^p$ and in this set we identify indistinguishable elements. It is a Banach space for the norm $\|M\|_{\mathcal{H}^p} = \mathbb{E}[|\sup_{t\in\mathbb{T}} M_t|^p]^{\frac{1}{p}}$, and $\mathcal{H}^p_0$ will denote the

Banach subspace of $\mathcal{H}^p$ containing the elements starting at zero.
If $\mathbb{T} = [0,T]$ for some $T \in \mathbb{R}_+^*$, a stopping time will be defined as a random variable with values in $[0,T] \cup \{+\infty\}$ such that for any $t \in [0,T]$, $\{\tau \leq t\} \in \mathcal{F}_t$. We define a **localizing sequence of stopping times** as an increasing sequence of stopping times $(\tau_n)_{n \geq 0}$ such that there exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$. Let Y be a process and τ a stopping time, we denote Y^τ the process $t \mapsto Y_{t \wedge \tau}$ which we call **stopped process**. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exist a localizing sequence $(\tau_n)_{n \geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$.
For any $M \in \mathcal{M}_{loc}$, we denote $[M]$ its **quadratic variation** and if moreover $M \in \mathcal{H}^2_{loc}$, $\langle M \rangle$ will denote its (predictable) **angular bracket**. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2} = \mathbb{E}[M_T N_T] = \mathbb{E}[\langle M, N \rangle_T]$ which makes it a Hilbert space. Two local martingales M, N will be said to be **strongly orthogonal** if MN is a local martingale starting in 0 at time 0. In $\mathcal{H}^2_{0,loc}$ this notion is equivalent to $\langle M, N \rangle = 0$.
If $M \in \mathcal{M}_{loc}$, and $p \in [1, \infty]$. We denote $L^p(M)$ the set of predictable processes H such that $\mathbb{E}\left[\left(\int_0^T H_r^2 d[M]_r\right)^{\frac{p}{2}}\right] < \infty$. This implies that $\int_0^\cdot H_r M_r$ belongs to $\mathcal{H}^p$.

3 BSDEs without driving martingale

In the whole present section we are given $T \in \mathbb{R}_+^*$, and a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}\right)$ fulfilling the usual conditions. Some proofs and intermediary results of the first part of this section are postponed to Appendix **??**.

Definition 3.1. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA \ll dB$) if for almost all ω, $dA(\omega) \ll dB(\omega)$ as Borel measures on $[0,T]$.*

We will say that dB and dA are mutually singular **in the sense of stochastic measures** *(written $dA \perp dB$) if for almost all ω, the Borel measures $dA(\omega)$ and $dB(\omega)$ are mutually singular.*

Let $B \in \mathcal{V}^+$. $dB \otimes d\mathbb{P}$ will denote the positive measure on $(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]))$ defined for any $F \in \mathcal{F} \otimes \mathcal{B}([0,T])$ by

$$dB \otimes d\mathbb{P}(F) = \mathbb{E}\left[\int_0^T \mathbb{1}_F(r,\omega) dB_r(\omega)\right].$$

A property which holds true everywhere except on a null set for this measure will be said to be true $dB \otimes d\mathbb{P}$ almost everywhere (a.e).

Proposition below admits a straightforward proof.

Proposition 3.2. *Let ϕ, ψ be two measurable mappings from $(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then if $\phi = \psi$ $dB \otimes d\mathbb{P}$ a.e, we have for $\mathbb{P}$ almost all ω that $(\phi(\omega) = \psi(\omega)\, dB(\omega)$ a.e.)*

The proof of Proposition below is in Appendix **??**.

Proposition 3.3. *For any A and B in $\mathcal{V}^{p,+}$, there exists a (non-negative $dB \otimes d\mathbb{P}$ a.e.) predictable process $\frac{dA}{dB}$ and a process in $\mathcal{V}^{p,+}$ $A^{\perp B}$ such that*

$$dA^{\perp B} \perp dB \text{ and } A = A^B + A^{\perp B} \text{ a.s.}$$

where $A^B = \int_0^{\cdot} \frac{dA}{dB}(r)dB_r$. The process $A^{\perp B}$ is unique and the process $\frac{dA}{dB}$ is unique $dB \otimes d\mathbb{P}$ a.e.
Moreover, there exists a predictable process K with values in $[0,1]$ (for every (ω,t)), such that $A^B = \int_0^{\cdot} \mathbb{1}_{\{K_r<1\}}dA_r$ and $A^{\perp B} = \int_0^{\cdot} \mathbb{1}_{\{K_r=1\}}dA_r$.

Definition 3.4. *The predictable process $\frac{dA}{dB}$ appearing in the statement of Proposition* **??** *will be called the* ***Radon-Nikodym derivative*** *of A by B.*

Remark 3.5. *Since for any $s<t$*

$$A_t - A_s = \int_s^t \frac{dA}{dB}(r)dB_r + A_t^{\perp B} - A_s^{\perp B} \text{ a.s.}$$

where $A^{\perp B}$ is increasing, it is clear that for any $s<t$.

$$\int_s^t \frac{dA}{dB}(r)dB_r \leq A_t - A_s \text{ a.s.}$$

Therefore that for any positive measurable process ϕ we have

$$\int_0^T \phi_r \frac{dA}{dB}(r)dB_r \leq \int_0^T \phi_r dA_r \text{ a.s.}$$

Notation 3.6. *Let A be in $\mathcal{V}$, we will denote A^+ and A^- the positive variation and negative variation parts of A, meaning the unique pair of elements $\mathcal{V}^+$ such that $A = A^+ - A^-$, see Proposition I.3.3 in [**?**] for their existence.*

Definition 3.7. *Let A be in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. We set $\frac{dA}{dB} := \frac{dA^+}{dB} - \frac{dA^-}{dB}$ and $A^{\perp B} := (A^+)^{\perp B} - (A^-)^{\perp B}$.*

The proof of the proposition below is also in Appendix **??**.

Proposition 3.8. *Let A_1 and A_2 be in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. Then, $\frac{d(A_1+A_2)}{dB} = \frac{dA_1}{dB} + \frac{dA_2}{dB}$ $dV \otimes d\mathbb{P}$ a.e. and $(A_1+A_2)^{\perp B} = A_1^{\perp B} + A_2^{\perp B}$.*

Proposition 3.9. *Let $M \in \mathcal{H}_0^2$, and let $V \in \mathcal{V}^{p,+}$. There exists a pair $(M^V, M^{\perp V})$ in $\mathcal{H}_0^2$ such that*

1. *$M = M^V + M^{\perp V}$;*
2. *$d\langle M^V \rangle \ll dV$;*
3. *$d\langle M^{\perp V} \rangle \perp dV$;*

4. $\langle M^V, M^{\perp V}\rangle = 0$.

Moreover, we have $\langle M^V\rangle = \langle M\rangle^V = \int_0^\cdot \frac{d\langle M\rangle}{dV}(r)dV_r$ *and* $\langle M^{\perp V}\rangle = \langle M\rangle^{\perp V}$ *and there exists a predictable process* K *with values in* $[0,1]$ *such that* $M^V = \int_0^\cdot \mathbb{1}_{\{K_r<1\}}dM_r$ *and* $M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_r=1\}}dM_r$.

Remark 3.10. *With those definitions, for* $M \in \mathcal{H}^2_0$ *it is clear that*

$$\langle M^V, M\rangle = \langle M^V, M^V\rangle = \int_0^\cdot \frac{d\langle M\rangle}{dV}(r)dV_r.$$

Definition 3.11. *Let* $V \in \mathcal{V}^{p,+}$. *We introduce two significant spaces related to* V.

1. $\mathcal{H}^{2,V} := \{M \in \mathcal{H}^2_0 | d\langle M\rangle \ll dV\}$;
2. $\mathcal{H}^{2,\perp V} := \{M \in \mathcal{H}^2_0 | d\langle M\rangle \perp dV\}$.

The proof of the proposition below is in Appendix **??**.

Proposition 3.12. $\mathcal{H}^{2,V}$ *and* $\mathcal{H}^{2,\perp V}$ *are orthogonal sub-Hilbert spaces of* $\mathcal{H}^2_0$ *and* $\mathcal{H}^2_0 = \mathcal{H}^{2,V} \oplus^{\perp} \mathcal{H}^{2,\perp V}$. *Moreover, any element of* $\mathcal{H}^{2,V}$ *is strongly orthogonal to any element of* $\mathcal{H}^{2,\perp V}$.

Remark 3.13. *All previous results extend when the filtration is indexed by* $\mathbb{R}_+$.

We are going to introduce here a new type of Backward Stochastic Differential Equation (BSDE) for which there is no need for having a particular martingale of reference.

We will denote $\mathcal{P}ro$ the σ-field generated by progressively measurable processes defined on $[0,T]\times\Omega$.

We consider the following:

1. a bounded process of reference $V \in \mathcal{V}^{c,+}$,
2. an $\mathcal{F}_T$-measurable random variable ξ called the **final condition**,
3. a **driver** $\hat{f} : ([0,T]\times\Omega)\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, measurable with respect to $\mathcal{P}ro \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Definition 3.14. *We will indicate by* $\mathcal{L}^0(dV\otimes d\mathbb{P})$ *the set of progressively measurable processes* ϕ *such that* $\int_0^T |\phi_r|dV_r$ *is finite a.s. and* $L^0(dV\otimes d\mathbb{P})$ *the quotient space of* $\mathcal{L}^0(dV\otimes d\mathbb{P})$ *with respect to the subspace of processes equal to zero* $dV\otimes d\mathbb{P}$ *a.e. The application which to a process associate its class will be denoted*

$$\left(\begin{array}{ccc} \phi & \longmapsto & \dot{\phi} \\ \mathcal{L}^0(dV\otimes d\mathbb{P}) & \longrightarrow & L^0(dV\otimes d\mathbb{P}) \end{array}\right).$$

We also define $\mathcal{L}^2(dV\otimes d\mathbb{P})$ *the set of progressively measurable processes* ϕ *such that* $\mathbb{E}[\int_0^T \phi_r^2 dV_r] < \infty$, *and the quotient space* $L^2(dV\otimes d\mathbb{P})$ *defined as* $L^0(dV\otimes d\mathbb{P})$. *More formally,* $L^2(dV\otimes d\mathbb{P})$ *corresponds to the classical* L^2 *space* $L^2([0,T]\times\Omega, \mathcal{P}ro, dV\otimes d\mathbb{P})$ *and is therefore complete for its usual norm.*

We will assume that $(\xi, \hat{f})$ verify the following hypothesis.

Hypothesis 3.15.

1. *$\xi \in L^2$;*
2. *$\hat{f}(\cdot,\cdot,0,0) \in \mathcal{L}^2(dV \otimes d\mathbb{P})$;*
3. *There exist positive constants K^Y, K^Z such that, $\mathbb{P}$ a.s.*
   - *(a) $\forall t, y, y', z$ we have $|\hat{f}(t,\cdot,y,z) - \hat{f}(t,\cdot,y',z)| \leq K^Y|y-y'|$;*
   - *(b) $\forall t, y, z, z'$ we have $|\hat{f}(t,\cdot,y,z) - \hat{f}(t,\cdot,y,z')| \leq K^Z|z-z'|$.*

We start with a lemma.

Lemma 3.16. *Let U_1 and U_2 be in $\mathcal{L}^0(dV \otimes d\mathbb{P})$ and such that $\dot{U}_1 = \dot{U}_2$. Let $F : [0,T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that $F((\cdot,\cdot),U_1)$ and $F(\cdot,\cdot,U_2)$ are in $\mathcal{L}^0(dV \otimes d\mathbb{P})$, then the processes $\int_0^{\cdot} F(r,\omega,U^1_r)dV_r$ and $\int_0^{\cdot} F(r,\omega,U^2_r)dV_r$ are indistinguishable.*

Proof. By Proposition **??**, there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for any $\omega \in \mathcal{N}^c$, $U^1(\omega) = U^2(\omega)$ $dV(\omega)$ a.e. So for any $\omega \in \mathcal{N}^c$, $F(\cdot,\omega,U^1(\omega)) = F(\cdot,\omega,U^2(\omega))$ $dV(\omega)$ a.e. This implies that for any $\omega \in \mathcal{N}^c$ and $u \in [0,T]$,
$\int_0^u F(r,\omega,U^1_r(\omega))dV_r(\omega) = \int_0^u F(r,\omega,U^2_r(\omega))dV_r(\omega)$. So $\int_0^{\cdot} F(r,\omega,U^1_r)dV_r$ and $\int_0^{\cdot} F(r,\omega,U^2_r)dV_r$ are indistinguishable processes. □

Remark 3.17. *In some of the following proofs, we will have to work with classes of processes. According to Lemma **??**, if $\dot{U}$ is an element of $L^2(dV \otimes d\mathbb{P})$ then we could define the integral process $\int_0^{\cdot} F(r,\omega,\dot{U}_r)dV_r$ as $\int_0^{\cdot} F(r,\omega,U_r)dV_r$, where U is a representative $\dot{U}$. Nevertheless we will rarely use the dot notation in the integral.*

We will start with a first formulation of our BSDE.

Definition 3.18. *We say that a couple $(\dot{Y}, M) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ is a solution of $\overline{BSDE}(\xi, \hat{f}, V)$ if there exists a cadlag representative Y of $\dot{Y}$ which verifies*

$$Y = \xi + \int_{\cdot}^{T} \hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}(r)}\right)dV_r - (M_T - M_\cdot) \qquad (3.1)$$

in the sense of indistinguishability.
*A couple $(Y, M) \in \mathcal{L}^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ verifying (**??**) will be said to be a solution of $BSDE(\xi, \hat{f}, V)$.*

Proposition 3.19. *$\overline{BSDE}(\xi, \hat{f}, V)$ has a solution iff $BSDE(\xi, \hat{f}, V)$ has a solution. Moreover, $\overline{BSDE}(\xi, \hat{f}, V)$ has a unique solution iff $BSDE(\xi, \hat{f}, V)$ has a unique solution.*

Proof. It is clear that if a couple (Y, M) solves $BSDE(\xi, \hat{f}, V)$ then $(\dot{Y}, M)$ solves $\overline{BSDE}(\xi, \hat{f}, V)$, and that if a couple $(\dot{Y}, M)$ solves $\overline{BSDE}(\xi, \hat{f}, V)$ via the cadlag representative Y, then (Y, M) solves $BSDE(\xi, \hat{f}, V)$. Therefore $\overline{BSDE}(\xi, \hat{f}, V)$ has a solution iff $BSDE(\xi, \hat{f}, V)$ has a solution.

Now let us suppose that $\overline{BSDE}(\xi, \hat{f}, V)$ has a unique solution. If (Y, M) and (Y', M') are two solutions of $BSDE(\xi, \hat{f}, V)$ then $(\dot{Y}, M) = (\dot{Y}', M')$ since they both solve $\overline{BSDE}(\xi, \hat{f}, V)$. Moreover, thanks to Lemma **??**, $\int_0^{\cdot} \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right) dV_r$ and $\int_0^{\cdot} \hat{f}\left(r, \cdot, Y'_r, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right) dV_r$ are indistinguishable, so by (**??**), Y and Y' are indistinguishable. So the solution of $BSDE(\xi, \hat{f}, V)$ is unique.

Conversely, if $BSDE(\xi, \hat{f}, V)$ has a unique solution and if $(\dot{Y}, M)$ and $(\dot{Y}', M')$ both solve $\overline{BSDE}(\xi, \hat{f}, V)$ via the cadlag representative Y and Y'. By definition, (Y, M) and (Y', M') both solve if $BSDE(\xi, \hat{f}, V)$ and are therefore equal. In particular $\dot{Y} = \dot{Y}'$ implying that the solution of $\overline{BSDE}(\xi, \hat{f}, V)$ is unique. □

Proposition 3.20. *If (Y, M) solves $BSDE(\xi, \hat{f}, V)$, and if we denote $\hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right)$ by $\hat{f}_r$, then for any $t \in [0, T]$ we have*

$$\left\{\begin{array}{rcl} Y_t & = & \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] \\ M_t & = & \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right], \end{array}\right.$$

a.s.

Proof. Since $Y_t = \xi + \int_t^T \hat{f}_r dV_r - (M_T - M_t)$ a.s. and Y being an adapted process and M a martingale, taking the expectation in (**??**) at time t, we directly get $Y_t = \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]$ and in particular that $Y_0 = \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]$. Since $M_0 = 0$, looking at the BSDE at time 0 we get

$$\begin{array}{rcl} M_T & = & \xi + \int_0^T \hat{f}_r dV_r - Y_0 \\ & = & \xi + \int_0^T \hat{f}_r dV_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]. \end{array}$$

Now evaluating again the the solution of the BSDE at time t we get the a.s. equalities

$$\begin{array}{rcl} M_t & = & Y_t - \left(\xi + \int_t^T \hat{f}_r dV_r\right) + M_T \\ & = & \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \left(\xi + \int_t^T \hat{f}_r dV_r\right) \\ & & + \left(\xi + \int_0^T \hat{f}_r dV_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]\right) \\ & = & \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] + \int_0^t \hat{f}_r dV_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right] \\ & = & \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]. \end{array}$$

□

We will proceed showing that $\overline{BSDE}(\xi, \hat{f}, V)$ has a unique solution, which, by Proposition **??**, implies well-posedness for $BSDE(\xi, \hat{f}, V)$. At this point we introduce a significant map Φ which will map $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ into itself. From now on, until Notation **??**, we fix a couple $(\dot{U}, N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ to which we will associate $(\dot{Y}, M)$ which, as we will show, will also belong to $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$. We will show that $(\dot{U}, N) \mapsto (\dot{Y}, M)$ is a contraction for a certain norm. In all the proofs below, $\dot{U}$ will only appear in integrals driven by dV, so as we have said in Remark **??**, we can consider that we are working with any element U of the class $\dot{U}$. This will however not be the case for $\dot{Y}$ for which we will have to pick a specific representative. Our strategy consists in starting by defining through Definition **??** a cadlag process Y, which will be said to be the *cadlag reference process*, associated with $(\dot{U}, N)$. Then we define $\dot{Y}$.

Proposition 3.21. *For any $t \in [0,T]$, $\int_t^T \hat{f}^2\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r$ is in L^1 and $\left(\xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r\right)$ is in L^2.*

Proof. By Jensen's inequality and thanks to the Lipschitz conditions on f in Hypothesis **??** and the fact that V is bounded, there exist positive constants C, C' such that, for any $t \in [0,T]$, we have

$$\begin{array}{ll} & \left(\int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r\right)^2 \\ \leq & C \int_t^T \hat{f}^2\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r \\ \leq & C' \left(\int_t^T \hat{f}^2\left(r, \cdot, 0, 0\right) dV_r + \int_t^T U_r^2 dV_r + \int_t^T \frac{d\langle N\rangle}{dV}(r) dV_r\right). \end{array} \tag{3.2}$$

Then by Remark **??** $\int_t^T \frac{d\langle N\rangle}{dV}(r) dV_r \leq (\langle N\rangle_T - \langle N\rangle_t)$ which belongs to L^1 since N is taken in $\mathcal{H}^2$. By Hypothesis **??**, $f(\cdot,\cdot,0,0)$ is in $\mathcal{L}^2(dV \otimes d\mathbb{P})$ and ξ is square integrable. $\dot{U}$ was also taken in $L^2(dV \otimes d\mathbb{P})$ so all the three terms in (**??**) are integrable and therefore $\left(\xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r\right)$ is in L^2, and $\int_t^T \hat{f}^2\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r$ in L^1. □

We can therefore state the following definition.

Definition 3.22. *Setting $\hat{f}_r = \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right)$. Let M be the cadlag version of the martingale*

$$t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right].$$

*M is square integrable by Proposition **??**. It admits a cadlag version taking into account Theorem 4 in Chapter IV of [**?**], since the filtration is complete and right-continuous. We denote by Y the cadlag process defined by*

$$Y_t = \xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r - (M_T - M_t).$$

This will be called the **cadlag reference process** *and we will often omit its dependence to $(\dot{U}, N)$.*

According to previous definition, it is not clear whether Y is adapted, however, we have the almost sure equalities

$$\begin{array}{rcl} Y_t & = & \xi + \int_t^T \hat{f}_r dV_r - (M_T - M_t) \\ & = & \xi + \int_t^T \hat{f}_r dV_r - \left(\xi + \int_0^T \hat{f}_r dV_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]\right) \\ & = & \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \int_0^t \hat{f}_r dV_r \\ & = & \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]. \end{array} \tag{3.3}$$

Moreover, Y is cadlag, so by Theorem 15 Chapter IV of [**?**], being adapted and cadlag, it is progressively measurable.

Proposition 3.23. *Y and M are square integrable processes.*

Proof. We already know that M is a square integrable martingale. As we have seen in Proposition **??**, $\left(\xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r\right)$ belongs to L^2 for any $t \in [0, T]$. So by (**??**) and the Jensen's inequality for conditional expectation we have

$$\begin{array}{rcl} \mathbb{E}\left[Y_t^2\right] & = & \mathbb{E}\left[\mathbb{E}\left[\xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r \middle| \mathcal{F}_t\right]^2\right] \\ & \leq & \mathbb{E}\left[\mathbb{E}\left[\left(\xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r\right)^2 \middle| \mathcal{F}_t\right]\right] \\ & = & \mathbb{E}\left[\left(\xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r\right)^2\right], \end{array}$$

which is finite. □

Lemma 3.24. *Let Y be a cadlag adapted process satisfying $\mathbb{E}\left[\underset{t\in[0,T]}{sup} Y_t^2\right] < \infty$ and M be a square integrable martingale, then there exists a constant $C > 0$ such that for any $\epsilon > 0$ we have*

$$\mathbb{E}\left[\underset{t\in[0,T]}{sup}\left|\int_0^t Y_{r-} dM_r\right|\right] \leq C\left(\frac{\epsilon}{2}\mathbb{E}\left[\underset{t\in[0,T]}{sup} Y_t^2\right] + \frac{1}{2\epsilon}\mathbb{E}\left[[M]_T\right]\right).$$

In particular, $\int_0^{\cdot} Y_{r-} dM_r$ is a uniformly integrable martingale.

Proof. By Burkholder-Davis-Gundy (shortened by BDG) and Cauchy-Schwarz (shortened by CS) inequalities, there exists $C > 0$ such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y_{r-}dM_r\right|\right] &\leq C\mathbb{E}\left[\sqrt{\int_0^T Y_{r-}^2\,d[M]_r}\right] \\
&\leq C\mathbb{E}\left[\sqrt{\sup_{t\in[0,T]} Y_t^2[M]_T}\right] \\
&\leq C\sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right]\mathbb{E}[[M]]_T} \\
&\leq C\left(\frac{\epsilon}{2}\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right]+\frac{1}{2\epsilon}\mathbb{E}\left[[M]_T\right]\right) \\
&< +\infty.
\end{aligned}$$

So $\int_0^{\cdot} Y_{r-}dM_r$ is a uniformly integrable local martingale, and therefore a martingale. □

Lemma 3.25. *Let Y be a cadlag adapted process and $M \in \mathcal{H}^2$. Assume the existence of a constant $C > 0$ and an L^1 random variable Z such that for any $t \in [0, T]$*

$$Y_t^2 \leq C\left(Z + \left|\int_0^t Y_{r-}dM_r\right|\right).$$

Then $\sup_{t\in[0,T]} |Y_t| \in L^2$.

Proof. For any stopping time τ we have

$$\sup_{t\in[0,\tau]} Y_t^2 \leq C\left(Z + \sup_{t\in[0,\tau]}\left|\int_0^t Y_{r-}dM_r\right|\right). \tag{3.4}$$

Since Y_{t-} is caglad and therefore locally bounded, (see Definition p164 in [?]) we define $\tau_n = \inf\{t > 0 : Y_{t-} \geq n\}$. It yields $\int_0^{\cdot\wedge\tau_n} Y_{r-}dM_r$ is in $\mathcal{H}^2$ since its angular bracket is equal to $\int_0^{\cdot\wedge\tau_n} Y_{r-}^2 d\langle M\rangle_r$ which is inferior to $n^2\langle M\rangle_T \in L^1$. By Doob's inequality we know that $\sup_{t\in[0,\tau_n]}\left|\int_0^t Y_{r-}dM_r\right|$ is L^2 and using (**??**), we get that $\sup_{t\in[0,\tau_n]} Y_t^2$ is L^1. By (**??**) applied with τ_n and taking expectation, we get

$$\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C'\left(1 + \mathbb{E}\left[\sup_{t\in[0,\tau_n]}\left|\int_0^t Y_{r-}dM_r\right|\right]\right), \tag{3.5}$$

for some C' which does not depend on n. By Lemma **??** applied to (Y^{τ_n}, M) there exists $C'' > 0$ such that for any $n \in \mathbb{N}^*$ and $\epsilon > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \quad \leq \quad C''\left(1+\tfrac{\epsilon}{2}\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right]+\tfrac{1}{2\epsilon}\mathbb{E}\left[[M]_T\right]\right).$$

Choosing $\epsilon = \frac{1}{C''}$, it follows that there exists $C_3 > 0$ such that for any $n > 0$,

$$\frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C_3\left(1+\mathbb{E}\left[[M]_T\right]\right) < \infty.$$

By Fatou's lemma, taking the limit in n we get the result. □

We come back to the process Y defined in Definition **??**.

Proposition 3.26. $\underset{t\in[0,T]}{sup} |Y_t| \in L^2$.

Proof. We will write $\hat{f}_r$ instead of $\hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}(r)}\right)$. Since $dY_r = -\hat{f}_r dV_r + dM_r$, by integration by parts formula we get

$$d(Y_r^2 e^{-V_r}) = -2e^{-V_r}Y_r\hat{f}_r dV_r + 2e^{-V_r}Y_{r-}dM_r + e^{-V_r}d[M]_r - e^{-V_r}Y_r^2 dV_r.$$

So integrating from 0 to some $t \in [0,T]$, we get

$$\begin{array}{rcl} Y_t^2 e^{-V_t} & = & Y_0^2 - 2\int_0^t e^{-V_r}Y_r\hat{f}_r dV_r + 2\int_0^t e^{-V_r}Y_{r-}dM_r \\ & & +\int_0^t e^{-V_r}d[M]_r - \int_0^t e^{-V_r}Y_r^2 dV_r \\ & \leq & Y_0^2 + \int_0^t e^{-V_r}Y_r^2 dV_r + \int_0^t e^{-V_r}\hat{f}_r^2 dV_r \\ & & +2\left|\int_0^t e^{-V_r}Y_{r-}dM_r\right| + \int_0^t e^{-V_r}d[M]_r - \int_0^t e^{-V_r}Y_r^2 dV_r \\ & = & Y_0^2 + \int_0^t e^{-V_r}\hat{f}_r^2 dV_r + \int_0^t e^{-V_r}d[M]_r + 2\left|\int_0^t e^{-V_r}Y_{r-}dM_r\right|. \end{array}$$

Setting $Z = Y_0^2 + \int_0^T e^{-V_r}\hat{f}_r^2 dV_r + \int_0^T e^{-V_r}d[M]_r$ we therefore have, for any $t \in [0,T]$

$$(Y_t e^{-V_t})^2 \leq Y_t^2 e^{-V_t} \leq Z + 2\left|\int_0^t e^{-V_r}Y_{r-}dM_r\right|.$$

Thanks to Propositions **??** and **??**, Z is integrable, so we can conclude by Lemma **??** applied to the process Ye^{-V}, and the fact that V is bounded.

□

Since Y is progressively measurable, $\sup_{t\in[0,T]} |Y_t| \in L^2$ and since V is bounded, it is clear that $Y \in \mathcal{L}^2(dV\otimes d\mathbb{P})$ and the corresponding class $\dot{Y}$ belongs to $L^2(dV\otimes d\mathbb{P})$. We recall that $M \in \mathcal{H}_0^2$ thanks to Proposition **??**.

Notation 3.27. *We denote by* Φ *the operator which associates to a couple* $(\dot{U},N)$ *the couple* $(\dot{Y},M)$.

$$\Phi : \begin{array}{rcl} L^2(dV\otimes d\mathbb{P})\times\mathcal{H}_0^2 & \longrightarrow & L^2(dV\otimes d\mathbb{P})\times\mathcal{H}_0^2 \\ (\dot{U},N) & \longmapsto & (\dot{Y},M). \end{array}$$

Proposition 3.28. *$(\dot{Y}, M) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ is a solution of $\overline{BSDE}(\xi, \hat{f}, V)$ iff it is a fixed point of Φ.*

Proof. Let $(\dot{Y}, M) = \Phi(\dot{U}, N)$ for a certain pair $(\dot{U}, N)$, and let us suppose that $(\dot{Y}, M) = (\dot{U}, N)$. The reference cadlag representative Y of $\dot{Y}$ in the sense of Definition **??** verifies
$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r - (M_T - M_\cdot)$ in the sense of indistinguishability. Since $M = N$ and $\dot{Y} = \dot{U}$, by Remark **??**,
$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right) dV_r - (M_T - M_\cdot)$ and $(\dot{Y}, M)$ solves $\overline{BSDE}(\xi, \hat{f}, V)$.

Reciprocally, let $(\dot{U}, N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ be a solution of $\overline{BSDE}(\xi, \hat{f}, V)$ and let $(\dot{Y}, M) = \Phi(\dot{U}, N)$. By definition of $\overline{BSDE}(\xi, \hat{f}, V)$, there exists a cadlag representative U of $\dot{U}$ verifying

$$U = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r - (N_T - N_\cdot).$$

Thanks to Proposition **??**, N is the cadlag version of
$t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}(r)}\right) dV_r \middle| \mathcal{F}_0\right]$,
but by Definition **??**, so is M. Therefore $M = N$. Again by Definition **??**,

$$\begin{aligned} Y &= \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\tfrac{d\langle N\rangle}{dV}(r)}\right) dV_r - (M_T - M_\cdot) \\ &= \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\tfrac{d\langle N\rangle}{dV}(r)}\right) dV_r - (N_T - N_\cdot) \\ &= U \end{aligned}$$

in the sense of indistinguishability, and in particular, $\dot{U} = \dot{Y}$, so
$(\dot{U}, N) = (\dot{Y}, M) = \Phi(\dot{U}, N)$ is a fixed point of Φ. □

Remark 3.29. *From now on, if $(\dot{Y}, M)$ is the image by Φ of a couple $(\dot{U}, N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$, by default, we will always refer to the cadlag representative Y of $\dot{Y}$ defined in Definition* **??**.

Proposition 3.30. *Let $\lambda \in \mathbb{R}$, let $(\dot{U}, N)$, $(\dot{U}', N')$ be in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$, let $(\dot{Y}, M)$, $(\dot{Y}', M')$ be their images by Φ and let Y, Y' be the cadlag representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition* **??**. *Then $\int_0^\cdot e^{\lambda V_r} Y_{r-} dM_r$, $\int_0^\cdot e^{\lambda V_r} Y'_{r-} dM'_r$, $\int_0^\cdot e^{\lambda V_r} Y_{r-} dM'_r$ and $\int_0^\cdot e^{\lambda V_r} Y'_{r-} dM_r$ are martingales.*

Proof. V is bounded and thanks to Proposition **??** we know that $\sup_{t\in[0,T]} |Y_t|$ and $\sup_{t\in[0,T]} |Y'_t|$ are L^2. Moreover since M and M' are square integrable, the statement yields therefore as a consequence of previous Lemma **??**. □

We will now show that Φ is a contraction for a certain norm. This will imply that $\overline{BSDE}(\xi, \hat{f}, V)$ has a unique solution in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ since this space is complete.

Definition 3.31. *For any $\lambda > 0$, we define the following norm on $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$:*

$$\|(\dot{Y}, M)\|^2_\lambda := \mathbb{E}\left[\int_0^T e^{\lambda V_r} Y_r^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle M\rangle_r\right].$$

Since V is bounded, these norms are all equivalent to the usual norm of this space, which corresponds to $\lambda = 0$.

Proposition 3.32. *There exists $\lambda > 0$ such that for any $(\dot{U}, N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$, $\left\|\Phi(\dot{U}, N)\right\|^2_\lambda \leq \frac{1}{2}\left\|(\dot{U}, N)\right\|^2_\lambda$. In particular, Φ is a contraction in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ for the norm $\|\cdot\|_\lambda$.*

Proof. Let $(\dot{U}, N)$ and $(\dot{U}', N')$ be two couples of $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$, let $(\dot{Y}, M)$ and $(\dot{Y}', M')$ be their images via Φ and let Y, Y' be the cadlag representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition **??**. We will write $\bar{Y}$ for $Y - Y'$ and we adopt a similar notation for other processes. We will also write
$\bar{f}_t := \hat{f}\left(t, \cdot, U_t, \sqrt{\frac{d\langle N\rangle}{dV}(t)}\right) - \hat{f}\left(t, \cdot, U'_t, \sqrt{\frac{d\langle N'\rangle}{dV}(t)}\right)$.

By additivity, we have $d\bar{Y}_t = -\bar{f}_t dV_t + d\bar{M}_t$. Since $\bar{Y}_T = \xi - \xi = 0$, applying the integration by parts formula to $\bar{Y}_t^2 e^{\lambda V_t}$ between 0 and T we get

$$\bar{Y}_0^2 - 2\int_0^T e^{\lambda V_r}\bar{Y}_r\bar{f}_r dV_r + 2\int_0^T e^{\lambda V_r}\bar{Y}_{r-} d\bar{M}_r + \int_0^T e^{\lambda V_r} d[\bar{M}]_r + \lambda\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r = 0.$$

Since, by Proposition **??**, the stochastic integral with respect to $\bar{M}$ is a real martingale, by taking the expectations we get

$$\mathbb{E}\left[\bar{Y}_0^2\right] - 2\mathbb{E}\left[\int_0^T e^{\lambda V_r}\bar{Y}_r\bar{f}_r dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle\bar{M}\rangle_r\right] + \lambda\mathbb{E}\left[\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r\right] = 0.$$

So by re-arranging and by using the Lipschitz condition on f stated in Hypothesis **??**, we get

$$
\begin{aligned}
& \lambda \mathbb{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & 2\mathbb{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r||\bar{f}_r| dV_r\right] \\
\leq\ & 2K^Y \mathbb{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r||\bar{U}_r| dV_r\right] \\
& +2K^Z \mathbb{E}\left[\int_0^T e^{\lambda V_r} |\bar{Y}_r| \left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right| dV_r\right] \\
\leq\ & (K^Y\alpha + K^Z\beta)\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{Y}_r|^2 dV_r\right] + \frac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] \\
+\ & \frac{K^Z}{\beta}\mathbb{E}\left[\int_0^T e^{\lambda V_r} \left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right],
\end{aligned}
$$

for any positive α and β. Then we pick $\alpha = 2K^Y$ and $\beta = 2K^Z$, which gives us

$$
\begin{aligned}
& \lambda \mathbb{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & 2((K^Y)^2 + (K^Z)^2)\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{Y}_r|^2 dV_r\right] \\
+\ & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda V_r}\left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right].
\end{aligned}
$$

We choose now $\lambda = 1 + 2((K^Y)^2 + (K^Z)^2)$ we get

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda V_r}\left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right].
\end{aligned} \tag{3.6}
$$

On the other hand, since by Proposition **??** we know that $\frac{d\langle N\rangle}{dV}\frac{d\langle N'\rangle}{dV} - \left(\frac{d\langle N,N'\rangle}{dV}\right)^2$ is a positive process, we have

$$
\begin{aligned}
\left|\sqrt{\frac{d\langle N\rangle}{dV}} - \sqrt{\frac{d\langle N'\rangle}{dV}}\right|^2 &= \frac{d\langle N\rangle}{dV} - 2\sqrt{\frac{d\langle N\rangle}{dV}}\sqrt{\frac{d\langle N'\rangle}{dV}} + \frac{d\langle N'\rangle}{dV} \\
&\leq \frac{d\langle N\rangle}{dV} - 2\frac{d\langle N,N'\rangle}{dV} + \frac{d\langle N'\rangle}{dV} \\
&= \frac{d\langle \bar{N}\rangle}{dV}\ dV\otimes d\mathbb{P}\ \text{a.e.}
\end{aligned} \tag{3.7}
$$

Therefore, since by Remark **??** we have $\int_0^\cdot e^{\lambda V_r}\frac{d\langle \bar{N}\rangle}{dV}(r)dV_r \leq \int_0^\cdot e^{\lambda V_r} d\langle \bar{N}\rangle_r$, then expression (**??**) implies

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^T e^{\lambda V_r} \bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{M}\rangle_r\right] \\
\leq\ & \frac{1}{2}\left(\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle \bar{N}\rangle_r\right]\right),
\end{aligned}
$$

which proves the contraction for the norm $\|\cdot\|_\lambda$. □

Corollary 3.33. *If* $(\xi, \hat{f})$ *verifies Hypothesis* **??** *then* $\overline{BSDE}(\xi, \hat{f}, V)$ *has a unique solution.*

Proof. The space $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ is complete and Φ defines on it a contraction for the norm $\|(\cdot,\cdot)\|_\lambda$ for some $\lambda > 0$, so Φ has a unique fixed point in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$. Then by Proposition **??**, $\overline{BSDE}(\xi, \hat{f}, V)$ has a unique solution. □

Theorem 3.34. *If $(\xi, \hat{f})$ verifies Hypothesis* **??** *then $BSDE(\xi, \hat{f}, V)$ has a unique solution.*

Proof. The proof follows directly from Corollary **??** and Proposition **??**. □

Remark 3.35. *Let (Y, M) be the solution of $BSDE(\xi, \hat{f}, V)$ and $\dot{Y}$ the class of Y in $L^2(dV \otimes d\mathbb{P})$. Thanks to Proposition* **??***, we know that $(\dot{Y}, M) = \Phi(\dot{Y}, M)$ and therefore by Propositions* **??** *and* **??** *that $\sup_{t\in[0,T]} |Y_t|$ is L^2 and that $\int_0^\cdot Y_{r-} dM_r$ is a real martingale.*

Remark 3.36. *Let $(\xi, \hat{f}, V)$ satisfying Hypothesis* **??***. Until now we have considered the related BSDE on the interval $[0, T]$. Without restriction of generality we can consider a BSDE on a restricted interval $[s, T]$ for some $s \in [0, T[$. The whole previous discussion and all the results expressed above trivially extend to this case. In particular there exists a unique couple of processes (Y^s, M^s), indexed by $[s, T]$ such that Y^s is adapted, cadlag and verifies $\mathbb{E}[\int_s^T (Y_r^s)^2 dV_r] < \infty$, such that M^s is a martingale starting at 0 in s and such that*

$$Y_\cdot^s = \xi + \int_\cdot \hat{f}\left(r, \cdot, Y_r^s, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right) dV_r - (M_T^s - M_\cdot^s),$$

in the sense of indistinguishability on $[s, T]$.
Moreover, if (Y, M) denotes the solution of $BSDE(\xi, \hat{f}, V)$ then $(Y, M_\cdot - M_s)$ and (Y^s, M^s) coincide on $[s, T]$. This follows by the uniqueness argument for the restricted BSDE to $[s, T]$.

The lemma below shows that, in order to verify that a couple (Y, M) is the solution of $BSDE(\xi, \hat{f}, V)$, it is not necessary to verify the square integrability of Y since it will be automatically fulfilled.

Lemma 3.37. *Let $(\xi, \hat{f}, V)$ verify Hypothesis* **??** *and consider $BSDE(\xi, \hat{f}, V)$ defined in Definition* **??***. Assume that there exists a cadlag adapted process Y with $Y_0 \in L^2$, and $M \in \mathcal{H}_0^2$ such that*

$$Y = \xi - \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}(r)}\right) dV_r - (M_T - M_\cdot), \qquad (3.8)$$

in the sense of indistinguishability. Then $\sup_{t\in[0,T]} |Y_t|$ is L^2. In particular, $Y \in \mathcal{L}^2(dV \otimes d\mathbb{P})$ and (Y, M) is the unique solution of $BSDE(\xi, \hat{f}, V)$.

*Similarly if (Y,M) verifies (**??**) on $[s,T]$ with $s<T$, if $Y_s\in L^2$, $M_s=0$ and if we denote (U,N) the unique solution of $BSDE(\xi,\hat{f},V)$, then (Y,M) and $(U,N.-N_s)$ are intistinguishable on $[s,T]$.*

Proof. Let $\lambda>0$ and $t\in[0,T]$. By integration by parts formula applied to $Y^2e^{-\lambda V}$ between 0 and t we get

$$\begin{aligned} Y_t^2e^{-\lambda V_t}-Y_0^2 &= -2\int_0^t e^{-\lambda V_r}Y_r\hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}(r)}\right)dV_r+2\int_0^t e^{-\lambda V_r}Y_{r-}dM_r \\ &\quad +\int_0^t e^{-\lambda V_r}d[M]_r-\lambda\int_0^t e^{-\lambda V_r}Y_r^2dM_r. \end{aligned}$$

By re-arranging the terms and using the Lipschitz conditions in Hypothesis **??**, we get

$$\begin{aligned} & Y_t^2e^{-\lambda V_t}+\lambda\int_0^t e^{-\lambda V_r}Y_r^2dV_r \\ \leq\ & Y_0^2+2\int_0^t e^{-\lambda V_r}|Y_r||\hat{f}|\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}(r)}\right)dV_r+2\left|\int_0^t e^{-\lambda V_r}Y_{r-}dM_r\right| \\ & +\int_0^t e^{-\lambda V_r}d[M]_r \\ \leq\ & Y_0^2+2\int_0^t e^{-\lambda V_r}|Y_r||\hat{f}|(r,\cdot,0,0)dV_r+2K^Y\int_0^t e^{-\lambda V_r}|Y_r|^2dV_r \\ & +2K^Z\int_0^t e^{-\lambda V_r}|Y_r|\sqrt{\frac{d\langle M\rangle}{dV}(r)}dV_r+2\left|\int_0^t e^{-\lambda V_r}Y_{r-}dM_r\right|+\int_0^t e^{-\lambda V_r}d[M]_r \\ \leq\ & Y_0^2+\int_0^t e^{-\lambda V_r}|\hat{f}|^2(r,\cdot,0,0)dV_r+(2K^Y+1+K^Z)\int_0^t e^{-\lambda V_r}|Y_r|^2dV_r \\ & +2\left|\int_0^t e^{-\lambda V_r}Y_{r-}dM_r\right|+\int_0^t e^{-\lambda V_r}d[M]_r. \end{aligned}$$

Picking $\lambda=2K^Y+1+K^Z$ this gives

$$\begin{aligned} Y_t^2e^{-\lambda V_t} &\leq Y_0^2+\int_0^t e^{-\lambda V_r}|\hat{f}|^2(r,\cdot,0,0)dV_r+K^Z\int_0^t e^{-\lambda V_r}\frac{d\langle M\rangle}{dV}(r)dV_r \\ &\quad +2\left|\int_0^t e^{-\lambda V_r}Y_{r-}dM_r\right|+\int_0^t e^{-\lambda V_r}d[M]_r. \end{aligned}$$

Since V is bounded, there is a constant $C>0$, such that for any $t\in[0,T]$

$$Y_t^2\leq C\left(Y_0^2+\int_0^T|\hat{f}|^2(r,\cdot,0,0)dV_r+\int_0^T\frac{d\langle M\rangle}{dV}(r)dV_r+[M]_T+\left|\int_0^t Y_{r-}dM_r\right|\right).$$

By Hypothesis **??** and since we assumed $Y_0\in L^2$ and $M\in\mathcal{H}^2$, the first four terms on the right hand side are integrable and we can conclude by Lemma **??**.

An analogous proof also holds on the interval $[s,T]$ taking into account Remark **??**. □

If the underlying filtration is Brownian and $V_t=t$, we can identify the solution of the BSDE with no driving martingale to the solution of a Brownian BSDE.

Let B be a 1-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $T\in\mathbb{R}_+^*$ and for any $t\in[0,T]$, let $\mathcal{F}_t^B$ denote the σ-field $\sigma(B_r|r\in[0,t])$ augmented with the $\mathbb{P}$-negligible sets.
In the stochastic basis $(\Omega,\mathcal{F},\mathcal{F}^B,\mathbb{P})$, let $V_t=t$ and $(\xi,\hat{f})$ satisfy Hypothesis **??**. Let (Y,M) be the unique solution of $BSDE(\xi,\hat{f},V)$, see Theorem **??**.

Proposition 3.38. *We have* $Y = U$, $M = \int_0^{\cdot} Z_r dB_r$*, where* (U, Z) *is the unique solution of the Brownian BSDE*

$$U = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, U_r, |Z_r|\right) dr - \int_{\cdot}^{T} Z_r dB_r, \tag{3.9}$$

in $\mathcal{L}^2(dt \otimes d\mathbb{P}) \times L^2(dt \otimes d\mathbb{P})$,

Proof. By Theorem 1.2 in [?], (**??**) admits a unique solution (U, Z) of progressively measurable processes such that $Z \in L^2(dt \otimes d\mathbb{P})$. It is known that $\sup\limits_{t\in[0,T]} |U_t| \in L^2$ and therefore that $U \in \mathcal{L}^2(dt \otimes d\mathbb{P})$, see Proposition 1.1 in [?] for instance. We define $N = \int_0^{\cdot} Z_r dB_r$. The couple (U, N) belongs to $\mathcal{L}^2(dt \otimes d\mathbb{P}) \times \mathcal{H}^2_0$. N verifies $\frac{d\langle N\rangle_r}{dr} = Z_r^2 \; dt \otimes d\mathbb{P}$ a.e. So by (**??**), the couple (U, N) verifies

$$U = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle_r}{dr}}\right) dr - (N_T - N_{\cdot}),$$

in the sense of indistinguishability. It therefore solves $BSDE(\xi, \hat{f}, V)$ and the assertion yields by uniqueness of the solution. □

4 The Markov Process and associated Martingale Problem

4.1 Martingale problems

In this section, we introduce the Markov process which will later be the forward process with which we will define some particular Forward BSDEs with no driving martingales. For details about the exact mathematical background that we use to define our Markov process, one can consult the Section **??** of the Appendix. We also introduce the martingale problem which this Markov process will be assumed to solve.

Let E be a Polish space and $T \in \mathbb{R}_+^*$ be a fixed horizon. From now on, $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ denotes the canonical space defined in Definition **??**. We consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition function measurable in time as defined in Definitions **??** and **??**, and for any $(s, x) \in [0, T] \times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ will denote the stochastic basis introduced in Definition **??** and which fulfills the usual conditions.

Remark 4.1. *All notions and results of this section extend to a time index equal to* $\mathbb{R}_+$.

We start by introducing a general notion of Martingale Problem as defined in Chapter XI of [?].

Definition 4.2. *Let χ be a family of stochastic processes defined on a filtered space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in\mathbb{T}})$. We say that a probability measure $\mathbb{P}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ solves the* ***martingale problem*** *associated to χ if under $\mathbb{P}$ all elements of χ are in $\mathcal{M}_{loc}$.*
We denote $\mathcal{MP}(\chi)$ the set of probability measures solving this martingale problem and $\mathcal{MP}_e(\chi)$ its set of extremal points meaning that if $\mathbb{P}$ belongs to $\mathcal{MP}_e(\chi)$, there can not exist distinct probability measures $\mathbb{Q}, \mathbb{Q}'$ in $\mathcal{MP}(\chi)$ and $\alpha \in]0,1[$ such that $\mathbb{P} = \alpha\mathbb{Q} + (1-\alpha)\mathbb{Q}'$.

We now want to introduce a Martingale problem associated to an operator. Our formalism will be very close to the one introduced by D.W. Stroock and S.R.S Varadhan in [**?**]. We will see in Remark **??** that both Definitions **??** and **??** are closely related.

Definition 4.3. *Let us consider*

1. *a domain $\mathcal{D}(a) \subset \mathcal{B}([0,T]\times E, \mathbb{R})$ which is a linear algebra;*
2. *a linear operator $a : \mathcal{D}(a) \longrightarrow \mathcal{B}([0,T]\times E, \mathbb{R})$;*
3. *a non-decreasing continuous function $V : [0,T] \to \mathbb{R}_+$ starting at 0.*

We say that a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega, \mathcal{F})$ solves the **martingale problem associated to** *$(\mathcal{D}(a), a, V)$ if, for any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ verifies*

(a) $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;

(b) for every $\phi \in \mathcal{D}(a)$, the process

$$\left(t \longmapsto \phi(t, X_t) - \phi(s,x) - \int_s^t a(\phi)(r, X_r)dV_r\right),\ t \in [s,T],$$

is a cadlag $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[s,T]})$-local martingale.

We say that the Martingale Problem is ***well-posed*** *if for any $(s,x)\in[0,T]\times E$, $\mathbb{P}^{s,x}$ is the only probability measure satisfying these two properties.*

Remark 4.4. *In other words, $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves the martingale problem associated to $(\mathcal{D}(a), a, V)$ if and only if, for any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x} \in \mathcal{MP}(\chi^{s,x})$ (see Definition **??**), where $\chi^{s,x}$ is the reunion of all processes*

$$\left\{t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dV_r\right)\middle|\phi \in \mathcal{D}(a)\right\},$$

and processes

$$\left\{t \mapsto \mathbb{1}_{\{r\}}(t)(X_t - x)\middle| r \in [0,s]\right\}.$$

Indeed for some $r \in [0,s]$, $t \mapsto \mathbb{1}_{\{r\}}(t)(X_t - x)$ is a cadlag local martingale iff $X_r = x$ a.s. so requiring that processes $t \mapsto \mathbb{1}_{\{r\}}(t)(X_t - x)$ are cadlag local martingales for every $r \in [0,s]$ is equivalent to requiring that $X_r = x$ a.s. for

every $r \in [0,s]$, and therefore since X and x are cadlag, it is equivalent to requiring item a) in Definition **??**.
$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves the well-posed martingale problem associated to* $(\mathcal{D}(a), a, V)$ *if and only if, for any* $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$, $\mathcal{MP}(\chi^{s,x}) = \{\mathbb{P}^{s,x}\}$.

Notation 4.5. *For every* $(s,x) \in [0,T] \times E$ *and* $\phi \in \mathcal{D}(a)$, *the process* $t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dV_r\right)$ *will be denoted* $M[\phi]^{s,x}$.

$M[\phi]^{s,x}$ is a cadlag $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[0,T]})$-local martingale equal to 0 on $[0,s]$, and by Proposition **??**, it is also a $(\mathbb{P}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]})$-local martingale.

In the sequel of the paper we will regularly assume the following.

Hypothesis 4.6. *The Markov class* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves a well-posed Martingale Problem associated to a triplet* $(\mathcal{D}(a), a, V)$ *in the sense of Definition* **??**.

The bilinear operator below was introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [**?**]), and popularized by P.A. Meyer in the study of homogeneous Markov processes, see for example Exposé II: L'opérateur carré du champs in [**?**].

Definition 4.7. *We set*

$$\Gamma : \begin{array}{rcl} \mathcal{D}(a)\times\mathcal{D}(a) & \rightarrow & \mathcal{B}([0,T]\times E) \\ (\phi,\psi) & \mapsto & a(\phi\psi) - \phi a(\psi) - \psi a(\phi). \end{array} \tag{4.1}$$

The operator Γ *is called the* ***square field operator***.

This operator will appear in the expression of the angular bracket of the local martingales that we have defined.

Proposition 4.8. *For any* $\phi \in \mathcal{D}(a)$ *and* $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x}$ *belongs to* $\mathcal{H}^2_{0,loc}$. *Moreover, for any* $(\phi,\psi) \in \mathcal{D}(a)\times\mathcal{D}(a)$ *and* $(s,x) \in [0,T]\times E$, *we have*

$$\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^{\cdot} \Gamma(\phi,\psi)(r,X_r)dV_r,$$

on the interval $[s,T]$, *in the stochastic basis* $(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x})$.

Proof. We fix some $(s,x) \in [0,T]\times E$ and the associated probability $\mathbb{P}^{s,x}$. For any ϕ,ψ in $\mathcal{D}(a)$, by integration by parts on $[s,T]$ we have

$$\begin{array}{rl} & M[\phi]^{s,x}M[\psi]^{s,x} \\ = & \int_s^{\cdot} M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^{\cdot} M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r + [M[\phi]^{s,x}, M[\psi]^{s,x}] \\ = & \int_s^{\cdot} M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^{\cdot} M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r + [\phi(\cdot,X_\cdot), \psi(\cdot,X_\cdot)] \\ = & \int_s^{\cdot} M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^{\cdot} M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r + \phi\psi(\cdot,X_\cdot) \\ & -\phi\psi(s,x) - \int_s^{\cdot} \phi(r^-,X_{r^-})d\psi(r,X_r) - \int_s^{\cdot} \psi(r^-,X_{r^-})d\phi(r,X_r). \end{array}$$

Since $\phi\psi$ belongs to $\mathcal{D}(a)$, we can use the decomposition of $\phi\psi(\cdot, X_\cdot)$ given by (b) in Definition **??** and

$$\begin{array}{ll} & M[\phi]^{s,x}M[\psi]^{s,x} \\ = & \int_s^\cdot M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r + \int_s^\cdot a(\phi\psi)(r, X_r)dV_r \\ & +M^{s,x}[\phi\psi] - \int_s^\cdot \phi a(\psi)(r, X_r)dV_r - \int_s^\cdot \psi a(\phi)(r, X_r)dV_r \\ & -\int_s^\cdot \phi(r^-, X_{r^-})dM^{s,x}[\psi]_r - \int_s^\cdot \psi(r^-, X_{r^-})dM^{s,x}[\phi]_r \\ = & \int_s^\cdot \Gamma(\phi,\psi)(r, X_r)dV_r + \int_s^\cdot M[\phi]^{s,x}_{r^-}dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-}dM[\phi]^{s,x}_r \\ & +M^{s,x}[\phi\psi] - \int_s^\cdot \phi(r^-, X_{r^-})dM^{s,x}[\psi]_r - \int_s^\cdot \psi(r^-, X_{r^-})dM^{s,x}[\phi]_r. \end{array} \quad (4.2)$$

Since V is continuous, this implies that $M[\phi]^{s,x}M[\psi]^{s,x}$ is a special semi-martingale with bounded variation predictable part $\int_s^\cdot \Gamma(\phi,\psi)(r, X_r)dV_r$. In particular taking $\phi = \psi$, we have on $[s, T]$

$$(M[\phi]^{s,x})^2 = \int_s^\cdot \Gamma(\phi,\phi)(r, X_r)dV_r + N^{s,x},$$

where $N^{s,x}$ is some local martingale. Since every continuous process is locally bounded, and a local martingale is locally integrable, then $(M[\phi]^{s,x})^2$ is locally integrable, meaning that $M[\phi]^{s,x}$ is in $\mathcal{H}^2_{0,loc}$.

Now coming back to any ϕ, ψ in $\mathcal{D}(a)$, since we know that $M[\phi]^{s,x}$, $M[\psi]^{s,x}$ belong to $\mathcal{H}^2_{0,loc}$ we can consider $\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle$ which, by definition, is the unique predictable process with bounded variation such that $M[\phi]^{s,x}M[\psi]^{s,x} - \langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle$ is a local martingale. So necessarily, taking (**??**) into account, $\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^\cdot \Gamma(\phi,\psi)(r, X_r)dV_r$. □

Taking $\phi = \psi$ in Proposition **??**, yields the following.

Corollary 4.9. *For any $(s, x) \in [0, T] \times E$ and $\phi \in \mathcal{D}(a)$, $M[\phi]^{s,x} \in \mathcal{H}^{2,V}_{loc}$.*

Remark 4.10. *By Proposition **??**, it is clear that any element of $\mathcal{H}^{2,\perp V}$ is strongly orthogonal to any element of $\mathcal{H}^{2,V}_{loc}$.*

We conclude this section showing that in our setup, $\mathcal{H}^2_0$ is always equal to $\mathcal{H}^{2,V}$. We need for this a theorem proven by J. Jacod and M. Yor which states (see e.g. Theorem 11.2 in [**?**]) the following.

Theorem 4.11. *Let χ be a set of processes defined on some fixed filtered space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in\mathbb{T}})$. If $\mathbb{P} \in \mathcal{MP}(\chi)$ then the following assertions are equivalent.*

1. *$\mathbb{P} \in \mathcal{MP}_e(\chi)$;*
2. *any $N \in \mathcal{H}^\infty_{0,loc}(\mathbb{P})$ strongly orthogonal to all elements of χ is equal to zero, and $\tilde{\mathcal{F}}_0$ is $\mathbb{P}$-trivial.*

Proposition 4.12. *Let $(s, x) \in [0, T] \times E$ and $\mathbb{P}^{s,x}$ be fixed. If $N \in \mathcal{H}^\infty_{0,loc}$ is strongly orthogonal to $M[\phi]^{s,x}$ for every $\phi \in \mathcal{D}(a)$ then it is necessarily equal to 0.*

Proof. In Hypothesis **??**, for any $(s,x) \in [0,T] \times E$ we have assumed that $\mathbb{P}^{s,x}$ was the unique element of $\mathcal{MP}(\chi^{s,x})$, where $\chi^{s,x}$ was introduced in Remark **??**. Therefore $\mathbb{P}^{s,x}$ is extremal in $\mathcal{MP}(\chi^{s,x})$. So thanks to Theorem **??**, we know that if an element N of $\mathcal{H}^{\infty}_{0,loc}$ is strongly orthogonal to all the $M[\phi]^{s,x}$ then it is equal to zero. □

Remark 4.13. *As announced before, the next step consists in proving $\mathcal{H}^2_0 = \mathcal{H}^{2,V}$. If Proposition **??** would hold replacing $\mathcal{H}^{\infty}_{0,loc}$ with $\mathcal{H}^2_0$, one could easily conclude. Indeed, let $N \in \mathcal{H}^2_0$. According to Proposition **??** we get the decomposition $N = N^V + N^{\perp V}$. It remains to show that $N^{\perp V} = 0$. Taking into account Definition **??**, by Remark **??** that process is strongly orthogonal to every element of $\mathcal{H}^{2,V}_{loc}$. By Corollary **??**, for every ϕ, $M[\phi]^{s,x}$ is in $\mathcal{H}^{2,V}_{loc}$, so $N^{\perp V}$ is orthogonal to all of them. If our conjecture concerning the possible extension of Proposition **??** would hold then the conclusion would follow.*

*Unfortunately the overmentioned conjecture about the extension of the validity of Proposition **??** is wrong in general. Indeed, according to Theorem 11.3 of [**?**], Theorem **??** would be wrong replacing its second item with $N \in \mathcal{H}^2_{0,loc}$ instead of $N \in \mathcal{H}^{\infty}_{0,loc}$.*

*In our situation, the special structure of $\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ permits to perform the announced step. This will be the object of Proposition **??**.*

Proposition 4.14. *If Hypothesis **??** is verified then for any $(s,x) \in [0,T] \times E$, in the stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$, we have $\mathcal{H}^2_0 = \mathcal{H}^{2,V}$.*

Proof. We fix $(s,x) \in [0,T] \times E$. It is enough to show the inclusion $\mathcal{H}^2_0 \subset \mathcal{H}^{2,V}$. We start considering a bounded martingale $N \in \mathcal{H}^{\infty}_0$ and showing that it is in $\mathcal{H}^{2,V}$. Since N belongs to $\mathcal{H}^2_0$, we can consider the corresponding $N^V, N^{\perp V}$ in $\mathcal{H}^2_0$, appearing in the statement of Proposition **??**. We show below that N^V and $N^{\perp V}$ are locally bounded, which will permit us to use Jacod-Yor theorem on $N^{\perp V}$.

Indeed, by Proposition **??** there exists a predictable process K such that $N^V = \int_s^{\cdot} \mathbb{1}_{\{K_r<1\}} dN_r$ and $N^{\perp V} = \int_s^{\cdot} \mathbb{1}_{\{K_r=1\}} dN_r$. So if N is bounded then it has bounded jumps; by previous way of characterizing N^V and $N^{\perp V}$, their jumps can be expressed $(\Delta N^V)_t = \mathbb{1}_{\{K_t<1\}} \Delta N_t$ and $(\Delta N^{\perp V})_t = \mathbb{1}_{\{K_t=1\}} \Delta N_t$ (see Theorem 8 Chapter IV.3 in [**?**]), so they also have bounded jumps which implies that they are locally bounded.

So $N^{\perp V}$ is in $\mathcal{H}^{\infty}_{0,loc}$ and by construction belongs to $\mathcal{H}^{2,\perp V}$, see Definition **??**. Since by Corollary **??**, all the $M[\phi]^{s,x}$ belong to $\mathcal{H}^{2,V}_{loc}$, then, by Remark **??**, $N^{\perp V}$ is strongly orthogonal to all the $M[\phi]^{s,x}$. Consequently, by Proposition **??**, $N^{\perp V}$ is equal to zero. This shows that $N = N^V$ which by construction belongs to $\mathcal{H}^{2,V}$, and consequently that $\mathcal{H}^{\infty}_0 \subset \mathcal{H}^{2,V}$, which concludes the proof when N is a bounded martingale.

We can conclude by density arguments as follows. Let $M \in \mathcal{H}^2_0$. For any integer $n \in \mathbb{N}^*$, we denote by M^n the martingale in $\mathcal{H}^{\infty}_0$ defined as the cadlag

version of $t \mapsto \mathbb{E}^{s,x}[((-n)\vee M_T\wedge n)|\mathcal{F}_t]$. Now $\left(M^n_T - M_T\right)^2 \underset{n\to\infty}{\longrightarrow} 0$ a.s. and this sequence is bounded by $4M_T^2$ which is an integrable r.v. So by the dominated convergence theorem
$\mathbb{E}^{s,x}\left[\left(M^n_T - M_T\right)^2\right] \underset{n\to\infty}{\longrightarrow} 0$. Then by Doob's inequality, $\sup\limits_{t\in[0,T]}(M^n_t - M_t) \overset{L^2}{\underset{n\to\infty}{\longrightarrow}} 0$ meaning that $M^n \overset{\mathcal{H}^2}{\underset{n\to\infty}{\longrightarrow}} M$. Since $\mathcal{H}^\infty_0 \subset \mathcal{H}^{2,V}$, then M^n belongs to $\mathcal{H}^{2,V}$ for any $n\geq 0$. Moreover $\mathcal{H}^{2,V}$ is closed in $\mathcal{H}^2$, since by Proposition **??**, it is a sub-Hilbert space. Finally we have shown that $M\in\mathcal{H}^{2,V}$. □

Remark 4.15. *Since V is continuous, it follows that every $(\mathbb{P}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]})$-square integrable martingale has a continuous angular bracket. By localization, the same assertion holds for local square integrable martingales.*

4.2 Extended operators and zero-potential sets

In this section, we work still on the canonical space $\left(\Omega,\mathcal{F},(X_t)_{t\in[0,T]},(\mathcal{F}_t)_{t\in[0,T]}\right)$ (see Definition **??**) for some $T\in\mathbb{R}^*_+$ and some Polish space E. We assume that we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to some transition function measurable in time (see Definitions **??** and **??**), which solves a well-posed martingale problem associated to a triplet $(\mathcal{D}(a),a,V)$, see Definition **??**.

We start with a small lemma which will be used several times in the next sections.

Lemma 4.16. *Let $(s,x)\in[0,T]\times E$ be fixed and let ϕ,ψ be two measurable processes. If ϕ and ψ are $\mathbb{P}^{s,x}$-modifications of each other, then they are equal $dV\otimes d\mathbb{P}^{s,x}$ a.e.*

Proof. By Fubini's theorem we can write

$$\mathbb{E}^{s,x}\left[\int_0^T \mathbb{1}_{\phi_t\neq\psi_t}dV_t\right] = \int_0^T \mathbb{P}^{s,x}(\phi_t\neq\psi_t)dV_t = 0,$$

since for any $t\in[0,T]$, $\phi_t=\psi_t$ $\mathbb{P}^{s,x}$ a.s. □

Definition 4.17. *For any $(s,x)\in[0,T]\times E$ we define the positive bounded **potential measures** $U(s,x,\cdot)$ on $([0,T]\times E,\mathcal{B}([0,T])\otimes\mathcal{B}(E))$ by*

$$U(s,x,\cdot):\begin{array}{rcl}\mathcal{B}([0,T])\otimes\mathcal{B}(E) & \longrightarrow & [0,V_T]\\ A & \longmapsto & \mathbb{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}}dV_t\right].\end{array}$$

Definition 4.18. *A Borel set $A\subset[0,T]\times E$ will be said to be* **of zero potential** *if, for any $(s,x)\in[0,T]\times E$ we have $U(s,x,A)=0$.*

Notation 4.19. *Let $p\in\mathbb{N}^*$. We define*
$\mathcal{L}^p_{s,x} := \mathcal{L}^p(U(s,x,\cdot)) = \left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}) : \mathbb{E}^{s,x}\left[\int_s^T |f|^p(r,X_r)dV_r\right]<\infty\right\}.$

That classical $\mathcal{L}^p$-space is equipped with the seminorm
$\|\cdot\|_{p,s,x} : f \mapsto \left(\mathbb{E}^{s,x}\left[\int_s^T |f(r,X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$. *We also define*
$\mathcal{L}^0_{s,x} := \left\{f \in \mathcal{B}([0,T]\times E,\mathbb{R}) : \int_s^T |f|(r,X_r)dV_r < \infty \quad \mathbb{P}^{s,x} \text{ a.s.}\right\}$.
We then define the intersection of those spaces:

1. $\mathcal{L}^p_X = \{f \in \mathcal{B}([0,T]\times E,\mathbb{R}) : \forall (s,x), \|f\|_{p,s,x} < \infty\}$ *if* $p \geq 1$*;*

2. $\mathcal{L}^0_X = \left\{f \in \mathcal{B}([0,T]\times E,\mathbb{R}) : \forall (s,x), \int_s^T |f|(t,X_t)dV_t < \infty \quad \mathbb{P}^{s,x} \text{ a.s.}\right\}$.

$\mathcal{N}$ *will denote the sub-space of* $\mathcal{B}([0,T]\times E,\mathbb{R})$ *containing all functions which are equal to 0,* $U(s,x,\cdot)$ *a.e. for every* (s,x)*.*
For any $p \in \mathbb{N}$*, we define the quotient space*

$$L^p_X = \mathcal{L}^p_X / \mathcal{N}.$$

If $p \geq 1$*,* L^p_X *can be equipped with the topology generated by the family of seminorms* $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ *which makes it a separate locally convex topological vector space, see Theorem 5.76 in [**?**].*

Proposition 4.20. *Let* f *and* g *be in* $\mathcal{L}^0_{s,x}$*. Then the following assertions are equivalent.*

1. *For any* $(s,x) \in [0,T]\times E$*, the processes* $\int_s^\cdot f(r,X_r)dV_r$ *and* $\int_s^\cdot g(r,X_r)dV_r$ *are indistinguishable under* $\mathbb{P}^{s,x}$*;*

2. f *and* g *are equal up to a set of zero potential.*

Of course in this case f *and* g *correspond to the same element of* L^0_X*.*

Proof. Let $\mathbb{P}^{s,x}$ be fixed. Evaluating the total variation of $\int_s^\cdot (f-g)(r,X_r)dV_r$ yields that $\int_s^\cdot f(r,X_r)dV_r$ and $\int_s^\cdot g(r,X_r)dV_r$ are indistinguishable if and only if $\int_s^T |f-g|(r,X_r)dV_r = 0$ a.s. Since that r.v. is non-negative, this is true if and only if
$\mathbb{E}^{s,x}\left[\int_s^T |f-g|(r,X_r)dV_r\right] = 0$ and therefore if and only if $U(s,x,N) = 0$, where N is the Borel subset of $[0,T]\times E$, defined by $\{(t,y) : f(t,y) \neq g(t,y)\}$. This concludes the proof of Proposition **??**.

□

We can now define our notion of **extended generator**.

Definition 4.21. *We first define the* ***extended domain*** $\mathcal{D}(\mathfrak{a})$ *as the set functions* $\phi \in \mathcal{B}([0,T]\times E,\mathbb{R})$ *for which there exists*
$\psi \in \mathcal{B}([0,T]\times E,\mathbb{R})$ *such that under any* $\mathbb{P}^{s,x}$ *the process*

$$\left(\phi(s\vee\cdot,X_\cdot) - \phi(s,x) - \int_s^{s\vee\cdot} \psi(r,X_r)dV_r\right) \tag{4.3}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}^2_0$*.*

Proposition 4.22. *Let ϕ be in $\mathcal{D}(\mathfrak{a})$. Then the function ψ satisfying the above condition is unique up to zero potential sets.*

Proof. Let ψ^1 and ψ^2 be two functions verifying the condition imposed by Definition **??**. We fix (s,x) and the related probability $\mathbb{P}^{s,x}$.
Then $\left(\phi(s\vee\cdot,X_\cdot)-\phi(s,x)-\int_s^{s\vee\cdot}\psi^1(r,X_r)dV_r\right)$ and
$\left(\phi(s\vee\cdot,X_\cdot)-\phi(s,x)-\int_s^{s\vee\cdot}\psi^2(r,X_r)dV_r\right)$, have both M^1,M^2 as cadlag modifications, which are in $\mathcal{H}^2_0$. So $\phi(s\vee\cdot,X_\cdot)$ has two cadlag modifications which are indistinguishable, and by uniqueness of the decomposition of special semi-martingales, $\int_s^{s\vee\cdot}\psi^1(r,X_r)dV_r$ and $\int_s^{s\vee\cdot}\psi^2(r,X_r)dV_r$ are indistinguishable. Since this is true under any $\mathbb{P}^{s,x}$, the two functions are equal up to a zero-potential set because of Proposition **??**. □

Definition 4.23. *Let ϕ be in $\mathcal{D}(\mathfrak{a})$ as in Definition* **??**. *We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process* (**??**) *in $\mathcal{H}^2_0$. This will not create any ambiguity with respect to Notation* **??**. *We can also define without ambiguity the operator*

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L^0_X \\ \phi & \longmapsto & \psi. \end{array}$$

At this point we can say that $\mathfrak{a}$ **extends** a in the following sense. If ϕ is in $\mathcal{D}(a)$ and such that $M[\phi]^{s,x}$ is square integrable for all $(s,x)\in[0,T]\times E$, then comparing Definitions **??** and **??**, we see that ϕ belongs to $\mathcal{D}(\mathfrak{a})$ and that $a(\phi)$ belongs to the class $\mathfrak{a}(\phi)$.

We now want to extend the square field operator $\Gamma(\cdot,\cdot)$ to $\mathcal{D}(\mathfrak{a})\times\mathcal{D}(\mathfrak{a})$.

Proposition 4.24. *Let ϕ and ψ be in $\mathcal{D}(\mathfrak{a})$, there exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T]\times E,\mathbb{R})$ which we will denote $\mathfrak{G}(\phi,\psi)$ such that under any $\mathbb{P}^{s,x}$, $\langle M[\phi]^{s,x},M[\psi]^{s,x}\rangle=\int_s^{\cdot}\mathfrak{G}(\phi,\psi)(r,X_r)dV_r$ on $[s,T]$, up to indistinguishability.*

Proof. Let ϕ and ψ be in $\mathcal{D}(\mathfrak{a})$, introduced in Definition **??**. By Remark **??**, there are square integrable MAFs $M[\phi]$ and $M[\psi]$ defined by
$M[\phi]^t_u=\phi(u,X_u)-\phi(t,X_t)-\int_t^u\mathfrak{a}(\phi)(r,X_r)dV_r$ and
$M[\psi]^t_u=\psi(u,X_u)-\psi(t,X_t)-\int_t^u\mathfrak{a}(\psi)(r,X_r)dV_r$, which admit, for any $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$, respectively $M[\psi]^{s,x}$ as cadlag versions under $\mathbb{P}^{s,x}$. The notion of square integrable MAF is introduced in Section **??** of the Appendix.
The existence of $\mathfrak{G}(\phi,\psi)$ therefore derives from Corollary **??**. By Proposition **??** that function is determined up to a zero-potential set. □

Remark 4.25. *$\mathfrak{G}$ therefore defines a bilinear operator from $\mathcal{D}(\mathfrak{a})\times\mathcal{D}(\mathfrak{a})$ to L^0_X which extends Γ in the following sense. If ϕ,ψ are in $\mathcal{D}(a)$ and such that $M[\phi]^{s,x}$, $M[\psi]^{s,x}$ are square integrable for all $(s,x)\in[0,T]\times E$, then ϕ,ψ belong to $\mathcal{D}(\mathfrak{a})$. Since, by Proposition* **??**, *under $\mathbb{P}^{s,x}$, on $[s,T]$, we have $\langle M[\phi]^{s,x},M[\psi]^{s,x}\rangle=\int_s^{\cdot}\Gamma(\phi,\psi)(r,X_r)dV_r$, then it is clear that $\Gamma(\phi,\psi)$ is in the class $\mathfrak{G}(\phi,\psi)$.*

Definition 4.26. *We will call $\mathfrak{G}$ the* ***extended square field operator***.

5 Pseudo-PDEs and associated Forward BSDEs with no driving martingale

In this section, we still consider $T \in \mathbb{R}_+^*$, a Polish space E and the associated canonical space $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$, see Definition **??**. We also consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition function measurable in time (see Definitions **??** and **??**) which solves a well-posed martingale problem associated to a triplet $(\mathcal{D}(a), a, V)$, see Definition **??** and Hypothesis **??**.

We will investigate here a specific type of BSDE with no driving martingale $BSDE(\xi, \hat{f}, V)$ which we will call **of forward type**, or **forward** BSDE, in the following sense.

1. The process V will be the (deterministic) function V introduced in Definition **??**;
2. the final condition ξ will only depend on the final value of the canonical process X_T;
3. the randomness of the driver $\hat{f}$ at time t will only appear via the value at time t of the forward process X. Given a function $f : [0,T] \times E \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we will set $\hat{f}(t, \omega, y, z) = f(t, X_t(\omega), y, z)$ for $t \in [0,T], \omega \in \Omega, y, z \in \mathbb{R}$.

That BSDE will be connected with the deterministic problem below.

Definition 5.1. *Let us consider the following data.*

1. *A measurable final condition $g \in \mathcal{B}(E, \mathbb{R})$;*
2. *a measurable nonlinear function $f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*

*We will call **Pseudo-Partial Differential Equation** (in short Pseudo-PDE) the following equation with final condition.*

$$\left\{ \begin{array}{rcll} a(u)(t,x) + f\left(t, x, u(t,x), \sqrt{\Gamma(u,u)(t,x)}\right) & = & 0 & \text{on } [0,T] \times E \\ u(T,\cdot) & = & g. & \end{array} \right. \tag{5.1}$$

*We will say that u is a **classical solution** of the Pseudo-PDE if it belongs to $\mathcal{D}(a)$ and verifies (**??**).*

Notation 5.2. *Equation (**??**) will be denoted $Pseudo-PDE(f,g)$.*

To be able to perform the connection between forward BSDEs and $Pseudo-PDE(f,g)$, we will assume some generalized moments conditions on X, and some growth conditions on the functions (f,g). Those will be related to two functions $\zeta, \eta \in \mathcal{B}(E, \mathbb{R}_+)$.

Hypothesis 5.3. *The Markov class will be said **to verify** $H^{mom}(\zeta,\eta)$ if*

1. *for any* $(s,x)\in[0,T]\times E$, $\mathbb{E}^{s,x}[\zeta^2(X_T)]$ *is finite;*

2. *for any* $(s,x)\in[0,T]\times E$, $\mathbb{E}^{s,x}\left[\int_0^T \eta^2(X_r)dV_r\right]$ *is finite.*

If $(E,\|\cdot\|)$ *is a separable Banach space, it is often useful to choose* $\zeta: x\mapsto \|x\|^p$, $\eta: x\mapsto\|x\|^q$ *for some* $p,q\in\mathbb{R}_+$, *see [?]. In that context we will write* $H^{mom}(p,q)$ *instead of* $H^{mom}(\zeta,\eta)$.

Hypothesis 5.4. *A couple of functions* $f\in\mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and* $g\in\mathcal{B}(E,\mathbb{R})$ *will be said* ***to verify*** $H(\zeta,\eta)$ *if there exist positive constants* K^Y,K^Z,C,C' *such that*

1. $\forall x: \quad |g(x)|\leq C(1+\zeta(x))$;

2. $\forall(t,x,y,y',z): \quad |f(t,x,y,z)-f(t,x,y',z)|\leq K^Y|y-y'|$;

3. $\forall(t,x,y,z,z'): \quad |f(t,x,y,z)-f(t,x,y,z')|\leq K^Z|z-z'|$;

4. $\forall(t,x): \quad |f(t,x,0,0)|\leq C'(1+\eta(x))$.

If $(E,\|\cdot\|)$ *is a separable Banach space and* $\zeta: x\mapsto\|x\|^p$, $\eta: x\mapsto\|x\|^q$ *for some* $p,q\in\mathbb{R}_+$, *we will write* $H(p,q)$ *instead of* $H(\zeta,\eta)$.

To the equation $Pseudo-PDE(f,g)$, we will associate the family of BSDEs with no driving martingale indexed by $(s,x)\in[0,T]\times E$ and defined on the interval $[0,T]$ and in the stochastic basis $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x}\right)$, given by

$$Y^{s,x}_t = g(X_T)+\int_t^T f\left(r,X_r,Y^{s,x}_r,\sqrt{\frac{d\langle M^{s,x}\rangle}{dV}(r)}\right)dV_r-(M^{s,x}_T-M^{s,x}_t). \quad (5.2)$$

Notation 5.5. *Equation* (**??**) *will be denoted* $FBSDE^{s,x}(f,g)$.

Remark 5.6. .

1. *If there exist* $\zeta,\eta\in\mathcal{B}(E,\mathbb{R}_+)$ *such that the Markov class verifies* $H^{mom}(\zeta,\eta)$ *and such that* (f,g) *verifies* $H(\zeta,\eta)$, *then Hypothesis* **??** *is verified for* (**??**). *By Theorem* **??**, *for any* (s,x), $FBSDE^{s,x}(f,g)$ *has a unique solution, in the sense of Definition* **??**.

2. *Even if the underlying process* X *admits no generalized moments, given a couple* (f,g) *such that* $f(\cdot,\cdot,0,0)$ *and* g *are bounded, the considerations of this section still apply. In particular the connection between the* $FBSDE^{s,x}(f,g)$ *and the corresponding* $Pseudo-PDE(f,g)$ *still exists.*

For the rest of this section, the positive functions ζ,η and the functions (f,g) appearing in $Pseudo-PDE(f,g)$ will be fixed, and we will assume that the Markov class verifies $H^{mom}(\zeta,\eta)$ and that (f,g) verify $H(\zeta,\eta)$.

Notation 5.7. *From now on,* $(Y^{s,x},M^{s,x})$ *will always denote the (unique) solution of* $FBSDE^{s,x}(f,g)$.

Remark 5.8. *Let $(s,x) \in [0,T] \times E$ be fixed. We know (see Proposition* **??***) that if $t < s$, $\mathcal{F}_t$ is $\mathbb{P}^{s,x}$-trivial. So since $Y^{s,x}$ and $M^{s,x}$ are adapted, they are deterministic on $[0,s[$. Moreover since $M^{s,x}$ belongs to $\mathcal{H}^2_0$, then it is equal to zero on $[0,s[$.*
We will not be interested in the value of $Y^{s,x}$ before s. However, we will later show that $Y^{s,x}_s$ is also deterministic. However, already at this stage, it is interesting to realize that on $[0,s]$, $Y^{s,x}$ is almost surely equal to the solution of the deterministic integral equation

$$Y^{s,x}_t = Y^{s,x}_s + \int_s^t f(r,x,Y^{s,x}_r,0)dV_r, \ t \in [0,s].$$

So, on $[0,s]$, $Y^{s,x}$ is almost surely to a deterministic function absolutely continuous with respect to V and solving the ODE (parametrized by x)

$$\frac{dY^{s,x}}{dV}(t) = -f(t,x,Y^{s,x}_t,0)dV_t.$$

The goal of our work is to understand if and how the solutions of equations $FBSDE^{s,x}(f,g)$ produce a solution of $Pseudo-PDE(f,g)$ and reciprocally.

We will start by showing that if $Pseudo-PDE(f,g)$ has a classical solution, then this one provides solutions to the associated $FBSDE^{s,x}(f,g)$.

Proposition 5.9. *We assume that there exists $u \in \mathcal{D}(a)$ such that*

$$\left\{\begin{array}{rcll} a(u)(s,x) + f(s,x,u(s,x),\sqrt{\Gamma(u,u)(s,x)}) &=& 0 & \text{on } [0,T]\times E \\ u(T,\cdot) &=& g, & \end{array}\right. \quad (5.3)$$

and we also assume the existence of a positive $C > 0$ such that for every $(s,x) \in [0,T] \times E$, $\sqrt{\Gamma(u,u)(s,x)} \leq C(1+\eta(x))$.
Then, for any $(s,x) \in [0,T]\times E$, if $M[u]^{s,x}$ is as in Notation **??** *and $(Y^{s,x},M^{s,x})$ is the unique solution of $FBSDE^{s,x}(f,g)$, then $(u(\cdot,X_\cdot), M[u]^{s,x}_\cdot - M[u]^{s,x}_s)$ and $(Y^{s,x},M^{s,x})$ are $\mathbb{P}^{s,x}$-indistinguishable on $[s,T]$.*

Proof. Let (s,x) be fixed. Since $u \in \mathcal{D}(a)$, the martingale problem in the sense of Definition **??** and (**??**) imply that, on $[s,T]$, under $\mathbb{P}^{s,x}$

$$\begin{array}{rl} & u(\cdot,X_\cdot) \\ = & u(T,X_T) - \int_\cdot^T a(u)(r,X_r)dV_r - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot) \\ = & g(X_T) + \int_\cdot^T f\left(r,X_r,u(r,X_r),\sqrt{\Gamma(u,u)(r,X_r)}\right) - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot) \\ = & g(X_T) + \int_\cdot^T f\left(r,X_r,Y_r,\sqrt{\frac{d\langle M[u]^{s,x}\rangle}{dV}(r)}\right)dV_r - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot), \end{array}$$

where the latter equality comes from Proposition **??**. Combining the growth assumption on $\Gamma(u,u)$ and $H^{mom}(\zeta,\eta)$ it follows that

$$\mathbb{E}^{s,x}\left[\langle M[u]^{s,x}\rangle_T\right] = \mathbb{E}^{s,x}\left[\int_s^T \Gamma(u,u)(r,X_r)dV_r\right] < \infty.$$

This means that $M[u]^{s,x} \in \mathcal{H}^2_0$, so by Lemma **??**, we have that $(u(\cdot, X_\cdot), M[u]^{s,x}_\cdot - M[u]^{s,x}_s)$ and $(Y^{s,x}, M^{s,x})$ are indistinguishable on $[s,T]$. □

We now want to adopt the converse point of view, and see what can be done starting with the solutions of the equations $FBSDE^{s,x}(f,g)$.
At this point we aim at showing that there exist Borel functions u and $v \geq 0$ such that for any $(s,x) \in [0,T] \times E$ we have for all $t \in [s,T]$, $Y^{s,x}_t = u(t, X_t)$ $\mathbb{P}^{s,x}$-a.s., and $\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot) \quad dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$.

The next significant result is Theorem **??**. An analogous result exists in the Brownian framework, see e.g. Theorem 4.1 in [**?**]. We start with a lemma.

Lemma 5.10. *Let $\tilde{f} \in \mathcal{B}([0,T] \times E, \mathbb{R})$ be such that for any $(s,x) \in [0,T] \times E$, $\tilde{f}(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$ belongs to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. Let, for any $(s,x) \in [0,T] \times E$, $(\tilde{Y}^{s,x}, \tilde{M}^{s,x})$ be the (unique by Theorem **??** and Remark **??**) solution of*

$$\tilde{Y}^{s,x}_t = g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r - (\tilde{M}^{s,x}_T - \tilde{M}^{s,x}_t), \quad t \in [s,T],$$

in $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$. Then there exist two functions u and $v \geq 0$ in $\mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x) \in [0,T] \times E$

$$\begin{cases} \forall t \in [s,T] : \tilde{Y}^{s,x}_t = u(t, X_t) \quad \mathbb{P}^{s,x}\, a.s. \\ \frac{d\langle \tilde{M}^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot) \quad dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T]. \end{cases}$$

Proof. We set

$$u(s,x) = \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T \tilde{f}\,(r, X_r)\, dV_r\right],$$

which is Borel by Proposition **??** and Lemma **??**. Therefore by (**??**) in Remark **??**, for a fixed $t \in [s,T]$ we have $\mathbb{P}^{s,x}$- a.s.

$$\begin{array}{rcl} u(t, X_t) & = & \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r \middle| \mathcal{F}_t\right] \\ & = & \mathbb{E}^{s,x}\left[\tilde{Y}^{s,x}_t + (\tilde{M}^{s,x}_T - \tilde{M}^{s,x}_t) | \mathcal{F}_t\right] \\ & = & \tilde{Y}^{s,x}_t, \end{array}$$

since $\tilde{M}^{s,x}$ is a martingale and $\tilde{Y}^{s,x}$ is adapted. Then the square integrable MAF (see Section **??**) defined by
$M^t_{t'} := u(t', X_{t'}) - u(t, X_t) + \int_t^{t'} \tilde{f}(r, X_r) dV_r$ has $\tilde{M}^{s,x}$ as cadlag version under $\mathbb{P}^{s,x}$, which guarantees the existence of the function v thanks to Corollary **??** setting $v = \sqrt{k}$. □

We now define the Picard iterations associated to the contraction defining the solution of a BSDE, see Notation **??**.

Notation 5.11. *For a fixed* $(s,x) \in [0,T] \times E$, $\Phi^{s,x}$ *will denote the contraction on* $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ *whose fixed point defines the solution of* $FBSDE^{s,x}(f,g)$*, see Definition* **??**. *In the sequel we will not distinguish between a couple* $(\dot{Y}, M)$ *in* $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ *and* (Y,M)*, where* Y *is the reference cadlag process of* $\dot{Y}$*, according to Definition* **??**.

We then convene the following.

1. $(Y^{0,s,x}, M^{0,s,x}) := (0,0)$;
2. $\forall k \in \mathbb{N}^* : (Y^{k,s,x}, M^{k,s,x}) := \Phi^{s,x}(Y^{k-1,s,x}, M^{k-1,s,x})$,

meaning that for $k \in \mathbb{N}^*$, $(Y^{k,s,x}, M^{k,s,x})$ *is the solution of the BSDE*

$$Y^{k,s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, Y^{k-1,s,x}, \sqrt{\frac{d\langle M^{k-1,s,x}\rangle}{dV}}(r)\right) dV_r - (M^{k,s,x}_T - M^{k,s,x}_\cdot). \tag{5.4}$$

Definition 5.12. *The processes* $(Y^{k,s,x}, M^{k,s,x})$ *will be called the* ***Picard iterations*** *of* $FBSDE^{s,x}(f,g)$

Proposition 5.13. *For each* $k \in \mathbb{N}$*, there exist functions* u_k *and* $v_k \geq 0$ *in* $\mathcal{B}([0,T] \times E, \mathbb{R})$ *such that for every* $(s,x) \in [0,T] \times E$

$$\left\{ \begin{array}{l} \forall t \in [s,T] : Y^{k,s,x}_t = u_k(t, X_t) \quad \mathbb{P}^{s,x} a.s. \\ \frac{d\langle M^{k,s,x}\rangle}{dV} = v_k^2(\cdot, X_\cdot) \quad dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T]. \end{array} \right. \tag{5.5}$$

Proof. We proceed by induction on k. It is clear that $(u_0, v_0) = (0,0)$ verifies the assertion for $k = 0$.
Now let us assume that functions u_{k-1}, v_{k-1} exist, for some integer $k \geq 1$, verifying (**??**) for k replaced with $k-1$.

By Lemma **??**, for every $(s,x) \in [0,T] \times E$, $(Y^{k-1,s,x}, Z^{k-1,s,x}) = (u_{k-1}, v_{k-1})(\cdot, X_\cdot)$ $dV \otimes \mathbb{P}^{s,x}$ a.e. on [s,T]. Therefore by (**??**), on $[s,T]$

$$Y^{k,s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, u_{k-1}(r, X_r), v_{k-1}(r, X_r)\right) dV_r - (M^{k,s,x}_T - M^{k,s,x}_\cdot).$$

For some fixed (s,x), since $\Phi^{s,x}$ maps $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ into itself (see Definition **??**), obviously all the Picard iterations belong to
$L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$. In particular, $Y^{k-1,s,x}$ and $\sqrt{\frac{d\langle M^{k-1,s,x}\rangle}{dV}}$ are in
$\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. So, by recurrence assumption on u_{k-1} and v_{k-1} , it follows that $u_{k-1}(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$ and $v_{k-1}(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. Combining $H^{mom}(\zeta, \eta)$ and the growth condition of f in $H(\zeta, \eta)$, $f(\cdot, X_\cdot, 0, 0)$ also belongs to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. Therefore thanks to the Lipschitz conditions on f assumed in $H(\zeta, \eta)$, $f\left(\cdot, X_\cdot, u_{k-1}(\cdot, X_\cdot), v_{k-1}(\cdot, X_\cdot)\right) \mathbb{1}_{[s,T]}$ is in $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$.

The existence of u_k and v_k now comes from Lemma **??** applied to $\tilde{f} := f(\cdot,\cdot,u_{k-1},v_{k-1})$. This establishes the induction step for a general k and allows to conclude the proof.

□

Remark 5.14. *For any $k \in \mathbb{N}^*$ we have*

1. $u_k \in \mathcal{D}(\mathfrak{a})$;
2. $v_k^2 = \mathfrak{G}(u_k,u_k)$;
3. $\mathfrak{a}(u_k) = -f(\cdot,\cdot,u_{k-1},v_{k-1})$.

Indeed for any $(s,x) \in [0,T]\times E$, under $\mathbb{P}^{s,x}$ for $t \in [s,T]$, we have $u_k(t,X_t) - u_k(s,x) = -\int_s^t f(\cdot,\cdot,u_{k-1},v_{k-1})(r,X_r)dV_r + (M_t^{k,s,x} - M_s^{k,s,x})$ *a.s. and we have* $\frac{d\langle M^{k,s,x}\rangle}{dV} = v_k^2(\cdot,X_\cdot)\ dV \otimes d\mathbb{P}^{s,x}$ *a.e. on* $[s,T]$.
*So from Definition **??** $u_k \in \mathcal{D}(\mathfrak{a})$ and $\mathfrak{a}(u_k) = -f(\cdot,\cdot,u_{k-1},v_{k-1})$ and by Definition **??**, $v_k^2 = \mathfrak{G}(u_k,u_k)$, which shows the statement.*

Remark **??** shows a first link between the BSDE, the martingale problem introduced in Hypothesis **??** and the Pseudo-PDE with extended operators.

Now we intend to pass to the limit in k. For any $(s,x)\in[0,T]\times E$, we have seen in Proposition **??** that $\Phi^{s,x}$ is a contraction in $\left(L^2(dV\otimes d\mathbb{P}^{s,x})\times\mathcal{H}^2_0, \|\cdot\|_\lambda\right)$ for some $\lambda > 0$, so we know that the sequence $(Y^{k,s,x},M^{k,s,x})$ converges to $(Y^{s,x},M^{s,x})$ in this topology.

The proposition below also shows an a.e. corresponding convergence, adapting the techniques of Corollary 2.1 in [?].

Proposition 5.15. *For any $(s,x)\in[0,T]\times E$, $Y^{k,s,x} \underset{k\to\infty}{\longrightarrow} Y^{s,x} \quad dV\otimes d\mathbb{P}^{s,x}$ a.e. and $\sqrt{\frac{d\langle M^{k,s,x}\rangle}{dV}} \underset{k\to\infty}{\longrightarrow} \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}} \quad dV\otimes d\mathbb{P}^{s,x}$ a.e.*

Proof. We fix (s,x) and the associated probability. In this proof, all s,x superscripts are dropped. We set $Z^k = \sqrt{\frac{d\langle M^k\rangle}{dV}}$ and $Z = \sqrt{\frac{d\langle M\rangle}{dV}}$. By Proposition **??**, there exists $\lambda > 0$ such that for any $k\in\mathbb{N}^*$

$$\begin{array}{ll} & \mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Y_r^{k+1}-Y_r^k|^2dV_r + \int_0^T e^{-\lambda V_r}d\langle M^{k+1}-M^k\rangle_r\right] \\ \leq & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Y_r^{k}-Y_r^{k-1}|^2dV_r + \int_0^T e^{-\lambda V_r}d\langle M^{k}-M^{k-1}\rangle_r\right], \end{array}$$

therefore

$$\begin{array}{ll} & \sum_{k\geq 0}\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Y_r^{k+1}-Y_r^k|^2dV_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda V_r}d\langle M^{k+1}-M^k\rangle_r\right] \\ \leq & \sum_{k\geq 0}\frac{1}{2^k}\left(\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Y_r^1|^2dV_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda V_r}d\langle M^1\rangle_r\right]\right) \\ < & \infty. \end{array}$$

We also have thanks to (**??**) that

$$\sum_{k\geq 0}\left(\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Y_r^{k+1}-Y_r^k|^2 dV_r\right]+\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Z_r^{k+1}-Z_r^k|^2 dV_r\right]\right)<\infty.$$

So by Fubini's theorem we have

$$\mathbb{E}\left[\int_0^T e^{-\lambda V_r}\left(\sum_{k\geq 0}(|Y_r^{k+1}-Y_r^k|^2+|Z_r^{k+1}-Z_r^k|^2)\right)dV_r\right]<\infty.$$

Consequently the sum

$$\sum_{k\geq 0}\left(|Y_r^{k+1}(\omega)-Y_r^k(\omega)|^2+|Z_r^{k+1}(\omega)-Z_r^k(\omega)|^2\right)$$

is finite on a set of full $dV\otimes d\mathbb{P}$ measure. So on this set of full measure, the sequence $(Y_t^{k+1}(\omega), Z_t^{k+1}(\omega))$ converges, and the limit is necessarily equal to $(Y_t(\omega), Z_t(\omega))$ $dV\otimes d\mathbb{P}$ a.e. because of the $L^2(dV\otimes d\mathbb{P})$ convergence that we have already established. ☐

Theorem 5.16. *There exist two functions u and $v\geq 0$ in $\mathcal{B}([0,T]\times E,\mathbb{R})$ such that for every $(s,x)\in[0,T]\times E$,*

$$\left\{\begin{array}{l}\forall t\in[s,T]: Y_t^{s,x}=u(t,X_t)\quad \mathbb{P}^{s,x}\ a.s.\\ \frac{d\langle M^{s,x}\rangle}{dV}=v^2(\cdot,X_\cdot)\quad dV\otimes d\mathbb{P}^{s,x}\ a.e.\ on\ [s,T].\end{array}\right. \tag{5.6}$$

Proof. We set $\bar{u}:=\limsup\limits_{k\in\mathbb{N}} u_k$ in the sense that for any $(s,x)\in[0,T]\times E$, $\bar{u}(s,x)=\limsup\limits_{k\in\mathbb{N}} u_k(s,x)$ and $v:=\limsup\limits_{k\in\mathbb{N}} v_k$. $\bar{u}$ and v are Borel functions. We know by Propositions **??**, **??** and Lemma **??** that for every $(s,x)\in[0,T]\times E$

$$\left\{\begin{array}{llll}u_k(\cdot,X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Y^{s,x} & dV\otimes d\mathbb{P}^{s,x}\text{ a.e. on }[s,T]\\ v_k(\cdot,X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Z^{s,x} & dV\otimes d\mathbb{P}^{s,x}\text{ a.e. on }[s,T],\end{array}\right.$$

where $Z^{s,x}:=\sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}$. Therefore, for some fixed $(s,x)\in[0,T]\times E$ and on the set of full $dV\otimes d\mathbb{P}^{s,x}$ measure on which these convergences hold we have

$$\left\{\begin{array}{l}\bar{u}(t,X_t(\omega))=\limsup\limits_{k\in\mathbb{N}} u_k(t,X_t(\omega))=\lim\limits_{k\in\mathbb{N}} u_k(t,X_t(\omega))=Y_t^{s,x}(\omega)\\ v(t,X_t(\omega))=\limsup\limits_{k\in\mathbb{N}} v_k(t,X_t(\omega))=\lim\limits_{k\in\mathbb{N}} v_k(t,X_t(\omega))=Z_t^{s,x}(\omega).\end{array}\right. \tag{5.7}$$

This shows in particular the existence of v and the validity of the second line of (**??**).
It remains to show the existence of u so that the first line of (**??**) holds. Thanks

to the $dV\otimes d\mathbb{P}^{s,x}$ equalities concerning v and $\bar{u}$ stated in (??), under every $\mathbb{P}^{s,x}$ we actually have

$$Y^{s,x}_{\cdot} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, \bar{u}(r,X_r), v(r,X_r)\right) dV_r - (M^{s,x}_T - M^{s,x}_{\cdot}). \quad (5.8)$$

Now (??) can be considered as a BSDE where the driver does not depend on y and z. For any $(s,x)\in[0,T]\times E$, $Y^{s,x}$ and $Z^{s,x}$ belong to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$, then by (??), so do $\bar{u}(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$ and $v(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$. Combining $H^{mom}(\zeta,\eta)$ and the Lipschitz condition on f assumed in $H(\zeta,\eta)$, $f(\cdot,X_\cdot,\bar{u}(\cdot,X_\cdot),v(\cdot,X_\cdot))\mathbb{1}_{[s,T]}$ also belongs to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$. We can therefore apply Lemma ?? to $\tilde{f} = f(\cdot,\cdot,\bar{u},v)$, and conclude on the existence of a Borel function u such that for every $(s,x)\in[0,T]\times E$, $Y^{s,x}$ is on $[s,T]$ a $\mathbb{P}^{s,x}$-version of $u(\cdot,X_\cdot)$.

□

Remark 5.17. *In particular, $Y^{s,x}_s = u(s,x)$ is deterministic and $M^{s,x}_s = Y^{s,x}_s - Y^{s,x}_0 + \int_0^s f\left(r,X_r,Y^{s,x}_r,0\right)dV_r$ is also deterministic and it is therefore equal to 0 since $M^{s,x}\in\mathcal{H}^2_0$, by Remark ??.*

Remark 5.18. *For any $(s,x)\in[0,T]\times E$, the stochastic convergence $(Y^{k,s,x},M^{k,s,x}) \xrightarrow[k\to\infty]{L^2(dV\otimes d\mathbb{P}^{s,x})\times\mathcal{H}^2} (Y^{s,x},M^{s,x})$ now has the functional counterpart*

$$\begin{cases} u_k & \xrightarrow[k\to\infty]{\|\cdot\|_{2,s,x}} & u \\ v_k & \xrightarrow[k\to\infty]{\|\cdot\|_{2,s,x}} & v, \end{cases}$$

which yields

$$\begin{cases} u_k & \xrightarrow[k\to\infty]{L^2_X} & u \\ v_k & \xrightarrow[k\to\infty]{L^2_X} & v, \end{cases}$$

where we recall that the locally convex topological space L^2_X was introduced in Notation ??.

Corollary 5.19. *For any $(s,x)\in[0,T]\times E$ and for any $t\in[s,T]$, the couple of functions (u,v) obtained in Theorem ?? verifies $\mathbb{P}^{s,x}$ a.s.*

$$u(t,X_t) = g(X_T) + \int_t^T f\left(r,X_r,u(r,X_r),v(r,X_r)\right)dV_r - (M^{s,x}_T - M^{s,x}_t),$$

where $M^{s,x}$ denotes the martingale part of the unique solution of $FBSDE^{s,x}(f,g)$.

Proof. The corollary follows from Theorem ?? and Lemma ??. □

We now introduce now a probabilistic notion of solution for $Pseudo{-}PDE(f,g)$.

Definition 5.20. *$u \in \mathcal{D}(\mathfrak{a})$ will be said to solve $Pseudo-PDE(f,g)$ in the* ***martingale sense*** *if*

$$\left\{\begin{array}{rcl} \mathfrak{a}(u) & = & -f(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}) \\ u(T,\cdot) & = & g. \end{array}\right. \tag{5.9}$$

Remark 5.21. *The first equation of (**??**) holds in L^0_X, hence up to a zero potential set. The second one is a pointwise equality.*

Theorem 5.22. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a Markov class associated to a transition function measurable in time (see Definitions **??** and **??**) which fulfills Hypothesis **??**, i.e. it is a solution of a well-posed martingale problem associated with the triplet $(\mathcal{D}(a),a,V)$. Moreover we suppose Hypothesis $H^{mom}(\zeta,\eta)$ for some positive ζ,η. Let $\mathfrak{a}$, $\mathfrak{G}$ be the extended operators defined in Definitions **??** and **??**. Let (f,g) be a couple verifying $H(\zeta,\eta)$. Let (u,v) be the functions defined in Theorem **??**.*

Then $u \in \mathcal{D}(\mathfrak{a})$, $v^2 = \mathfrak{G}(u,u)$ and u solves $Pseudo-PDE(f,g)$ in the ***martingale sense****.*

Proof. For any $(s,x) \in [0,T]\times E$, by Corollary **??**, for $t \in [s,T]$, we have

$$u(t,X_t)-u(s,x) = -\int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dV_r+(M^{s,x}_t-M^{s,x}_s) \quad \mathbb{P}^{s,x} \text{ a.s.}$$

so by Definition **??**, $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and
$M[u]^{s,x} = M^{s,x}_\cdot - M^{s,x}_s$.
Moreover by Theorem **??** we have $\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot,X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$, so by Proposition **??** it follows $v^2 = \mathfrak{G}(u,u)$ and therefore, the L^2_X equality $\mathfrak{a}(u) = -f(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)})$, which establishes the first line of (**??**).
Concerning the second line, we have for any $x \in E$,
$u(T,x) = u(T,X_T) = g(X_T) = g(x)$ $\mathbb{P}^{T,x}$ a.s. so $u(T,\cdot) = g$ (in the deterministic pointwise sense). □

Remark 5.23. *The equality $\mathfrak{a}(u) = -f(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)})$ takes place in L^2_X.*

So in the most general setup, the function u constructed by the $FBSDE^{s,x}(f,g)$ solves $Pseudo-PDE(f,g)$ in the martingale sense. However, a priori, the extended operators have no analytical meaning.

In the companion paper [?] we will show that the couple (u,v) also solves $Pseudo-PDE(f,g)$ in more specific analytical way. However, the notion of Pseudo-PDE related to extended operators stated above has two interesting features. First, any classical solution (see Corollary **??** below) of $Pseudo-PDE(f,g)$ is also a solution in the martingale sense, secondly the solution of (**??**) is unique, see Theorem **??** below.

Corollary 5.24. *We assume that there exists $u' \in \mathcal{D}(a)$ such that*

$$\left\{\begin{array}{rcll} a(u')(s,x)+f(s,x,u'(s,x),\sqrt{\Gamma(u',u')(s,x)}) & = & 0 & \text{on } [0,T]\times E \\ u'(T,\cdot) & = & g. & \end{array}\right.$$

We also suppose the existence of a positive $C > 0$ such that for every $(s,x) \in [0,T] \times E$, $\sqrt{\Gamma(u',u')(s,x)} \leq C(1+\eta(x))$. Then $u' \in \mathcal{D}(\mathfrak{a})$ and u' solves $Pseudo-PDE(f,g)$ in the martingale sense.

Proof. We recall that for a given (s,x), $(Y^{s,x}, M^{s,x})$ denotes the (unique) solution of $FBSDE^{s,x}(f,g)$. By Proposition **??** we know that for any $(s,x) \in [0,T] \times E$, $(u'(\cdot, X_\cdot), M[u']^{s,x}_\cdot - M[u']^{s,x}_s)$ and $(Y^{s,x}, M^{s,x})$ are $\mathbb{P}^{s,x}$-indistinguishable on $[s,T]$, so by Corollary **??** and Definition **??**, it is clear that $u' \in \mathcal{D}(\mathfrak{a})$ with $\mathfrak{a}(u') = -f(\cdot,\cdot,u,v)$, where u, v are the functions built in Theorem **??**. We know by definition of u that for any $(s,x) \in [0,T] \times E$, $Y^{s,x}$ is also a $\mathbb{P}^{s,x}$-version of $u(\cdot, X_\cdot)$ on $[s,T]$. So $u'(\cdot, X_\cdot)$ and $u(\cdot, X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications on $[s,T]$; by Lemma **??** and Proposition **??**, $u = u'$ up to a zero potential set. Moreover by Proposition **??**, under any $\mathbb{P}^{s,x}$,

$$\int_s^\cdot \Gamma(u',u')(r,X_r)dV_r = \langle M[u']^{s,x} \rangle = \langle M^{s,x} \rangle = \int_s^\cdot v^2(r,X_r)dV_r,$$

so $v^2 = \Gamma(u',u')$ up to a zero potential set, and u' solves $Pseudo-PDE(f,g)$ in the martingale sense. □

Theorem 5.25. *The problem* (**??**) *admits a unique solution.*

Proof. Existence has been the object of Theorem **??**.

Let u and u' be two elements of $\mathcal{D}(\mathfrak{a})$ solving (**??**) and let $(s,x) \in [0,T] \times E$ be fixed. By Definition **??** and Remark **??**, the process $u(\cdot, X_\cdot)$ (respectively $u'(\cdot, X_\cdot)$) under $\mathbb{P}^{s,x}$ admits a cadlag modification $U^{s,x}$ (respectively $U'^{s,x}$) on $[s,T]$, which is a special semi-martingale with decomposition

$$\begin{array}{rcl} U^{s,x} & = & u(s,x) + \int_s^\cdot \mathfrak{a}(u)(r,X_r)dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r, X_r, u(r,X_r), \sqrt{\mathfrak{G}(u,u)}(r,X_r)\right) dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r, X_r, U^{s,x}, \sqrt{\mathfrak{G}(u,u)}(r,X_r)\right) dV_r + M[u]^{s,x}, \end{array} \tag{5.10}$$

where the third equality of (**??**) comes from Lemma **??**. Of course we have similarly $U'^{s,x} = u'(s,x) - \int_s^\cdot f\left(r, X_r, U'^{s,x}, \sqrt{\mathfrak{G}(u',u')}(r,X_r)\right) dV_r + M[u']^{s,x}$). $M[u]^{s,x}$ and $M[u']^{s,x}$ were introduced at Definition **??**: they belong to $\mathcal{H}^2_0$, and by Proposition **??**, $\langle M[u]^{s,x} \rangle = \int_s^\cdot \mathfrak{G}(u,u)(r,X_r)dV_r$ (respectively $\langle M[u']^{s,x} \rangle = \int_s^\cdot \mathfrak{G}(u',u')(r,X_r)dV_r$). Moreover since $u(T,\cdot) = u'(T,\cdot) = g$, then $u(T,X_T) = u'(T,X_T) = g(X_T)$ a.s. then the couples $(U^{s,x}, M[u]^{s,x})$ and $(U'^{s,x}, M[u']^{s,x})$ both verify the equation

$$Y_\cdot = g(X_T) + \int_\cdot^T f\left(r, X_r, Y_r, \sqrt{\frac{d\langle M \rangle}{dV}(r)}\right) dV_r - (M_T - M_\cdot) \tag{5.11}$$

on $[s,T]$.

Even though we do not have a priori information on the square integrability of

$U^{s,x}$ and $U'^{s,x}$, we know that $M[u]^{s,x}$ and $M[u']^{s,x}$ are in $\mathcal{H}^2$ and equal to zero at time s, and that $U^{s,x}_s$ and $U'^{s,x}_s$ are deterministic so L^2. By Lemma **??** and the fact that $(U^{s,x}, M[u]^{s,x})$ and $(U'^{s,x}, M[u']^{s,x})$ solve the BSDE in the weaker sense (**??**), it is sufficient to conclude that both solve $FBSDE^{s,x}(f,g)$ on $[s,T]$. By Theorem **??** and Remark **??** the two couples are $\mathbb{P}^{s,x}$-indistinguishable. This implies that $u(\cdot, X_\cdot)$ and $u'(\cdot, X_\cdot)$ are modifications one of the other on $[s,T]$, and by Lemma **??** that
$\int_s^\cdot u(r,X_r)dV_r = \int_s^\cdot u'(r,X_r)dV_r$ in the sense of indistinguishability. Since this is true under any $\mathbb{P}^{s,x}$, then by Proposition **??**, u and u' are equal up to a zero-potential set. So they correspond to the same element of $\mathcal{D}(\mathfrak{a})$. □

We now conclude this section by showing that if the function u built with the BSDEs is in the initial domain $\mathcal{D}(a)$ then it is a classical solution of $Pseudo-PDE(f,g)$, up to a zero-potential set. We still assume that for some positive ζ, η, the Markov class verifies $H^{mom}(\zeta,\eta)$ and that the functions (f,g) appearing in (**??**) verify $H(\zeta,\eta)$.

Theorem 5.26. *Let (u,v) be the functions built via the solutions of $FBSDE^{s,x}(f,g)$ by Theorem **??**. If $u \in \mathcal{D}(a)$ then u is a classical solution of*

$$\left\{\begin{array}{rcll} a(u)(s,x) + f(s,x,u(s,x),\sqrt{\Gamma(u,u)(s,x)}) &=& 0 & \text{on } [0,T]\times E \\ u(T,\cdot) &=& g, & \end{array}\right.$$

up to a zero-potential set, meaning that the first equality holds up to a set of zero potential. Moreover $v = \sqrt{\Gamma(u,u)}$ up to a zero potential set.

Proof. We fix $(s,x) \in [0,T]\times E$ and the corresponding probability $\mathbb{P}^{s,x}$. We denote $(Y^{s,x}, M^{s,x})$ the unique solution of $FBSDE^{s,x}(f,g)$. Corollary **??** implies, for any $t \in [s,T]$ the a.s. equality

$$u(t,X_t) = u(s,x) + \int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dV_r + M^{s,x}_t.$$

The martingale problem related to Definition **??** gives, on $[s,T]$,

$$u(\cdot,X_\cdot) = u(s,x) + \int_s^\cdot a(u)(r,X_r)dV_r + (M[u]^{s,x}_\cdot - M[u]^{s,x}_s).$$

So on $[s,T]$, the processes $\int_s^\cdot f(r,X_r,u(r,X_r),v(r,X_r))dV_r + M^{s,x}$ and $\int_s^\cdot a(u)(r,X_r)dV_r + (M[u]^{s,x}_\cdot - M[u]^{s,x}_s)$ are modifications of each other. Since they are cadlag, they are indistinguishable. By uniqueness of the decomposition of a special semimartingale, it yields that $\int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dV_r$ is indistinguishable from $\int_s^t a(u)(r,X_r)dV_r$ and $M^{s,x}$ is indistinguishable from $M[u]^{s,x} - M[u]^{s,x}_s$. Since this holds under any $\mathbb{P}^{s,x}$, by Proposition **??**, $a(u) = f(\cdot,\cdot,u,v)$ up to a zero-potential set.
Moreover, by evaluating the angular brackets of $M^{s,x}$ and $M^{s,x}[u]$, by Proposition **??** and Theorem **??**, then, under any $\mathbb{P}^{s,x}$, $\int_s^\cdot v^2(r,X_r)dV_r$ and $\int_s^\cdot \Gamma(u,u)(r,X_r)dV_r$

are indistinguishable. Therefore by Proposition **??**, $v^2 = \Gamma(u,u)$ up to a zero-potential. So $a(u) + f\left(\cdot,\cdot,u,\sqrt{\Gamma(u,u)}\right) = 0$ up to a zero-potential set.
We also have under every $\mathbb{P}^{T,x}$ that $u(T,x) = u(T,X_T) = g(X_T) = g(x)$ a.s. so $u(T,\cdot) = g$. □

6 Upcoming applications

In the companion paper [**?**], several examples shall be studied. The examples below fit in the framework of Section **??**.

6.1 Jump Diffusions

The first class of processes that falls into the abstract set up which we studied are Markovian jump diffusions, which include continuous diffusions. Such processes may be defined as solving Martingale problems with operators of type

$$\begin{array}{rcl} a(\phi) & = & \partial_t\phi + \frac{1}{2}\sum\limits_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_ix_j}\phi + \sum\limits_{i\leq d}\mu_i\partial_{x_i}\phi \\ & & + \int\left(\phi(\cdot,\cdot+y) - \phi(\cdot,y) - \frac{1}{1+\|y\|^2}\sum\limits_{i\leq d}y_i\partial_{x_i}\phi\right)K(\cdot,\cdot,dy). \end{array}$$

On the domain $\mathcal{D}(a) = \mathcal{C}^{1,2}_b([0,T]\times\mathbb{R}^d)$, the set of real continuous bounded functions on $[0,T]\times\mathbb{R}^d$ which are differentiable in the first variable with bounded continuous derivative, and twice differentiable in the second variable with bounded continuous derivatives.
Here μ is a Borel function with values in $\mathbb{R}^d$ and σ is a Borel function with values in $M_d(\mathbb{R})$, the set of matrices of size d. K is a Lévy kernel, meaning that for every $(t,x)\in[0,T]\times\mathbb{R}^d$, $K(t,x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d\backslash\{0\}$ verifying $\int\frac{\|y\|^2}{1+\|y\|^2}K(t,x,dy) < \infty$ and for every Borel set $A\in\mathcal{B}(\mathbb{R}^d\backslash\{0\})$,
$(t,x)\longmapsto\int_A\frac{\|y\|^2}{1+\|y\|^2}K(t,x,dy)$ is Borel.

Martingale problems associated to such operators were first studied by D.W. Stroock in [**?**]. Its Theorem 4.3 states the following.

Theorem 6.1. *If μ is bounded Borel, σ is bounded continuous and takes values in the set of invertible matrices $Gl_d(\mathbb{R})$ of size d, and if for any $A\in\mathcal{B}(\mathbb{R}^d\backslash\{0\})$, $(t,x)\longmapsto\int_A\frac{y}{1+\|y\|^2}K(t,x,dy)$ is bounded continuous, then for every (s,x) there exists a unique probability $\mathbb{P}^{s,x}$ on the canonical space (defined in **??**) such that $\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot a(\phi)(r,X_r)dr$ is a local martingale for any $\phi\in\mathcal{D}(a)$ and $\mathbb{P}^{s,x}(\forall t\in[0,s], X_t = x) = 1$. Moreover the family $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ defines a Markov class which is Feller continuous (in the sense of Definition **??**).*

The last point was the object of the first remark after Theorem 4.3 in [**?**].

In this context, $\mathcal{D}(a)$ is an algebra and for ϕ, ψ in $\mathcal{D}(a)$, the square field operator is given by

$$\Gamma(\phi,\psi) = \sum_{i,j\le d} (\sigma\sigma^\intercal)_{i,j}\partial_{x_i}\phi\partial_{x_j}\psi + \int (\phi(\cdot,\cdot+y)-\phi)(\psi(\cdot,\cdot+y)-\psi)K(\cdot,\cdot,dy).$$

If the underlying process is the d-dimensional Brownian motion (meaning $\mu = 0$, $\sigma = Id$, $K = 0$) then for any $\phi \in \mathcal{D}(a)$ we have $\Gamma(\phi,\phi) = \|\nabla\phi\|^2$, which is at the origin of the terminology "square field operator".
We will provide mild solutions (in some cases, viscosity solutions) to Pseudo PDEs of type

$$\begin{cases} \partial_t u + Lu + f\left(\cdot,\cdot,u,\sqrt{\Gamma(u,u)}\right) = 0 & \text{on } [0,T]\times\mathbb{R}^d \\ u(T,\cdot) = g, \end{cases}$$

where Lu denotes

$$\frac{1}{2}Tr[\nabla^2 u\sigma\sigma^\intercal] + \langle\mu,\nabla u\rangle + \int \left(u(\cdot,\cdot+y) - u(\cdot,y) - \frac{1}{1+\|y\|^2}\langle y,\nabla u\rangle\right)K(\cdot,\cdot,dy),$$

where μ,σ,K verify the conditions of Theorem **??** and f,g verify $H(\zeta,\eta)$ for some ζ,η.

6.2 α-stable Lévy processes

D. Stroock only studied jump diffusions with non degenerate diffusion part, but one can also be interested in pure-jumps processes. We shall therefore also study a typical example of pure jump process, the α-stable Lévy process. The associated operator will involve the fractional Laplace $(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in]0,2[$, see Chapter 3 in [**?**] for an introduction. Let $d \in \mathbb{N}^*$ and $\mathcal{C}^2_b(\mathbb{R}^d)$ denote the set of twice continuously differentiable functions which are bounded with bounded derivatives. On $\mathcal{C}^2_b(\mathbb{R}^d)$ this operator can be defined by

$$(-\Delta)^{\frac{\alpha}{2}}(\phi)(x) = c_{\alpha,d}PV\int_{\mathbb{R}^d}\frac{(\phi(x+y)-\phi(x))}{\|y\|^{d+\alpha}}dy,$$

where $c_{\alpha,d}$ is a constant only depending on α and d and PV is a notation for principal value. On the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, this operator can also be defined as a Fourier multiplier

$$(-\Delta)^{\frac{\alpha}{2}}(f) = \mathcal{F}^{-1}(\|\xi\|^\alpha\mathcal{F}(\phi)),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ the invert Fourier transform, see Proposition 3.3 in [**?**].

Theorem 1.2 in [**?**] states that a certain martingale problem associated to $-(-\Delta)^{\frac{\alpha}{2}}$ is well posed. Its solution defines a Markov process often called the **α-stable (rotationally invariant) Lévy process**.

We shall therefore be interested in the operator $a : \phi \longmapsto \partial_t\phi - (-\Delta)^{\frac{\alpha}{2}}\phi$. The associated square field operator Γ^α will be given by the formula

$$\Gamma^\alpha(\phi,\psi)(t,x) = c_{\alpha,d} PV \int \frac{(\phi(t,x+y)-\phi(t,x))(\psi(t,x+y)-\psi(t,x))}{\|y\|^{d+\alpha}} dy,$$

and we will provide mild solutions (in some cases, viscosity solutions) to Pseudo PDEs of type

$$\left\{ \begin{array}{l} \partial_t u - (-\Delta)^{\frac{\alpha}{2}} u + f\left(\cdot,\cdot,u,\left(PV\int \frac{(u(\cdot,\cdot+y)-u)^2}{\|y\|^{d+\alpha}}dy\right)^{\frac{1}{2}}\right) = 0 \quad \text{on } [0,T]\times\mathbb{R}^d \\ u(T,\cdot) = g. \end{array}\right.$$

6.3 Diffusions with distributional drift

Finally, our set-up goes beyond Markovian semi-martingales, so we shall study an example of Markovian process which is a Dirichlet process, and not necessarily a semi-martingale.

Concerning this example, we will use the formalism and results obtained by [**?**], see also [**?**, **?**] and references therein for more recent developments.

Let $b,\sigma : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $\sigma > 0$. By a mollifier, we intent a function in the class of Schwartz $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)dx = 1$. We denote

$$\Phi_n(x) = n\Phi(nx),\ \sigma_n^2 = \sigma^2 * \Phi_n,\ b_n = b * \Phi_n.$$

We then define on $\mathcal{C}^2(\mathbb{R})$ the sequence of operators $L_n g = \frac{\sigma_n^2}{2}g'' + b_n' g'$.
$f \in \mathcal{C}^1(\mathbb{R})$ is said to be a solution to $Lf = \dot{l}$ where $\dot{l} \in \mathcal{C}^0$, if for any mollifier Φ, there are sequences (f_n) in $\mathcal{C}^2$, $(\dot{l}_n)$ in $\mathcal{C}^0$ such that

$$L_n f_n = (\dot{l}_n),\ f_n \xrightarrow{\mathcal{C}^1} f,\ \dot{l}_n \xrightarrow{\mathcal{C}^0} \dot{l}.$$

Under some conditions on b and σ, there exists a unique solution to

$$Lf(x) = \dot{l},\ f \in \mathcal{C}^1,\ f(0) = x_0,\ f'(0) = x_1,$$

for any $\dot{l} \in \mathcal{C}^0$, $x_0, x_1 \in \mathbb{R}$. $\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1$ such that there exists some $\dot{l} \in \mathcal{C}^0$ with $Lf = \dot{l}$ and can be shown to be an algebra.

In [**?**] the authors show that there exists a unique solution to a martingale problem associated to $(\mathcal{D}_L, L)$, and that this solution defines a Markovian Dirichlet process.

We will be interested in some operator $a : \phi \longmapsto \partial_t\phi + L\phi$ with associated square field operator $\Gamma : (\phi,\psi) \longmapsto \sigma^2\partial_x\phi\partial_x\psi$, and we shall provide mild solutions (in some cases, viscosity solutions) to semi-linear parabolic PDEs with distributional drift of type

$$\left\{ \begin{array}{l} \partial_t u + Lu + f(\cdot,\cdot,u,\sigma|\partial_x u|) = 0 \quad \text{on } [0,T]\times\mathbb{R} \\ u(T,\cdot) = g. \end{array}\right.$$

We mention some notes about bibliography. In the case of dimension 1 [?] considered BSDEs with terminal condition in connection with a PDE of elliptic type. On the other hand in dimension d, the recent preprint [?] considers from a different point of view a class of BSDEs involving a distributional drift.

Appendices

A Markov classes

We believe that the content of this Appendix is very close to standard material in the theory of Markov processes. However, most of the Markov processes literature concerns time-homogeneous processes, and most of the articles or books about time-dependent processes do not emphasize the measurability issues which we will extensively use.
So, for the comfort of the reader, we have decided to recall some well-known definitions and to give the exact mathematical background that we will use when defining a "non-homogeneous Markov process" and to prove (or possibly re-prove) some basic properties which are close to those in classical textbooks. Most of the definitions that we will introduce in this section are inspired from chapter VI of [?], which considers a more general setting, not necessarily useful for our purposes.

The first definition refers to the canonical space that one can find in [?], see paragraph 12.63.

Notation A.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space), and $\mathcal{B}(E)$ its Borel σ-field. E will be called the* ***state space****.*

We consider $T \in \mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}(E)$ the Skorokhod space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T (e.g. cadlag). For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.

On the measurable space $(\Omega, \mathcal{F})$, we introduce the measurable ***canonical process***

$$X : \begin{array}{ccc} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E, \mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$.

$\big(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ *will be called the* ***canonical space*** *(associated to* T *and* E*).*

We recall that since E is Polish, then $\mathbb{D}(E)$ can be equipped with a Skorokhod distance which makes it a Polish metric space (see Theorem 5.6 in chapter 3 of [**?**], and for which the Borel σ-field is $\mathcal{F}$ (see Proposition 7.1 in chapter 3 of [**?**]). This in particular implies that $\mathcal{F}$ is separable, as the Borel σ-field of a separable metric space.

Remark A.2. *Previous definitions and all the results of this Appendix, extend to a time interval equal to* $\mathbb{R}_+$*.*

Definition A.3. *The function*

$$p: \begin{array}{rcl} (s,x,t,A) & \longmapsto & p(s,x,t,A) \\ {[0,T]\times E\times[0,T]\times\mathcal{B}(E)} & \longrightarrow & [0,1], \end{array}$$

will be called ***transition function*** *if, for any* s,t *in* $[0,T]$*,* $x\in E$*,* $A\in\mathcal{B}(E)$*, it verifies*

1. $x\mapsto p(s,x,t,A)$ *is Borel,*
2. $B\mapsto p(s,x,t,B)$ *is a probability measure on* $(E,\mathcal{B}(E))$*,*
3. *if* $t\leq s$ *then* $p(s,x,t,A)=\mathbb{1}_A(x)$*,*
4. *if* $s<t$*, for any* $u>t$*:*

$$\int_E p(s,x,t,dy)p(t,y,u,A) = p(s,x,u,A).$$

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition A.4. *A transition function* p *for which the first item is reinforced supposing that* $(s,x)\longmapsto p(s,x,t,A)$ *is Borel for any* t,A*, will be said* ***measurable in time****.*

Remark A.5. *Let* p *be a transition function which is measurable in time. By approximation by step functions, one can easily show that, for any Borel function* ϕ *from* E *to* $\mathbb{R}$ *then* $(s,x)\mapsto\int\phi(y)p(s,x,t,dy)$ *is Borel, provided previous integral makes sense. In this paper we will only consider transition functions which are measurable in time.*

Definition A.6. *A* ***canonical Markov class*** *associated to a transition function* p *is a set of probability measures* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *defined on the measurable space* $(\Omega,\mathcal{F})$ *and verifying for any* $t\in[0,T]$ *and* $A\in\mathcal{B}(E)$

$$\mathbb{P}^{s,x}(X_t\in A) = p(s,x,t,A), \tag{A.1}$$

and for any $s\leq t\leq u$

$$\mathbb{P}^{s,x}(X_u\in A|\mathcal{F}_t) = p(t,X_t,u,A) \quad \mathbb{P}^{s,x}\ a.s. \tag{A.2}$$

(**??**) *will be called the* ***Markov property****.*

Remark A.7. *By approximation by step functions, it follows that for any Borel function ϕ, $x \in E$ and $0 \le s \le t \le u \le T$*

$$
\begin{aligned}
\mathbb{E}^{s,x}[\phi(X_t)] &= p(s,x,u,\phi) := \int_E p(s,x,t,dy)\phi(y), \\
\mathbb{E}^{s,x}[\phi(X_u)|\mathcal{F}_t] &= p(t,X_t,u,\phi) = \int_E p(t,X_t,u,dy)\phi(y),
\end{aligned}
\tag{A.3}
$$

provided previous integrals make sense. Moreover, from (**??**) *and Definition* **??** *it follows that for any $(s,x) \in [0,T] \times E$ and $t \le s$*

$$\mathbb{P}^{s,x}(X_t = x) = 1. \tag{A.4}$$

Since X is cadlag, we even have $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$.

Proposition A.8. *For any event $F \in \mathcal{F}$, $(s,x) \longmapsto \mathbb{P}^{s,x}(F)$ is Borel. For any random variable Z, if the function $(s,x) \longmapsto \mathbb{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty,\infty]$), then it is Borel.*

Proof. We start by assuming that F is of the form $\bigcap\limits_{i \le n} \{X_{t_i} \in A_i\}$, where $n \in \mathbb{N}^*$, $0 \le t_1 < \cdots < t_n \le T$ and $A_1, \cdots, A_n$ are Borel sets of E, and we denote by Π the set of such events.
In this proof we will make use of monotone class arguments. See for instance Section 4.3 in [**?**] for the definitions of π-systems and λ-systems and for this version of the monotone class theorem, also called the Dynkin's lemma.
We remark that Π is a π-system (see Definition 4.9 in [**?**]) generating $\mathcal{F}$. For such events, applying (**??**) and (**??**) successively, we can explicitly compute $\mathbb{P}^{s,x}(F)$.
We compute $\mathbb{P}^{s,x}(F)$, when (s,x) belongs to $[t_{i^*-1} - 1, t_{i^*}[\times E$ for some $0 < i^* \le n+1$, where by convention, $t_0 = 0$. On $[t_n,T] \times E$, the same computation can be performed. We will show below that those restricted functions are Borel, the general result will follow by concatenation.
We have

$$
\begin{aligned}
& \mathbb{P}^{s,x}(F) \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x) \mathbb{E}^{s,x}\left[\prod_{j=i^*}^{n} \mathbb{1}_{A_i}(X_{t_i})\right] \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x) \mathbb{E}^{s,x}\left[\prod_{j=i^*}^{n-1} \mathbb{1}_{A_i}(X_{t_i}) \mathbb{E}^{s,x}[\mathbb{1}_{A_n}(X_{t_n})|\mathcal{F}_{t_{n-1}}]\right] \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x) \mathbb{E}^{s,x}\left[\prod_{j=i^*}^{n-1} \mathbb{1}_{A_i}(X_{t_i}) p(t_{n-1}, X_{t_{n-1}}, t_n, A_n)\right] \\
= & \cdots \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x) \int \left(\prod_{j=i^*+1}^{n} \mathbb{1}_{A_j}(x_j) p(t_{j-1}, x_{j-1}, t_j, dx_j)\right) \mathbb{1}_{A_{i^*}}(x_{i^*}) p(s,x,t_{i^*}, dx_{i^*}),
\end{aligned}
$$

which indeed is Borel in (s,x) thank to Definition **??** and Remark **??**.
We can extend this result to any event F by the monotone class theorem. Indeed, let Λ be the set of elements F of $\mathcal{F}$ such that $(s,x)\mapsto \mathbb{P}^{s,x}(F)$ is Borel. For any two events F^1, F^2, in Λ with $F^1\subset F^2$, since for any (s,x), $\mathbb{P}^{s,x}(F^2\backslash F^1)=\mathbb{P}^{s,x}(F^2)-\mathbb{P}^{s,x}(F^1)$, $(s,x)\mapsto \mathbb{P}^{s,x}(F^2\backslash F^1)$ is still Borel. For any increasing sequence $(F^n)_{n\geq 0}$ of elements of Λ, $\mathbb{P}^{s,x}(\bigcup\limits_{n\in\mathbb{N}}F^n)=\lim\limits_{n\rightarrow\infty}\mathbb{P}^{s,x}(F^n)$ so $(s,x)\mapsto \mathbb{P}^{s,x}(\bigcup\limits_{n\in\mathbb{N}}F^n)$ is still Borel, therefore Λ is a λ-system containing the π-system Π which generates $\mathcal{F}$. So by the monotone class theorem, $\Lambda=\mathcal{F}$.

Concerning the second statement of the proposition, if $Z\geq 0$, there exists an increasing sequence $(Z_n)_{n\geq 0}$ of simple functions on Ω converging pointwise to Z, and thank to the first statement of the Proposition, for each of these functions, $(s,x)\mapsto\mathbb{E}^{s,x}[Z_n]$ is Borel. Therefore since by monotonic convergence, $\mathbb{E}^{s,x}[Z_n]\underset{n\rightarrow\infty}{\longrightarrow}\mathbb{E}^{s,x}[Z]$, then $(s,x)\mapsto\mathbb{E}^{s,x}[Z]$ is Borel as the pointwise limit of Borel functions. For a general Z one just has to consider its decomposition $Z=Z^+-Z^-$ where Z^+ and Z^- are positive. □

Lemma A.9. *Let V be a continuous non-decreasing function on $[0,T]$ and $f\in\mathcal{B}([0,T]\times E)$ be such that for every (s,x), $\mathbb{E}^{s,x}[\int_s^T|f(r,X_r)|dV_r]<\infty$, then $(s,x)\longmapsto\mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$ is Borel.*

Proof. We will in fact show that on the set $\{(s,x,t)\in[0,T]\times E\times[0,T]:s\leq t\}$, the function $(s,x,t)\mapsto\mathbb{E}^{s,x}[\int_t^T f(r,X_r)dV_r]$ (which takes finite values thanks to the integrability assumed for f) is Borel. The Lemma will follow by composing with the measurable function $(s,x,t)\mapsto(s,x,s)$.
Let $t\in[0,T]$ be fixed, then by Proposition **??**, $(s,x)\mapsto\mathbb{E}^{s,x}[\int_t^T f(r,X_r)dV_r]$ is Borel. Let $(s,x)\in[0,T]\times E$ be fixed and $t_n\underset{n\rightarrow\infty}{\longrightarrow}t$ be a converging sequence in $[s,T]$. Since V is continuous, $\int_{t_n}^T f(r,X_r)dV_r\underset{n\rightarrow\infty}{\longrightarrow}\int_t^T f(r,X_r)dV_r$ a.s. And since this sequence is uniformly bounded by the L^1 r.v. $\int_s^T|f(r,X_r)|dV_r$, by dominated convergence theorem, the same convergence holds under the expectation. This implies that $t\mapsto\mathbb{E}^{s,x}[\int_t^T f(r,X_r)dV_r]$ is continuous. By Lemma 4.51 in [**?**], $(s,x,t)\mapsto\mathbb{E}^{s,x}[\int_t^T f(r,X_r)dV_r]$ is therefore jointly Borel, and by composition, so is $(s,x)\longmapsto\mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$. □

Proposition A.10. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a canonical Markov class associated to a transition function measurable in time, let $f\in\mathcal{B}([0,T]\times E,\mathbb{R})$ be such that for any (s,x,t), $\mathbb{E}^{s,x}[|f(t,X_t)|]<\infty$ then $(s,x,t)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is Borel.*

Proof. We start by showing it for $f\in\mathcal{C}_b([0,T]\times E,\mathbb{R})$. X is cadlag so $t\longmapsto f(t,X_t)$ also is. So for any fixed $(s,x)\in[0,T]\times E$ if we take a converging sequence $t_n\underset{n\rightarrow\infty}{\longrightarrow}t^+$(resp. t^-), an easy application of the Lebesgue dominated convergence theorem shows that $t\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is cadlag. On the other hand, by Proposition **??**, for a fixed t, $(s,x) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. Therefore by Theorem 15 Chapter IV of [?], $(s,x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is jointly Borel.

In order to extend the result to any $f \in \mathcal{B}_b([0,T] \times E, \mathbb{R})$, we consider the subset $\mathcal{I}$ of functions $f \in \mathcal{B}_b([0,T] \times E)$ such that $(s,x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. Then $\mathcal{I}$ is a linear space stable by uniform convergence and by monotone convergence and containing $\mathcal{C}_b([0,T] \times E)$ which is stable by multiplication and generates the Borel σ-field $\mathcal{B}([0,T]) \otimes \mathcal{B}(E)$. So by Theorem 21 in Chapter I of [?], $\mathcal{I} = \mathcal{B}_b([0,T] \times E)$. This Theorem is sometimes called the functional monotone class theorem.

Now for any positive Borel function f, we can set $f_n = f \wedge n$ which is bounded Borel. Since by monotonic convergence, $\mathbb{E}^{s,x}[f_n(t,X_t)]$ tends to $\mathbb{E}^{s,x}[f(t,X_t)]$, then $(s,x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel as the pointwise limit of Borel functions. Finally for a general f it is enough to decompose it into $f = f^+ - f^-$ where f^+, f^- are positive functions. □

Definition A.11. *Let $\mathbb{P}$ be a probability on $(\Omega, \mathcal{F})$. A set $N \subset \Omega$ is said to be $\mathbb{P}$-**negligible** (or $\mathbb{P}$-**null**) if it is included in a measurable set $N' \in \mathcal{F}$ such that $\mathbb{P}(N') = 0$. We denote $\mathcal{N}_{\mathbb{P}}$ the set of $\mathbb{P}$-negligible sets. We call $\mathbb{P}$-**completion** of $\mathcal{F}$ the σ-field generated by $\mathcal{F} \cup \mathcal{N}_{\mathbb{P}}$ which we denote $\mathcal{F}^{\mathbb{P}}$.*

*Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, we call $\mathbb{P}$-**closure** of $\mathcal{G}$ the σ-field generated by $\mathcal{G} \cup \mathcal{N}_{\mathbb{P}}$ which we denote $\mathcal{G}^{\mathbb{P}}$.*

Remark A.12. *Thanks to Remark 32.b) in Chapter II of [?], we have an equivalent definition of the $\mathbb{P}$-completion of $\mathcal{F}$ which can be characterized by the following property: $B \in \mathcal{F}^{\mathbb{P}}$ if and only if there exist $F \in \mathcal{F}$ and a $\mathbb{P}$-negligible set N such that $F \backslash N \subset B \subset F \cup N$. Moreover, $\mathbb{P}$ can be extended to a probability on $\mathcal{F}^{\mathbb{P}}$ by setting $\mathbb{P}(B) = \mathbb{P}(F)$.*

Similarly, if $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, the $\mathbb{P}$-closure of $\mathcal{G}$ can be defined by the condition: $B \in \mathcal{G}^{\mathbb{P}}$ if and only if there exist $G \in \mathcal{G}$ and a $\mathbb{P}$-negligible set N such that $G \backslash N \subset B \subset G \cup N$.

*The procedure above for defining completion and closure corresponds to the one in [?], however we believe that the ones given in Definition **??** are more standard nowadays.*

Definition A.13. *A probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P})$ is said to be **complete** if any $\mathbb{P}$-negligible set belongs to $\tilde{\mathcal{F}}$.*

For any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, if we build the $\mathbb{P}$-completion of $\mathcal{F}$ and extend $\mathbb{P}$ to it as explained in Remark **??**, then $(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P})$ is a complete probability space. Similarly, if we build the $\mathbb{P}$-closure of a sub-σ-field $\mathcal{G}$ and consider the restriction of $\mathbb{P}$ to it (after having been extended to $\mathcal{F}^{\mathbb{P}}$), then $(\Omega, \mathcal{G}^{\mathbb{P}}, \mathbb{P})$ is a complete probability space. We will sometimes just say that $\mathcal{G}^{\mathbb{P}}$ is complete.

Definition A.14. *For any $(s,x) \in [0,T] \times E$ we will consider the (s,x)-* ***completion*** $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ *of the stochastic basis* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ *by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}_t^{s,x}$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$ for every $t \in [0,T]$.*

We remark that, for any $(s,x) \in [0,T] \times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ is a stochastic basis fulfilling the usual conditions.

Proposition A.15. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathbb{P}$ a probability on $(\Omega, \mathcal{F})$ and $\mathcal{G}^{\mathbb{P}}$ the $\mathbb{P}$-closure of $\mathcal{G}$. Let $Z^{\mathbb{P}}$ be a real $\mathcal{G}^{\mathbb{P}}$-measurable random variable. There exists a $\mathcal{G}$-measurable random variable Z such that $Z = Z^{\mathbb{P}}$ $\mathbb{P}$-a.s.*

Proof. By Remark **??**, if $G^{\mathbb{P}} \in \mathcal{G}^{\mathbb{P}}$, there exist $G \in \mathcal{G}$ and a $\mathbb{P}$-negligible set N such that $G\backslash N \subset G^{\mathbb{P}} \subset G \cup N$ so

$$\begin{cases} \mathbb{1}_{G\backslash N} & \leq & \mathbb{1}_{G^{\mathbb{P}}} & \leq & \mathbb{1}_{G\cup N}\, \mathbb{P} \text{ a.s.} \\ \mathbb{1}_{G\backslash N} & \leq & \mathbb{1}_{G} & \leq & \mathbb{1}_{G\cup N}\, \mathbb{P} \text{ a.s.} \end{cases}$$

but since N is $\mathbb{P}$-negligible then $\mathbb{1}_{G\backslash N} = \mathbb{1}_{G\cup N}\, \mathbb{P}$ a.s. so all the previous indicators are $\mathbb{P}$ a.s. equal. Setting $Z = \mathbb{1}_G$, the assertion is true for such a r.v. Taking linear combinations, the assertion still holds if $Z^{\mathbb{P}}$ is a simple function.

If $Z^{\mathbb{P}}$ is any positive $\mathcal{G}^{\mathbb{P}}$-measurable function, there exists a sequence of positive $\mathcal{G}^{\mathbb{P}}$-measurable simple functions $\left(Z_n^{\mathbb{P}}\right)_{n\geq 0}$ such that $Z_n^{\mathbb{P}}$ tends to $Z^{\mathbb{P}}$ pointwise. For any $n \geq 0$, there exists a $\mathcal{G}$-measurable step function Z_n (which can be taken positive) such that $Z_n = Z_n^{\mathbb{P}}$ $\mathbb{P}$ a.s. We set $Z = \liminf\limits_{n\geq 0} Z_n$. It is finite for any ω since $Z_n \geq 0$ for all n, and Z is $\mathcal{G}$-measurable. There exists a set of $\mathbb{P}$-full measure on which for any n, $Z_n^{\mathbb{P}}(\omega) = Z_n(\omega)$ and on which $Z_n^{\mathbb{P}}(\omega)$ tends to $Z^{\mathbb{P}}(\omega)$, so on this set of full measure, $Z(\omega) = Z^{\mathbb{P}}(\omega)$.
For a general Z one needs to decompose it into $Z = Z^+ - Z^-$ where Z^+ and Z^- are positive. □

Proposition A.16. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a canonical Markov class. Let $(s,x) \in [0,T] \times E$ be fixed, Z be a random variable and $t \in [s,T]$, then $\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}_t^{s,x}]$ $\mathbb{P}^{s,x}$ a.s.*

Proof. $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable and therefore $\mathcal{F}_t^{s,x}$-measurable. Moreover, let $G^{s,x} \in \mathcal{F}_t^{s,x}$, by Remark **??**, there exists $G \in \mathcal{F}_t$ such that $\mathbb{P}^{s,x}(G \cup G^{s,x}) = \mathbb{P}^{s,x}(G \backslash G^{s,x})$ implying $\mathbb{1}_G = \mathbb{1}_{G^{s,x}}$ $\mathbb{P}^{s,x}$ a.s. So

$$\begin{aligned} \mathbb{E}^{s,x}\left[\mathbb{1}_{G^{s,x}}\mathbb{E}^{s,x}[Z|\mathcal{F}_t]\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{G}\mathbb{E}^{s,x}[Z|\mathcal{F}_t]\right] \\ &= \mathbb{E}^{s,x}\left[\mathbb{1}_{G}Z\right] \\ &= \mathbb{E}^{s,x}\left[\mathbb{1}_{G^{s,x}}Z\right], \end{aligned}$$

where the second equality occurs because of the definition of $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]$. □

So often when considering conditional expectations, we will drop the (s,x) superscript, except if we want to emphasize some specific measurability condition.

Remark A.17. *In this section we focus on some measurability issues which are of fundamental importance in the paper.*
Let us consider an event F which belongs to $\bigcap_{(s,x)\in[0,T]\times E} \mathcal{F}^{s,x}$. *In spite of the validity of Proposition* **??**, *it is hard to believe that the function $(s,x) \mapsto P^{s,x}(F)$ is is still Borel.*

In the literature of (generally homogeneous) Markov processes, one replaces instead the Borel σ-field by the so called σ-field of universally measurable sets, see e.g. Property 11 in Chapter XIV.1 of [?*]).*

We believe that this idea does not adapt very well to the context of time-inhomogeneous Markov processes, for which the probabilities are also indexed by the starting time. For that reason we have decided, as much as possible, to reduce the framework to events and r.v. which are measurable with respect to the uncompleted σ-field $\mathcal{F}$.

Remark A.18. *Our setup is a clear restriction of the most general setup described in [*?*] for the following reasons.*

1. *The time interval $[0,T]$ is deterministic and the probability measures $p(s,x,t,\cdot)$ have unit mass, which means that we consider processes with no explosion;*
2. *the state space E does not depend on the time;*
3. *we impose in our definition that trajectories must be cadlag;*
4. *we do not consider branching points;*
5. *the $p(\cdot,\cdot,t,A)$ are Borel in (s,x) and not just in x which simplifies considerations about measurability that we will have to make.*

Remark A.19. *Concerning the latter difference, in most of the literature, a transition function is defined as being only Borel in the space variable. With such a definition, one can do as in [*?*] and define the notion of p-measurability which makes all the functions $(s,x) \longmapsto p(s,x,t,A)$ p-measurable. However we prefer not to work in this setup in order to avoid useless complications since in all our examples we will indeed have measurability in the time variable.*

Notation A.20. *On $\mathbb{D}(E)$ we denote by $\mathcal{J}_1$ the Skorokhod topology, and we denote $\mathcal{P}(\mathbb{D}(E))$ the set of Borel probability measures on $\mathbb{D}(E)$ which we equip with the topology of weak convergence of measures with respect to $\mathcal{J}_1$.*

Definition A.21. *When considering a canonical Markov class, we say that $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ is* ***Feller continuous*** *if*

$$\begin{array}{ccc} (s,x) & \longmapsto & \mathbb{P}^{s,x} \\ [0,T]\times E & \longrightarrow & \mathcal{P}(\mathbb{D}(E)) \end{array}$$

is continuous.

We will show that when $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ is Feller continuous then the transition function is necessarily measurable in time. But first we need the following Lemma.

Lemma A.22. *Let $t\in[0,T]$, $\epsilon>0$ and $f\in\mathcal{C}_b(E,\mathbb{R})$ then*

$$F^{t,\epsilon,f}: \begin{array}{ccc} \omega & \longrightarrow & \frac{1}{\epsilon}\int_t^{t+\epsilon} f(\omega(r))dr \\ (\mathbb{D},J_1) & \longrightarrow & \mathbb{R}, \end{array}$$

is continuous.

Proof. Let $\omega_n \underset{n\rightarrow\infty}{\longmapsto} \omega$ in $(\mathbb{D}(E),J^1)$ then $\omega_n(s) \underset{n\rightarrow\infty}{\longmapsto} \omega(s)$ on all continuity points of ω, see Proposition 5.2 in chapter 3 of [**?**]. Since ω only has a countable number of jumps (see Lemma 5.1 in chapter 3 of [**?**]) then ω_n tends to ω Lebesgue a.e. So the continuity of f and dominated convergence theorem imply that $\int_t^{t+\epsilon} f(\omega_n(r))dr \underset{n\rightarrow\infty}{\longmapsto} \int_t^{t+\epsilon} f(\omega(r))dr$. ☐

Proposition A.23. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a canonical Markov class associated to a transition function p. If $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ is Feller continuous then p is measurable in time.*

Proof. By Lemma **??**, for any $t\in[0,T]$, $\epsilon>0$ and $f\in\mathcal{C}_b(E,\mathbb{R})$, $F^{t,\epsilon,f}$ is a continuous functional and it is clearly bounded so by definition of the weak convergence of measures, $(s,x)\longmapsto \mathbb{E}^{s,x}\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} f(X_r)dr\right]$ is continuous and therefore Borel. Then for some fixed (s,x), since f is continuous and X cadlag, we have $\frac{1}{\epsilon}\int_t^{t+\epsilon} f(X_r)dr \underset{\epsilon\rightarrow 0}{\longrightarrow} f(X_t)$ $\mathbb{P}^{s,x}$-a.s. Since f is bounded, by dominated convergence we get $\mathbb{E}^{s,x}\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} f(X_r)dr\right] \underset{\epsilon\rightarrow 0}{\longrightarrow} \mathbb{E}^{s,x}[f(X_t)]$ and therefore $(s,x)\longmapsto \mathbb{E}^{s,x}[f(X_t)]$ is Borel as pointwise limit of Borel functions. Now this can be extended to any bounded Borel function f by using as in Proposition **??** the functional version of the monotone class theorem, i.e. Theorem 21 in Chapter I of [**?**]. Since by (**??**) $p(s,x,t,A)=\mathbb{E}^{s,x}[\mathbb{1}_A(X_t)]$ for any s,x,t,A, we have shown that p is measurable in time. ☐

Notation A.24. *For any $t\in[0,T]$ we denote $\mathcal{F}_{t,T}:=\sigma(X_r|r\geq t)$ and $\mathcal{F}^{s,x}_{t,T}$ its $\mathbb{P}^{s,x}$-closure.*

For any $0\leq t\leq u<T$ we will denote $\mathcal{F}_{t,u}:=\bigcap_{n\geq 0}\sigma(X_r|r\in[t,u+\frac{1}{n}])$ and $\mathcal{F}^{s,x}_{t,u}$ will stand for its $\mathbb{P}^{s,x}$-closure. The filtration $(\mathcal{F}_{t,u})_{u\in[t,T]}$ will sometimes be denoted $\mathcal{F}_{t,\cdot}$.

Remark A.25. *The above σ-fields fulfill the properties below.*

1. *For any $0\leq t\leq u<T$, $\mathcal{F}_{t,u}=\mathcal{F}_u\cap\mathcal{F}_{t,T}$;*
2. *for any $t\geq 0$, $\mathcal{F}_t\vee\mathcal{F}_{t,T}=\mathcal{F}$;*

3. *for any* $(s,x) \in [0,T] \times E$*, the two first items remain true when considering the* $\mathbb{P}^{s,x}$*-closures of all the* σ*-fields;*

4. *for any* $t \geq 0$, $\Pi := \{F = F_t \cap F^t_T | (F_t, F^t_T) \in \mathcal{F}_t \times \mathcal{F}_{t,T}\}$ *is a* π*-system generating* $\mathcal{F}$*.*

Remark A.26. *Formula 1.7 in Chapter 6 of [?] states that for any* $A \in \mathcal{F}_{t,T}$ *yields*

$$\mathbb{P}^{s,x}(A|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(A) = \mathbb{P}^{s,x}(A|X_t) \quad \mathbb{P}^{s,x} a.s. \tag{A.5}$$

Property (**??**) *will also often be called* ***Markov property****.*

Definition A.27. *If* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P})$ *is a probability space and* $\mathcal{G}$ *is a sub-*σ*-field of* $\tilde{\mathcal{F}}$*, we say that* $\mathcal{G}$ *is* $\mathbb{P}$***-trivial*** *if for any element* G *of* $\mathcal{G}$*, then* $\mathbb{P}(G)$ *is equal to 0 or 1.*

We now show that in our setup, a canonical Markov class verifies the **Blumenthal 0-1 law** in the following sense.

Proposition A.28. *Let* $(s,x) \in [0,T] \times E$ *and* $F \in \mathcal{F}_{s,s}$ *then* $\mathbb{P}^{s,x}(F)$ *is equal to* 1 *or to* 0*. In other words,* $\mathcal{F}_{s,s}$ *is* $\mathbb{P}^{s,x}$*-trivial.*

Proof. Let $F \in \mathcal{F}_{s,s}$ as introduced in Notation **??**.
Since by Remark **??**, $\mathcal{F}_{s,s} = \mathcal{F}_s \cap \mathcal{F}_{s,T}$, then F belongs to $\mathcal{F}_s$ so by conditioning we get

$$\begin{array}{rcl} \mathbb{E}^{s,x}[\mathbb{1}_F] & = & \mathbb{E}^{s,x}[\mathbb{1}_F \mathbb{1}_F] \\ & = & \mathbb{E}^{s,x}[\mathbb{1}_F \mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_s]] \\ & = & \mathbb{E}^{s,x}[\mathbb{1}_F \mathbb{E}^{s,X_s}[\mathbb{1}_F]], \end{array}$$

where the latter equality comes from (**??**) because $F \in \mathcal{F}_{s,T}$. But by (**??**), $X_s = x$ $\mathbb{P}^{s,x}$ a.s. so

$$\begin{array}{rcl} \mathbb{E}^{s,x}[\mathbb{1}_F] & = & \mathbb{E}^{s,x}[\mathbb{1}_F \mathbb{E}^{s,x}[\mathbb{1}_F]] \\ & = & \mathbb{E}^{s,x}[\mathbb{1}_F]^2. \end{array}$$

□

Proposition A.29. *For any* $(s,x) \in]0,T] \times E$ *and* $t \in [0,s[$, $\mathcal{F}_t$ *is* $\mathbb{P}^{s,x}$*-trivial.*

Proof. Let (s,x) and t be fixed. We recall that, by (**??**), X is almost surely equal to the constant x on $[0,s]$. $\mathcal{F}_t$ is generated by the π-system Π_t composed of events of type $\bigcap_{i\leq n}\{X_{t_i} \in A_i\}$ where $n \in \mathbb{N}$, $t_1, \cdots, t_n$ are in $[0,t]$, and $A_1, \cdots, A_n$ are Borel subsets of E. Let Λ_t be the set of events of $\mathcal{F}_t$ of probability one or zero. Λ_t is clearly a λ-system and by (**??**) contains Π_t. By the monotone class theorem, it follows $\Lambda_t = \mathcal{F}_t$. □

B Non-homogeneous Additive Functionals

In this section, we introduce the notion of non-homogeneous Additive Functional that we use in the paper. This looks to be a good compromise between the notion of Additive Functional associated to a stochastic system introduced by E.B. Dynkin (see for example [**?**]) and the more popular notion of homogeneous Additive Functional studied extensively, for instance by C. Dellacherie and P.A. Meyer in [**?**] Chapter XV. This section of the Appendix consists in extending some essential results stated in [**?**] Chapter XV to our setup.

In this section we will use the notation $\Delta := \{(t,u) \in [0,T]^2 | t \leq u\}$. As in Appendix **??**, $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ will still denote the canonical space (associated to T and E).

Definition B.1. *On $(\Omega, \mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened AF) as a random-field indexed by Δ with values in $\mathbb{R}$ $A := (A^t_u)_{(t,u)\in\Delta}$ verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, A^t_u is $\mathcal{F}_{t,u}$-measurable, see Definition **??**;*
2. *for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $\mathcal{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$,*
$$A^t_u = A^{s,x}_u - A^{s,x}_t \quad \mathbb{P}^{s,x} \text{ a.s.}$$

We denote by A^t the ($\mathcal{F}_{t,\cdot}$-adapted) process $u \mapsto A^t_u$ indexed by $[t,T]$. For any $(s,x) \in [0,t] \times E$, $A^{s,x}_\cdot - A^{s,x}_t$ is a $\mathbb{P}^{s,x}$-version of A^t on $[t,T]$.
*$A^{s,x}$ will be called the **cadlag version of** A **under** $\mathbb{P}^{s,x}$.*

*An AF will be called a **non-homogeneous Martingale Additive Functional** (shortened MAF) if under any $\mathbb{P}^{s,x}$ its cadlag version is a martingale.*

More generally an AF will be said to verify a certain property (being non-negative, increasing, of bounded variation, square integrable, having L^1 terminal value) if under any $\mathbb{P}^{s,x}$ its cadlag version verifies it.

*Finally, given two increasing AF A and B, A will be said to be **absolutely continuous with respect to** B if for any $(s,x) \in [0,T] \times E$, $dA^{s,x} \ll dB^{s,x}$ in the sense of stochastic measures.*

Remark B.2. *The set of AFs (respectively of AFs with bounded variation, of AFs with L^1 terminal value, of MAFs, of square integrable MAFs) is a linear space.*

Remark B.3. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ and set*
*$M[\phi]^t_u = \phi(u, X_u) - \phi(t, X_t) - \int_t^u \mathfrak{a}(\phi)(r, X_r) dV_r$. Then $(M[\phi]^t_u)_{(t,u)\in\Delta}$ is an example of square integrable MAF and its cadlag version under $\mathbb{P}^{s,x}$ is $M[\phi]^{s,x}$, as defined in Definition **??**.*

In this section for a given MAF $(M^t_u)_{(t,u)\in\Delta}$ we will be able to exhibit two AF, denoted respectively by $([M]^t_u)_{(t,u)\in\Delta}$ and $(\langle M\rangle^t_u)_{(t,u)\in\Delta}$, which will play respectively the role of a quadratic variation and an angular bracket of it. Moreover we will show that the Radon-Nikodym derivative of the mentioned angular bracket of a MAF with respect to our reference function V is a time-dependent function of the underlying process.

Proposition B.4. *Let $(M^t_u)_{(t,u)\in\Delta}$ be a square integrable MAF, and for any $(s,x)\in[0,T]\times E$, $[M^{s,x}]$ be the quadratic variation of its cadlag version $M^{s,x}$ under $\mathbb{P}^{s,x}$. Then there exists an AF which we will call $([M]^t_u)_{(t,u)\in\Delta}$ and which, for any $(s,x)\in[0,T]\times E$, has $[M^{s,x}]$ as cadlag version under $\mathbb{P}^{s,x}$.*

Proof. We adapt Theorem 16 Chapter XV in [**?**] to a non homogeneous set-up but the reader must keep in mind that our definition of Additive Functional is different from the one related to the homogeneous case.

For the whole proof $t<u$ will be fixed. We consider a sequence of subdivisions of $[t,u]$: $t=t^k_1<t^k_2<\cdots<t^k_k=u$ such that $\min\limits_{i<k}(t^k_{i+1}-t^k_i)\underset{k\to\infty}{\longrightarrow}0$. Let $(s,x)\in[0,t]\times E$ with corresponding probability $\mathbb{P}^{s,x}$. For any k, we have $\sum\limits_{i<k}\left(M^{t^k_i}_{t^k_{i+1}}\right)^2=\sum\limits_{i<k}(M^{s,x}_{t^k_{i+1}}-M^{s,x}_{t^k_i})^2$ $\mathbb{P}^{s,x}$ a.s., so by definition of quadratic variation we know that

$$\sum_{i<k}\left(M^{t^k_i}_{t^k_{i+1}}\right)^2\underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}}[M^{s,x}]_u-[M^{s,x}]_t. \tag{B.1}$$

In the sequel we will construct an $\mathcal{F}_{t,u}$-measurable random variable $[M]^t_u$ such that for any $(s,x)\in[0,t]\times E$,
$\sum_{i\leq k}\left(M^{t^k_i}_{t^k_{i+1}}\right)^2\underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}}[M]^t_u$. In that case $[M]^t_u$ will then be $\mathbb{P}^{s,x}$ a.s. equal to $[M^{s,x}]_u-[M^{s,x}]_t$.

Let $x\in E$. Since M is a MAF, for any k, $\sum\limits_{i<k}\left(M^{t^k_i}_{t^k_{i+1}}\right)^2$ is $\mathcal{F}_{t,u}$-measurable and therefore $\mathcal{F}^{t,x}_{t,u}$-measurable. Since $\mathcal{F}^{t,x}_{t,u}$ is complete, the limit in probability of this sequence, $[M^{t,x}]_u-[M^{t,x}]_t$, is still $\mathcal{F}^{t,x}_{t,u}$-measurable. By Proposition **??**, there is an $\mathcal{F}_{t,u}$-measurable variable which depends on (t,x), that we call $a_t(x,\omega)$ such that

$$a_t(x,\omega)=[M^{t,x}]_u-[M^{t,x}]_t,\mathbb{P}^{t,x}\text{ a.s.} \tag{B.2}$$

We will show below that there is a jointly measurable version of
$(x,\omega)\mapsto a_t(x,\omega)$.

For any integer $n\geq 0$, we set $a^n_t(x,\omega):=n\wedge a_t(x,\omega)$ which is in particular limit in probability of $n\wedge\sum\limits_{i\leq k}\left(M^{t^k_i}_{t^k_{i+1}}\right)^2$ under $\mathbb{P}^{t,x}$.

For any integers k, n and any $x \in E$, we define the finite positive measures $\mathbb{Q}^{k,n,x}$, $\mathbb{Q}^{n,x}$ and $\mathbb{Q}^{x}$ on $(\Omega, \mathcal{F}_{t,u})$ by

1. $\mathbb{Q}^{k,n,x}(F) := \mathbb{E}^{t,x}\left[\mathbb{1}_F\left(n \wedge \sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2\right)\right]$;
2. $\mathbb{Q}^{n,x}(F) := \mathbb{E}^{t,x}[\mathbb{1}_F\,(a_t^n(x,\omega))]$;
3. $\mathbb{Q}^{x}(F) := \mathbb{E}^{t,x}[\mathbb{1}_F\,(a_t(x,\omega))]$.

When k and n are fixed, for any fixed F, by Proposition **??**,
$x \longmapsto \mathbb{E}^{t,x}\left[F\left(n \wedge \sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2\right)\right]$, is Borel.

Then $n \wedge \sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{t,x}}{\longrightarrow}} a_t^n(x,\omega)$, and this sequence is uniformly bounded by the constant n, so the convergence takes place in L^1, therefore $x \longmapsto \mathbb{Q}^{n,x}(F)$ is also Borel as the pointwise limit in k of the functions $x \longmapsto \mathbb{Q}^{k,n,x}(F)$. Similarly, $a_t^n(x,\omega) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} a_t(x,\omega)$ and is non-decreasing, so by monotone convergence theorem, being a pointwise limit in n of the functions $x \longmapsto \mathbb{Q}^{n,x}(F)$, the function $x \longmapsto \mathbb{Q}^{x}(F)$ is Borel.
We recall that $\mathcal{F}$ is separable. By Theorem 58 Chapter V in [**?**], the two properties above and the fact that, for any x, we also have (by item 3. above) $\mathbb{Q}^x \ll \mathbb{P}^{t,x}$, allows to show the existence of a jointly measurable (for $\mathcal{B}(E)\otimes\mathcal{F}_{t,u}$) version of $(x,\omega) \mapsto a_t(x,\omega)$, that we recall to be densities of $\mathbb{Q}^x$ with respect to $\mathbb{P}^{t,x}$. That version will still be denoted by the same symbol.

We can now set $[M]^t_u(\omega) = a_t(X_t(\omega),\omega)$, which is a correctly defined $\mathcal{F}_{t,u}$-measurable random variable. For any x, taking into account (**??**), we have the equalities

$$[M]^t_u = a_t(x,\omega) = [M^{t,x}]_u - [M^{t,x}]_t \;\; \mathbb{P}^{t,x}\text{a.s.} \tag{B.3}$$

We will in fact prove that

$$[M]^t_u = [M^{s,x}]_u - [M^{s,x}]_t \;\; \mathbb{P}^{s,x} \text{ a.s.}, \tag{B.4}$$

holds for every $(s,x) \in [0,t] \times E$, and not just in the case $s = t$ that we have just established in (**??**).

We proceed proving the validity of (**??**) also for a fixed $s < t$ and $x \in E$. We show that under any $\mathbb{P}^{s,x}$, $[M]^t_u$ is the limit in probability of $\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2$. Indeed, let $\epsilon > 0$, the event $\left\{\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2 - [M]^t_u\right| > \epsilon\right\}$ belongs to $\mathcal{F}_{t,T}$ so by conditioning and using the Markov property (**??**) we have

$$\begin{array}{rl} & \mathbb{P}^{s,x}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right) \\ = & \mathbb{E}^{s,x}\left[\mathbb{P}^{s,x}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\middle|\mathcal{F}_t\right)\right] \\ = & \mathbb{E}^{s,x}\left[\mathbb{P}^{t,X_t}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)\right]. \end{array}$$

For any fixed y, by (**??**) and (**??**), $\mathbb{P}^{t,y}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)$ tends to zero when k goes to infinity, so a.s. under the probability $\mathbb{P}^{s,x}$, it yields $\mathbb{P}^{t,X_t}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)$ tends to zero when k goes to infinity. Since this sequence is dominated by the constant 1, that convergence still holds under the expectation thanks to the dominated convergence theorem.

So we have built an $\mathcal{F}_{t,u}$-measurable variable $[M]^t_u$ such that under any $\mathbb{P}^{s,x}$ with $s\leq t$, $[M^{s,x}]_u-[M^{s,x}]_t=[M]^t_u$ a.s. and this concludes the proof. □

We will now extend the result about quadratic variation to the angular bracket of MAFs. The next result can be seen as an extension of Theorem 15 Chapter XV in [**?**] to a non-homogeneous context.

Proposition B.5. *Let $(B^t_u)_{(t,u)\in\Delta}$ be an increasing AF with L^1 terminal value, for any $(s,x)\in[0,T]\times E$, let $B^{s,x}$ be its cadlag version under $\mathbb{P}^{s,x}$ and let $A^{s,x}$ be the predictable dual projection of $B^{s,x}$ in $(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x})$. Then there exists an increasing AF with L^1 terminal value $(A^t_u)_{(t,u)\in\Delta}$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of A is $A^{s,x}$.*

Proof. The first half of the demonstration will consist in showing that for any $(s,x)\in[0,T]\times E$ and $s\leq t<u$, $(A^{s,x}_u-A^{s,x}_t)$ is $\mathcal{F}^{s,x}_{t,u}$-measurable.

We start by recalling a property of the predictable dual projection which we will have to extend slightly.
Let us fix (s,x) and the corresponding stochastic basis
$(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x})$. For any $F\in\mathcal{F}^{s,x}$, let $N^{s,x,F}$ be the cadlag version of the martingale,
$r\longmapsto\mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_r]$. Then for any $0\leq t\leq u\leq T$, the predictable projection of the process $r\mapsto\mathbb{1}_F\mathbb{1}_{[t,u[}(r)$ is $r\mapsto N^{s,x,F}_{r^-}\mathbb{1}_{[t,u[}(r)$, see the proof of Theorem 43 Chapter VI in [**?**]. Therefore by definition of the dual predictable projection (see Definition 73 Chapter VI in [**?**]) we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u-A^{s,x}_t)\right]=\mathbb{E}^{s,x}\left[\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r\right], \qquad \text{(B.5)}$$

for any $F\in\mathcal{F}^{s,x}$.

We will now prove four technical lemmas which in a sense extend this property, and will permit us to work with a good common version of the random variable $\int_t^u N^{s,x,F}_{r^-} dB^{s,x}_r$ not depending on (s,x).

For the rest of the proof, $0 \leq t < u \leq T$ are fixed.

Lemma B.6. *Let $F \in \mathcal{F}_{t,T}$. There exists an $\mathcal{F}_{t,\cdot}$-adapted process called N^F indexed on $[t,T]$ and taking values in $[0,1]$ for all ω, such that for any $(s,x) \in [0,t] \times E$, the cadlag martingale $N^{s,x,F}$ defined above is a $\mathbb{P}^{s,x}$-version of N^F on $[t,T]$.*
Moreover if F belongs to $\mathcal{F}^t_u$ for some $u \in [t,T]$ then N^F_r is $\mathcal{F}_{t,u}$-measurable for any $r \in [t,T]$.

Proof. We start proving the first assertion of the lemma. We fix $r \in [t,T]$; it is enough to show that for any $F \in \mathcal{F}_{t,T}$ there exists an $\mathcal{F}_{t,r}$-measurable random variable N^F_r taking values in $[0,1]$ such that for every $(s,x) \in [0,t] \times E$, $N^F_r = N^{s,x,F}_r \quad \mathbb{P}^{s,x}$ a.s.
We denote by Λ the set of elements $F \in \mathcal{F}_{t,T}$ for which there exists an $\mathcal{F}_{t,r}$-measurable random variable N^F_r such that for every $(s,x) \in [0,t] \times E$,
$N^F_r = N^{s,x,F}_r$ $\mathbb{P}^{s,x}$ a.s., and by Π the π-system of elements $F \in \mathcal{F}_{t,T}$ of type $F_{t,r} \cap F_{r,T}$ where $F_{t,r} \in \mathcal{F}_{t,r}$ and $F_{r,T} \in \mathcal{F}_{r,T}$. By Remark **??**, Π generates $\mathcal{F}_{t,T}$.

We fix $F = F_{t,r} \cap F_{r,T} \in \Pi$ and $(s,x) \in [0,t] \times E$. Thanks to (**??**) we have the a.s. equalities

$$\begin{aligned} \mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_r] &= \mathbb{1}_{F_{t,r}} \mathbb{P}^{s,x}(F_{r,T}|\mathcal{F}_r) \\ &= \mathbb{1}_{F_{t,r}} \mathbb{P}^{r,X_r}(F_{r,T}). \end{aligned}$$

For this F, $N^F_r := \mathbb{1}_{F_{t,r}} \mathbb{P}^{r,X_r}(F_{r,T})$ does indeed not depend on (s,x), is $\mathcal{F}_{t,r}$-measurable and non-negative. This proves that $\Pi \subset \Lambda$.
For any $F \in \Lambda$, passing from N^F_r to $0 \vee N^F_r \wedge 1$ does not change the $\mathbb{P}^{s,x}$ a.s equalities since all r.v. $N^{s,x,F}_r$ belong a.s. to $[0,1]$, so up to this additional step, N^F_r can always be assumed to take values in $[0,1]$ for all ω.
We now show that Λ is a λ-system. If $F^1, F^2 \in \Lambda$ with $F^1 \subset F^2$, setting $N^{F^1 \backslash F^2}_r := N^{F^1}_r - N^{F^2}_r$, we have $F^1 \backslash F^2 \in \Lambda$. If $(F^n)_{n \in \mathbb{N}}$ is an increasing sequence of elements of Λ, setting $N^{\cup F^n}_r := \underset{n}{\liminf} N^{F^n}_r$, which exists since all terms are positive. $F := \underset{n}{\cup} F^n \in \Lambda$: indeed, for every (s,x) the monotone convergence theorem allows to show that $N^F_r = N^{s,x,F}_r$ $\mathbb{P}^{s,x}$ a.s. So Λ is a λ-system containing the π-system Π which generates $\mathcal{F}$, it is therefore equal to $\mathcal{F}$ by the monotone class theorem. This shows the first part of the lemma.

Concerning the second part, assume that $F \in \mathcal{F}^t_u$. Either $u \leq r$, then $F \in \mathcal{F}^t_r$, so $F \in \Pi$ and $N^F_r = \mathbb{1}_F$ is indeed $\mathcal{F}^t_u$-measurable.

In the case $r \leq u$, since we have shown that N^F_r is $\mathcal{F}_{t,r}$-measurable, then it is also $\mathcal{F}^t_u$-measurable. In both cases, N^F_r is $\mathcal{F}_{t,u}$. □

Lemma B.7. *Let $F \in \mathcal{F}_{t,T}$. There exists an $\mathcal{F}_{t,T}$-measurable random variable which we will call $\int_t^u N^F_{r^-} dB_r$ such that for any $(s,x) \in [0,t] \times E$,*

$\int_t^u N^F_{r^-} dB_r = \int_t^u N^{s,x,F}_{r^-} dB^{s,x}_r$ $\mathbb{P}^{s,x}$ *a.s. If moreover* $F \in \mathcal{F}_{t,u}$ *then* $\int_t^u N^F_{r^-} dB_r$ *is* $\mathcal{F}_{t,u}$*-measurable.*

Remark B.8. *By definition, the process* N^F *in the statement of Lemma* **??** *and the r.v.* $\int_t^u N^F_{r^-} dB_r$ *will not depend on any* (s,x)*.*

Proof. In some sense we wish to integrate $r \mapsto N^F_{r^-}$ against B^t for fixed ω. However first we do not know a priori if the paths $r \mapsto N^F_r$ and $r \mapsto B^t_r$ are measurable, second $r \mapsto N^F_r$ may not have a left limit and B^t may be not of bounded variation. So it is not clear if $\int_t^u N^F_{r^-} dB^t_r$ makes sense for any ω. Moreover under a certain $\mathbb{P}^{s,x}$, $N^{F,s,x}$ and $B^{s,x}_{\cdot} - B^{s,x}_t$ are only versions of N^F and B^t and not indistinguishable to them. Even if we could compute the overmentioned integral, it would not be clear if $\int_t^u N^F_{r^-} dB^t_r = \int_t^u N^{s,x,F}_{r^-} dB^{s,x}_r$ $\mathbb{P}^{s,x}$ a.s.

We start by some considerations about B, setting $W_{tu} := \{\omega : \sup\limits_{r \in [t,u] \cap \mathbb{Q}} B^t_r < \infty\}$ which is $\mathcal{F}_{t,u}$-measurable, and for $r \in [t,u]$

$$\bar{B}^t_r(\omega) := \begin{cases} \sup\limits_{\substack{t \le v < r \\ v \in \mathbb{Q}}} B^t_v(\omega) \text{ if } \omega \in W_{tu} \\ 0 \text{ otherwise.} \end{cases}$$

$\bar{B}^t$ is an increasing, finite (for all ω) process. In general, it is neither a measurable nor an adapted process; however for any $r \in [t,u]$, $\bar{B}^t_r$ is still $\mathcal{F}_{t,u}$-measurable. Since it is increasing, it has right and left limits at each point for every ω, so we can define the process $\tilde{B}^t$ indexed on $[t,u]$ below:

$$\tilde{B}^t_r := \lim_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} \bar{B}^t_v, r \in [t,u], \tag{B.6}$$

when $u \in]t,T[$ and $\tilde{B}^t_T := B^t_T$ if $u = T$. Therefore $\tilde{B}^t$ is an increasing, cadlag process. It is constituted by $\mathcal{F}_{t,u}$-measurable random variables, and by Theorem 15 Chapter IV of [**?**], $\tilde{B}^t$ is a also a measurable process (indexed by $[t,u]$).

We can show that $\tilde{B}^t$ is $\mathbb{P}^{s,x}$-indistinguishable from $B^{s,x}_{\cdot} - B^{s,x}_t$ for any $(s,x) \in [0,t] \times E$. Indeed, let (s,x) be fixed. Since $B^{s,x}_{\cdot} - B^{s,x}_t$ is a version of B^t and $\mathbb{Q}$ being countable, there exists a $\mathbb{P}^{s,x}$-null set $\mathcal{N}$ such that for all $\omega \in \mathcal{N}^c$ and $r \in \mathbb{Q} \cap [t,u]$, $B^{s,x}_r(\omega) - B^{s,x}_t(\omega) = B^t_r(\omega)$. Therefore for any $\omega \in \mathcal{N}^c$ and $r \in [t,u]$,

$$\tilde{B}^t_r(\omega) = \lim_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} \sup_{\substack{t \le w < v \\ w \in \mathbb{Q}}} B^t_w(\omega) = \lim_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} \sup_{\substack{t \le w < v \\ w \in \mathbb{Q}}} B^{s,x}(\omega)_w - B^{s,x}(\omega)_t = B^{s,x}(\omega)_r - B^{s,x}(\omega)_t,$$

where the latter equality comes from the fact that $B^{s,x}(\omega)$ is cadlag and increasing. So we have constructed an increasing finite cadlag (for all ω) process and so the path $r \mapsto \tilde{B}^t(\omega)$ is a Lebesgue integrator on $[t,u]$ for each ω.

We fix now $F \in \mathcal{F}_{t,T}$ and we discuss some issues related to N^F . Since it is positive, we can start defining the process $\bar{N}$, for index values $r \in [t,T[$ by $\bar{N}^F_r := \liminf_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} N^F_v$, and setting $\bar{N}^F_T := N^F_T$. This process is (by similar arguments as for $\tilde{B}^t$ defined in (**??**)), $\mathbb{P}^{s,x}$-indistinguishable to $N^{s,x,F}$ for all $(s,x) \in [0,t] \times E$. By Lemma **??**, for any $r \geq t$, N^F_r is $\mathcal{F}_{t,T}$-measurable (and even $\mathcal{F}_{t,u}$-measurable if $F \in \mathcal{F}_{t,u}$), so $\bar{N}^F_r$ will also be $\mathcal{F}_{t,T}$-measurable for any $r \geq t$ (also $\mathcal{F}_{t,u}$-measurable if $F \in \mathcal{F}_{t,u}$). However, $\bar{N}^F$ is not necessarily cadlag for every ω, and also not necessarily a measurable process.

We subsequently define

$W'_{tu} := \{\omega \in \Omega |$there exists a cadlag function f such that $\bar{N}^F(\omega) = f$ on $[t,u] \cap \mathbb{Q}\}$.

By Theorem 18 b) in Chapter IV of [**?**], W'_{tu} is $\mathcal{F}_{t,u}$-measurable so we can define on $[t,u]$ $\tilde{N}^F_r := \bar{N}^F_r \mathbb{1}_{W'_{tu}}$. $\tilde{N}^F$ is no longer $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, however, it is now cadlag for all ω and therefore a measurable process by Theorem 15 Chapter IV of [**?**]. The r.v. $\tilde{N}^F_r$ are still $\mathcal{F}_{t,T}$-measurable (even $\mathcal{F}_{t,u}$-measurable if $F \in \mathcal{F}_{t,u}$), and $\tilde{N}^F$ is still $\mathbb{P}^{s,x}$-indistinguishable to $N^{s,x,F}$ on $[t,u]$ for any $(s,x) \in [0,t] \times E$.

Finally we can define $\int_t^u N^F_{r^-} dB_r := \int_t^u \tilde{N}^F_{r^-} d\tilde{B}^t_r$ which is $\mathbb{P}^{s,x}$ a.s. equal to $\int_t^u N^{s,x,F}_{r^-} dB^{s,x}_r$ for any $(s,x) \in [0,t] \times E$.

Moreover, since $\tilde{N}^F$ and $\tilde{B}$ are both measurable with respect to $\mathcal{B}([t,u]) \otimes \mathcal{F}_{t,T}$ (even $\mathcal{B}([t,u]) \otimes \mathcal{F}_{t,u}$ if $F \in \mathcal{F}_{t,u}$), then $\int_t^u N^F_{r^-} dB_r$ is $\mathcal{F}_{t,T}$-measurable (even $\mathcal{F}_{t,u}$-measurable if $F \in \mathcal{F}_{t,u}$). □

The lemma below is a conditional version of the property (**??**).

Lemma B.9. *For any $(s,x) \in [0,t] \times E$ and $F \in \mathcal{F}^{s,x}_{t,T}$ we have $\mathbb{P}^{s,x}$-a.s.*

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F (A^{s,x}_u - A^{s,x}_t) | \mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-} dB_r \middle| \mathcal{F}_t\right].$$

Proof. Let s,x,F be fixed. By definition of conditional expectation, we need to show that for any $G \in \mathcal{F}_t$ we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{1}_F (A^{s,x}_u - A^{s,x}_t)\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-} dB_r \middle| \mathcal{F}_t\right]\right] \text{ a.s.}$$

For $r \in [t,u]$ we have $\mathbb{E}^{s,x}[\mathbb{1}_{F\cap G} | \mathcal{F}_r] = \mathbb{1}_G \mathbb{E}^{s,x}[\mathbb{1}_F | \mathcal{F}_r]$ a.s. therefore the cadlag versions of these processes are indistinguishable on $[t,u]$ and the random variables $\int_t^u N^{G\cap F}_{r^-} dB_r$ and $\mathbb{1}_G \int_t^u N^F_{r^-} dB_r$ as defined in Lemma **??** are a.s. equal. So by the non conditional property of dual predictable projection (**??**) we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{1}_F (A^{s,x}_u - A^{s,x}_t)\right] & = & \mathbb{E}^{s,x}\left[\int_t^u N^{G\cap F}_{r^-} dB_r\right] \\ & = & \mathbb{E}^{s,x}\left[\mathbb{1}_G \int_t^u N^F_{r^-} dB_r\right] \\ & = & \mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-} dB_r | \mathcal{F}_t\right]\right], \end{array}$$

which concludes the proof. □

Lemma B.10. *For any* $(s,x) \in [0,t] \times E$ *and* $F \in \mathcal{F}_{t,T}$ *we have* $\mathbb{P}^{s,x}$*-a.s.,*

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|X_t\right].$$

Proof. By Lemma **??** we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right].$$

By Lemma **??**, $\int_t^u N_{r^-}^F dB_r$ is $\mathcal{F}_{t,T}$ measurable so the Markov property (**??**) implies

$$\mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| X_t\right],$$

therefore $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right]$ is a.s. equal to a $\sigma(X_t)$-measurable r.v and so is a.s. equal to $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|X_t\right]$. □

We now finally prove the first important issue of the proof of Proposition **??**, which states that

$$\forall (s,x) \in [0,t] \times E,\ (A_u^{s,x} - A_t^{s,x}) \text{ is } \mathcal{F}_{t,u}^{s,x}\text{−measurable.} \tag{B.7}$$

By definition, a predictable dual projection is adapted so we already know that $(A_u^{s,x} - A_t^{s,x})$ is $\mathcal{F}_u^{s,x}$-measurable, therefore by Remark **??**, it is enough to show that it is also $\mathcal{F}_{t,T}^{s,x}$-measurable.
So we are going to show that

$$A_u^{s,x} - A_t^{s,x} = \mathbb{E}^{s,x}\left[A_u^{s,x} - A_t^{s,x}|\mathcal{F}_{t,T}\right]\ \mathbb{P}^{s,x} \text{ a.s.} \tag{B.8}$$

For this we will show that

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F \mathbb{E}^{s,x}\left[A_u^{s,x} - A_t^{s,x}|\mathcal{F}_{t,T}\right]\right], \tag{B.9}$$

for any $F \in \mathcal{F}$. We will prove (**??**) for $F \in \mathcal{F}$ event of the form $F = F_t \cap F_{t,T}$ with $F_t \in \mathcal{F}_t$ and $F_{t,T} \in \mathcal{F}_{t,T}$.
By Remark **??**, such events form a π-system Π which generates $\mathcal{F}$.

Consequently, by the monotone class theorem, (**??**) will remain true for any $F \in \mathcal{F}$ and even in $\mathcal{F}^{s,x}$ since $\mathbb{P}^{s,x}$-null set will not impact the equality. This will imply (**??**) so that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,T}^{s,x}$-measurable.
At this point, as we have anticipated, we prove (**??**) for a fixed $F = F_t \cap F_{t,T} \in \Pi$. By Lemma **??** we have

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|X_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]|X_t\right]\right],
\end{aligned}$$

where the latter equality holds since $\sigma(X_t) \subset \mathcal{F}_{t,T}$.

Now since $\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_{t,T}\right]$ is $\mathcal{F}_{t,T}$-measurable, the Markov property (??) allows us to substitute the conditional σ-field $\sigma(X_t)$ with $\mathcal{F}_t$ and obtain

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_{t,T}\right]|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_{t,T}\right]|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_{t,T}\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{1}_{F_{t,T}}\mathbb{E}^{s,x}\left[(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_{t,T}\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_F\mathbb{E}^{s,x}\left[(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_{t,T}\right]\right].
\end{aligned}$$

This concludes the proof of (??), therefore (??) holds so that $A^{s,x}_u - A^{s,x}_t$ is $\mathcal{F}^{s,x}_{t,u}$-measurable. This concludes the first part of the proof of Proposition ??.
We pass to the second part of the proof of Proposition ?? where we will show that for given $0 < t < u$ there is an $\mathcal{F}_{t,u}$-measurable r.v. A^t_u such that for every $(s,x) \in [0,t] \times E$, $(A^{s,x}_u - A^{s,x}_t) = A^t_u \quad \mathbb{P}^{s,x}$ a.s.

Similarly to what we did with the quadratic variation in Proposition ??, we start by noticing that for any $x \in E$, since $(A^{t,x}_u - A^{t,x}_t)$ is $\mathcal{F}^{t,x}_{t,u}$-measurable, there exists by Proposition ?? an $\mathcal{F}_{t,u}$-measurable r.v. $a(x,\omega)$ such that

$$a(x,\omega) = A^{t,x}_u - A^{t,x}_t \quad \mathbb{P}^{t,x} \text{ a.s.} \tag{B.10}$$

As in the proof of Proposition ??, we will show the existence of a jointly-measurable version of $(x,\omega) \mapsto a(x,\omega)$.
For every $x \in E$ we define on $\mathcal{F}_{t,u}$ the positive measure

$$\mathbb{Q}^x : F \longmapsto \mathbb{E}^{t,x}\left[\mathbb{1}_F(A^{t,x}_u - A^{t,x}_t)\right] = \mathbb{E}^{t,x}\left[\mathbb{1}_F a(x,\omega)\right]. \tag{B.11}$$

By Lemma ?? and (??), for every $F \in \mathcal{F}_{t,u}$ we have

$$\mathbb{Q}^x(F) = \mathbb{E}^{t,x}\left[\int_t^u N^F_{r^-}dB_r\right], \tag{B.12}$$

and we recall that $\int_t^u N^F_{r^-}dB_r$ does not depend on x. So by Proposition ?? $x \longmapsto \mathbb{Q}^x(F)$ is Borel for any F. Moreover, for any x, $\mathbb{Q}^x \ll \mathbb{P}^{t,x}$. Again by Theorem 58 Chapter V in [?], there exists a version $(x,\omega) \mapsto a(x,\omega)$ measurable for $\mathcal{B}(E) \otimes \mathcal{F}_{t,u}$ of the related Radon-Nikodym densities.

We can now set $A^t_u := a(X_t,\omega)$ which is then an $\mathcal{F}_{t,u}$-measurable r.v.
Given (??) and (??), we have

$$A^t_u = a(X_t,\omega) = a(x,\omega) = A^{t,x}_u - A^{t,x}_t \quad \mathbb{P}^{t,x} \text{ a.s.} \tag{B.13}$$

We now set $s < t$ and $x \in E$ and we want to show that we still have
$A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s. So, as above, we consider $F \in \mathcal{F}_{t,u}$ and, thanks to (??) we compute

$$
\begin{array}{rcl}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)\right] & = & \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r\right] \\
& = & \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r|\mathcal{F}_t\right]\right] \\
& = & \mathbb{E}^{s,x}\left[\mathbb{E}^{t,X_t}\left[\int_t^u N^F_{r^-}dB_r\right]\right] \\
& = & \mathbb{E}^{s,x}\left[\mathbb{E}^{t,X_t}\left[\mathbb{1}_F A^t_u\right]\right] \\
& = & \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_F A^t_u|\mathcal{F}_t\right]\right] \\
& = & \mathbb{E}^{s,x}\left[\mathbb{1}_F A^t_u\right].
\end{array}
\tag{B.14}
$$

Indeed, concerning the fourth equality we recall that, by (**??**), (**??**) and (**??**), we have $\mathbb{E}^{t,x}\left[\int_t^u N^F_{r^-}dB_r\right] = \mathbb{E}^{t,x}\left[\mathbb{1}_F A^t_u\right]$ for all x, so this equality becomes an a.s. equality whatever random variable we plug into x. The third and fifth equalities come from the Markov property (**??**) since $\int_t^u N^F_{r^-}dB_r$ and A^t_u are $\mathcal{F}_{t,T}$-measurable.
Then, adding $\mathbb{P}^{s,x}$-null sets does not change the validity of (**??**), so we have for any $F \in \mathcal{F}^{s,x}_{t,u}$ that $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F A^t_u\right]$.
Finally, since we had shown in the first half of the proof that $A^{s,x}_u - A^{s,x}_t$ is $\mathcal{F}^{s,x}_{t,u}$-measurable, and since A^t_u also has, by construction, the same measurability property, we can conclude that $A^{s,x}_u - A^{s,x}_t = A^t_u$ $\mathbb{P}^{s,x}$ a.s.

Since this holds for any $t \leq u$ and $(s,x) \in [0,t] \times E$, $(A^t_u)_{(t,u)\in\Delta}$ is the desired AF, which ends the proof of Proposition **??**. □

Corollary B.11. *Let M, M' be two square integrable MAFs, let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under $\mathbb{P}^{s,x}$. Then there exists a bounded variation AF with L^1 terminal condition denoted $\langle M, M'\rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M, M'\rangle$ is $\langle M^{s,x}, M'^{s,x}\rangle$. If $M = M'$ the AF $\langle M, M'\rangle$ will be denoted $\langle M\rangle$ and is increasing.*

Proof. If $M = M'$, the corollary comes from the combination of Propositions **??** and **??**, and the fact that the angular bracket of a square integrable martingale is the dual predictable projection of its quadratic variation.
Otherwise, it is clear that $M + M'$ and $M - M'$ are square integrable MAFs, so we can consider the increasing MAFs $\langle M - M'\rangle$ and $\langle M + M'\rangle$. We introduce the AF

$$\langle M, M'\rangle = \frac{1}{4}(\langle M + M'\rangle - \langle M - M'\rangle),$$

which by polarization has cadlag version $\langle M^{s,x}, M'^{s,x}\rangle$ under $\mathbb{P}^{s,x}$. $\langle M, M'\rangle$ is therefore a bounded variation AF with L^1 terminal condition. □

We are now going to study the Radon-Nikodym derivative of an increasing continuous AF with respect to our reference measure dV. The next result can be seen as an extension of Theorem 13 Chapter XV in [**?**] in a non-homogeneous setup.

Proposition B.12. *Let A be a positive, non-decreasing AF absolutely continuous with respect to V, and for any $(s,x) \in [0,T[\times E$ let $A^{s,x}$ be the cadlag version of A under $\mathbb{P}^{s,x}$. There exists a Borel function $h \in \mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x) \in [0,T] \times E$, $A^{s,x} = \int_s^{\cdot} h(r,X_r)dV_r$, in the sense of indistinguishability.*

Proof. We set

$$C_u^t = A_u^t + (V_u - V_t) + (u - t), \tag{B.15}$$

which is an AF with cadlag versions

$$C_t^{s,x} = A_t^{s,x} + V_t + t, \tag{B.16}$$

and we start by showing the statement for A and C instead of A and V. We introduce the intermediary function C so that for any $u > t$ that $\frac{A_u^{s,x} - A_t^{s,x}}{C_u^{s,x} - C_t^{s,x}} \in [0, 1]$; that property will be used extensively in connections with the application of dominated convergence theorem.

Since $A^{s,x}$ is non-decreasing for any $(s, x) \in [0, T] \times E$, A can be taken positive (in the sense that $A_u^t(\omega) \geq 0$ for any $(t, u) \in \Delta$ and $\omega \in \Omega$) by considering A^+ (defined by $(A^+)_u^t(\omega) := A_u^t(\omega)^+$) instead of A.

On $[0, T[$ we set

$$\begin{aligned} K_t &= \liminf_{n\to\infty} \frac{A^t_{t+\frac{1}{n}}}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)} \\ &= \lim_{n\to\infty} \inf_{p\geq n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)} \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}. \end{aligned} \tag{B.17}$$

By positivity, this liminf always exists and belongs to $[0, 1]$ since the sequence belongs to $[0, 1]$. For any $(s, x) \in [0, T] \times E$, since for all $t \geq s$ and $n \geq 0$, $A^t_{t+\frac{1}{n}} = A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s., then $K^{s,x}$ defined by $K_t^{s,x} := \liminf_{n\to\infty} \frac{A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t}{C^{s,x}_{t+\frac{1}{n}} - C^{s,x}_t}$ is a $\mathbb{P}^{s,x}$-version of K, for $t \in [s, T[$.
By Lebesgue Differentiation theorem (see Theorem 12 Chapter XV in [**?**] for a version of the theorem with a general atomless measure), for any (s, x), for $\mathbb{P}^{s,x}$-almost all ω, since $dC^{s,x}(\omega)$ is absolutely continuous with respect to $dA^{s,x}(\omega)$, $K^{s,x}(\omega)$ is a density of $dA^{s,x}(\omega)$ with respect to $dC^{s,x}(\omega)$.

We now show that there exists a Borel function k in $\mathcal{B}([0, T[\times E, \mathbb{R})$ such that under any $\mathbb{P}^{s,x}$, $k(t, X_t)$ is on $[s, T[$ a version of K (and therefore of $K^{s,x}$).
For any $t \in [0, T[$, K_t is measurable with respect to $\bigcap_{n\geq 0} \mathcal{F}_{t,t+\frac{1}{n}} = \mathcal{F}_{t,t}$ by construction, taking into account Definition **??**. So for any $(t, x) \in [0, T] \times E$, by Proposition **??**,

$$K_t = \text{constant} =: k(t, x), \mathbb{P}^{t,x}\text{a.s.} \tag{B.18}$$

For any integers (n, m), we define $k^{n,m}$ by

$$(t,x) \mapsto \mathbb{E}^{t,x}\left[\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right],$$

and k^n by

$$(t,x) \mapsto \mathbb{E}^{t,x}\left[\inf_{p\geq n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right],$$

We start showing that $\tilde{k}^{n,m}$ defined by

$$\begin{array}{rcl} (s,x,t) & \longmapsto & \mathbb{E}^{s\wedge t,x}\left[\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}\right], \\ {[0,T]\times E\times[0,T[} & \longrightarrow & [0,1], \end{array} \quad \text{(B.19)}$$

is jointly Borel.

If we fix t, then by Proposition **??** $(s,x) \longmapsto \mathbb{E}^{s,x}\left[\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}\right]$ is a Borel function, so by composing with $(s,x) \mapsto (s\wedge t, x)$, then $(s,x) \mapsto \tilde{k}^{n,m}(s,x,t)$ is Borel. Moreover, if we fix $(s,x) \in [0,T[\times E$ we show below that $t \mapsto \tilde{k}^{n,m}(s,x,t)$ is continuous, which by Lemma 4.51 in [**?**] implies the joint measurability of $\tilde{k}^{n,m}$.

To show that mentioned continuity property, we first remark that $\tilde{k}^{n,m}(s,x,\cdot)$ is constant on $[0,s]$; moreover $A^{s,x}$ is continuous $\mathbb{P}^{s,x}$ a.s. V is continuous, and the minimum of a finite number of continuous functions remains continuous. Let $t_q \underset{q\to\infty}{\longrightarrow} t$ be a converging sequence in $[s,T[$. Then $\min_{n\leq p\leq m} \frac{A^{s,x}_{t_q+\frac{1}{p}}-A^{s,x}_{t_q}}{A^{s,x}_{t_q+\frac{1}{p}}-A^{s,x}_{t_q}+\frac{1}{p}+(V_{t_q+\frac{1}{p}}-V_{t_q})}$ tends a.s. to $\min_{n\leq p\leq m} \frac{A^{s,x}_{t+\frac{1}{p}}-A^{s,x}_{t}}{A^{s,x}_{t+\frac{1}{p}}-A^{s,x}_{t}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}$ when q tends to infinity. Since for any $s \leq t \leq u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s., then $\frac{A^{t_q}_{t_q+\frac{1}{p}}}{A^{t_q}_{t_q+\frac{1}{p}}+\frac{1}{p}+(V_{t_q+\frac{1}{p}}-V_{t_q})}$ tends a.s. to $\frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}$. All those terms being smaller than one, by dominated convergence theorem, the mentioned convergence also holds under the expectation, hence the announced continuity related to $\tilde{k}^{n,m}$ is established.

Since $k^{n,m}(t,y) = \tilde{k}^{n,m}(t,t,y)$, by composition we can deduce that for any n, m, $k^{n,m}$ is Borel. By the dominated convergence theorem, $k^{n,m}$ tends pointwise to k^n when m goes to infinity so k^n are also Borel for every n. Finally, keeping in mind (**??**) nd (**??**) we have $\mathbb{P}^{t,x}$ a.s.

$$k(t,x) = K_t = \lim_{n\to\infty} \inf_{p\geq n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}.$$

Taking the expectation and again by the dominated convergence theorem, k^n (defined in (**??**)) tends pointwise to k when n goes to infinity so k is Borel.

We now show that, for any $(s,x) \in [0,T] \times E$, $k(\cdot, X_\cdot)$ is a $\mathbb{P}^{s,x}$-version of K on $[s,T[$.
By (**??**), we know that for any $t \in [0,T]$, $x \in E$, we have $K_t = k(t,x) = k(t,X_t)$ $\mathbb{P}^{t,x}$-a.s., and we prove below that for any $t \in [0,T]$, $(s,x) \in [0,t] \times E$, we have $K_t = k(t,X_t)$ $\mathbb{P}^{s,x}$-a.s.
Let $t \in [0,T]$ be fixed. Since A is an AF, for any n, $\frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}$ is $\mathcal{F}_{t,t+\frac{1}{n}}$-measurable.
So the event $\left\{\liminf\limits_{n\rightarrow\infty} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)} = k(t,X_t)\right\}$ belongs to $\mathcal{F}_{t,T}$ and by Markov property (**??**), for any $(s,x) \in [0,t] \times E$, we get

$$\begin{array}{rcl} \mathbb{P}^{s,x}(K_t = k(t,X_t)) & = & \mathbb{E}^{s,x}[\mathbb{P}^{s,x}\left(K_t = k(t,X_t)|\mathcal{F}_t\right)] \\ & = & \mathbb{E}^{s,x}[\mathbb{P}^{t,X_t}\left(K_t = k(t,X_t)\right)] \\ & = & 1. \end{array}$$

For any (s,x), the process $k(\cdot, X_\cdot)$ is therefore on $[s,T[$ a $\mathbb{P}^{s,x}$-modification of K and therefore of $K^{s,x}$.

But it is not yet clear that it provides another density of $dA^{s,x}$ with respect to $dC^{s,x}$, which was defined at (**??**).
Considering that $(t,u,\omega) \mapsto V_u - V_t$ also defines a positive non-decreasing AF absolutely continuous with respect to C, defined in (**??**), we proceed similarly as at the beginning of the proof, replacing the AF A with V.

Let the process K' be defined by

$$K'_t = \liminf_{n\rightarrow\infty} \frac{V_{t+\frac{1}{n}} - V_t}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)},$$

and for any(s,x), let $K'^{s,x}$ be defined on $[s,T[$ by

$$K'^{s,x}_t = \liminf_{n\rightarrow\infty} \frac{V_{t+\frac{1}{n}} - V_t}{A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)}.$$

Then, for any (s,x), $K'^{s,x}$ on $[s,T[$ is a $\mathbb{P}^{s,x}$-version of K', and it constitutes a density of $dV(\omega)$ with respect to $dC^{s,x}(\omega)$ on $[s,T[$, for almost all ω. One shows then the existence of a Borel function k' such that for any (s,x), $k'(\cdot, X_\cdot)$ is a $\mathbb{P}^{s,x}$-version of K' and a modification of $K'^{s,x}$ on $[s,T[$.
So for any (s,x), under $\mathbb{P}^{s,x}$, we can write

$$\left\{\begin{array}{rcl} A^{s,x}_\cdot & = & \int_s^\cdot K^{s,x}_r dC^{s,x}_r \\ V_\cdot - V_s & = & \int_s^\cdot K'^{s,x}_r dC^{s,x}_r \end{array}\right.$$

Now since $dA^{s,x} \ll dV$, for a fixed ω, the set $\{r \in [s,T] | K'^{s,x}_r(\omega) = 0\}$ is negligible with respect to dV so also for $dA^{s,x}(\omega)$ and therefore we can write

$$\begin{array}{rcl} A^{s,x} & = & \int_s^\cdot K^{s,x}_r dC^{s,x}_r \\ & = & \int_s^\cdot \frac{K^{s,x}_r}{K'^{s,x}_r} \mathbb{1}_{\{K'^{s,x}_r \neq 0\}} K'^{s,x}_r dC^{s,x}_r \\ & + & \int_s^\cdot \mathbb{1}_{\{K'^{s,x}_r = 0\}} dA^{s,x}_r \\ & = & \int_s^\cdot \frac{K^{s,x}_r}{K'^{s,x}_r} \mathbb{1}_{\{K'^{s,x}_r \neq 0\}} dV_r, \end{array}$$

where we use the convention that for any two functions ϕ, ψ then $\frac{\phi}{\psi}\mathbb{1}_{\psi \neq 0}$ is defined by by

$$\frac{\phi}{\psi}\mathbb{1}_{\{\psi \neq 0\}}(x) = \left\{ \begin{array}{l} \frac{\phi(x)}{\psi(x)} \text{ if } \psi(x) \neq 0 \\ 0 \text{ if } \psi(x) = 0. \end{array} \right.$$

We set now $h := \frac{k}{k'}\mathbb{1}_{\{k'_r \neq 0\}}$ which is Borel, and clearly for any (s,x), $h(t, X_t)$ is a $\mathbb{P}^{s,x}$-version of $H^{s,x} := \frac{K^{s,x}}{K'^{s,x}}\mathbb{1}_{\{K'^{s,x} \neq 0\}}$ on $[s,T[$. So by Lemma **??**, $H^{s,x}_t = h(t, X_t)$ $dV \otimes d\mathbb{P}^{s,x}$ a.e. and finally we have shown that under any $\mathbb{P}^{s,x}$, $A^{s,x}_\cdot - A^{s,x}_s = \int_s^\cdot h(r, X_r)dV_r$ on $[s,T[$. Without change of notations we extend h to $[0,T] \times E$ by zero for $t = T$. Since $A^{s,x}$ is continuous on $[s,T]$ $\mathbb{P}^{s,x}$-a.s. previous equality extends to T.

□

Corollary B.13. *Let M and M' be two square integrable MAFs and let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under a fixed $\mathbb{P}^{s,x}$. There exists a Borel function $k \in \mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x) \in [0,T] \times E$,*

$$\langle M^{s,x}, M'^{s,x} \rangle = \int_s^\cdot k(r, X_r)dV_r.$$

In particular if M is a square integrable MAF and $M^{s,x}$ its cadlag version under a fixed $\mathbb{P}^{s,x}$, there exists a Borel function $k \in \mathcal{B}([0,T] \times E, \mathbb{R})$ (which can be taken positive) such that for any $(s,x) \in [0,T] \times E$,

$$\langle M^{s,x} \rangle = \int_s^\cdot k(r, X_r)dV_r.$$

Proof. In the case $M = M'$, by Corollary **??** there is an increasing AF $\langle M \rangle$ whose cadlag version under a fixed $\mathbb{P}^{s,x}$ is $\langle M^{s,x} \rangle$. Since by Proposition **??**, under any $\mathbb{P}^{s,x}$, $d\langle M^{s,x} \rangle \ll dV$, then $\langle M \rangle$ is absolutely continuous with respect to V, in the sense of Definition **??**.
The statement can therefore be deduced from Proposition **??**.
The case $M \neq M'$ is obtained by polarization, considering the square integrable MAFs $M + M'$ and $M - M'$. □

C Technicalities related to Section ??

Proof of Proposition **??**.

Since we have $dA \ll dA + dB$ in the sense of stochastic measures with A, B predictable, there exists a predictable positive process K such that

$$A = \int_0^{\cdot} K_s dA_s + \int_0^{\cdot} K_s dB_s, \tag{C.1}$$

up to indistinguishability, see Proposition I.3.13 in [?].
Now there exists $\mathcal{N}$ such that $\mathbb{P}(\mathcal{N}) = 0$ and such that for any $\omega \in \mathcal{N}^c$ we have

$$0 \leq \int_0^{\cdot} K_s(\omega) dB_s(\omega) = \int_0^{\cdot} (1 - K_s(\omega)) dA_s(\omega),$$

so $K(\omega) \leq 1$ $dA(\omega)$ a.e. on $\mathcal{N}^c$. Therefore if we set $E(\omega) = \{t : K_t(\omega) = 1\}$ and $F(\omega) = \{t : K_t(\omega) < 1\}$ then $E(\omega)$ and $F(\omega)$ are disjoint Borel sets and $dA(\omega)$ has all its mass in $E(\omega) \cup F(\omega)$ so we can decompose $dA(\omega)$ within these two sets.

We therefore define the processes

$$\begin{aligned} A^{\perp B} &= \int_0^{\cdot} \mathbb{1}_{\{K_s = 1\}} dA_s; \\ A^B &= \int_0^{\cdot} \mathbb{1}_{\{K_s < 1\}} dA_s. \end{aligned}$$

$A^{\perp B}$ and A^B are both in $\mathcal{V}^{p,+}$, and $A = A^{\perp B} + A^B$. In particular the (stochastic) measures $dA^{\perp B}(\omega)$ and $dA^B(\omega)$ fulfill

$$\begin{aligned} dA^{\perp B}(\omega)(G) &= dA(\omega)(E(\omega) \cap G); \\ dA^B(\omega)(G) &= dA(\omega)(F(\omega) \cap G). \end{aligned} \tag{C.2}$$

We remark $dA^{\perp B} \perp dB$ in the sense of stochastic measures. Indeed, fixing $\omega \in \mathcal{N}^c$, for $t \in E(\omega)$, $K_t(\omega) = 1$, so $\int_{E(\omega)} dA(\omega) = \int_{E(\omega)} dA(\omega) + \int_{E(\omega)} dB(\omega)$ implying that $\int_{E(\omega)} dB(\omega) = 0$. Since for any $\omega \in \mathcal{N}^c$, $dB(\omega)\,(E(\omega)) = 0$ while $dA^{\perp B}(\omega)$ has all its mass in $E(\omega)$, which gives this first result.

Now let us prove $dA^B \ll dB$ in the sense of stochastic measure.
Let $\omega \in \mathcal{N}^c$, and let $G \in \mathcal{B}([0, T])$, such that $\int_G dB(\omega) = 0$. Then

$$\begin{aligned} \int_G dA^B(\omega) &= \int_{G \cap F(\omega)} dA(\omega) \\ &= \int_{G \cap F(\omega)} K(\omega) dA(\omega) + \int_{G \cap F(\omega)} K(\omega) dB(\omega) \\ &= \int_{G \cap F(\omega)} K(\omega) dA(\omega). \end{aligned}$$

So $\int_{G \cap F(\omega)} (1 - K(\omega)) dA(\omega) = 0$, but $(1 - K(\omega)) > 0$ on $F(\omega)$.
So $dA^B(\omega)(G) = 0$. Consequently for every $\omega \in \mathcal{N}^c$, $dA^B(\omega) \ll dB(\omega)$ and so that $dA^B \ll dB$.

Now, since K is positive and $K(\omega) \leq 1 \; dA(\omega)$ a.e. for almost all ω, we can replace K by $K \wedge 1$ which is still positive predictable and verifies (**??**); therefore we can consider that $K \in [0,1]$ for all (ω, t).

We remark that for $\mathbb{P}$ almost all ω the decomposition $A^{\perp B}$ and A^B is unique because of the corresponding uniqueness of the decomposition in the Lebesgue-Radon-Nikodym theorem for each fixed $\omega \in \mathcal{N}^c$.

Since $dA^B \ll dB$, again by Proposition I.3.13 in [**?**], there exists a predictable positive process that we will call $\frac{dA}{dB}$ such that $A^B = \int_0^\cdot \frac{dA}{dB} dB$ and which is only unique up to $dB \otimes d\mathbb{P}$ null sets. □

Proof of Proposition **??**.
If A_1 ans A_2 belong to $\mathcal{V}^{p,+}$, by additivity we can write
$A_1 + A_2 = \int_0^\cdot (\frac{dA_1}{dB} + \frac{dA_2}{dB}) dB + (A_1^{\perp B} + A_2^{\perp B})$. Since previous integral is dominated by dB and the second term is singular to dB, the result follows by the uniqueness property of Proposition **??**.
For general processes in $\mathcal{V}^p$, the result follows from Definition **??**. Indeed we have

$$\begin{aligned} A_1 + A_2 &= (A_1^- - A_1^-) + (A_2^+ - A_2^-) \\ &= (A_1^+ + A_2^+) - (A_1^- + A_2^-), \end{aligned}$$

so by Definition **??**, $\frac{d(A_1+A_2)}{dB} = \frac{d(A_1^+ + A_2^+)}{dB} - \frac{d(A_1^- + A_2^-)}{dB}$ and by the linearity of $\frac{d}{dB}$ which has already been proven at the beginning when A_1 and A_2 are increasing processes, we have

$$\begin{aligned} \frac{d(A_1+A_2)}{dB} &= \frac{dA_1^+}{dB} + \frac{dA_2^+}{dB} - \frac{dA_1^-}{dB} - \frac{dA_2^-}{dB} \\ &= \left(\frac{dA_1^+}{dB} - \frac{dA_1^-}{dB} \right) + \left(\frac{dA_2^+}{dB} - \frac{dA_2^-}{dB} \right), \end{aligned}$$

which by Definition **??** is equal to $\frac{dA_1}{dB} + \frac{dA_2}{dB}$. □

Proposition C.1. *Let M and M' be two local martingales in $\mathcal{H}^2_{loc}$ and let $V \in \mathcal{V}^{p,+}$. We have*

$$\frac{d\langle M \rangle}{dV} \frac{d\langle M' \rangle}{dV} - \left(\frac{d\langle M, M' \rangle}{dV} \right)^2 \geq 0 \quad dV \otimes d\mathbb{P} \, a.e.$$

Proof. The proof is very similar to the Cauchy-Schwarz inequality proof except that we have to be careful with "almost everywhere equalities".
Let $q \in \mathbb{Q}$. Since $\langle M + qM' \rangle$ is an increasing process starting at zero, then by Proposition **??**, we have $\frac{d\langle M + qM' \rangle}{dV} \geq 0 \quad dV \otimes d\mathbb{P}$ a.e.
By the linearity property stated in Proposition **??**, we have

$$\frac{d\langle M + qM' \rangle}{dV} = \frac{d\langle M \rangle}{dV}(\omega, t) + 2q \frac{d\langle M, M' \rangle}{dV} + q^2 \frac{d\langle M' \rangle}{dV} \quad dV \otimes d\mathbb{P} \text{ a.e.}$$

So there exists a set $\mathcal{N}^q$ such that $dV \otimes d\mathbb{P}(\mathcal{N}^q) = 0$ and

$$\forall (\omega, t) \in (\mathcal{N}^q)^c : \; \frac{d\langle M \rangle}{dV}(\omega, t) + 2q \frac{d\langle M, M' \rangle}{dV}(\omega, t) + q^2 \frac{d\langle M' \rangle}{dV}(\omega, t) \geq 0.$$

Now if we set $\mathcal{N} = \bigcup_{q\in\mathbb{Q}} \mathcal{N}^q$ we still have that $dV \otimes d\mathbb{P}(\mathcal{N}) = 0$ and for all $(\omega, t) \in (\mathcal{N})^c$,

$$x \longmapsto \frac{d\langle M\rangle}{dV}(\omega,t) + 2x\frac{d\langle M, M'\rangle}{dV}(\omega,t) + x^2\frac{d\langle M'\rangle}{dV}(\omega,t)$$

is a polynome which is positive on $\mathbb{Q}$ and therefore on $\mathbb{R}$ by continuity of polynomes. So by looking at its discriminant, we deduce that

$$\forall(\omega,t) \in \mathcal{N}^c : \, 4\left(\frac{d\langle M, M'\rangle}{dV}(\omega,t)\right)^2 - 4\frac{d\langle M\rangle}{dV}(\omega,t)\frac{d\langle M'\rangle}{dV}(\omega,t) \le 0.$$

□

Proof of Proposition **??**.
Since the angular bracket $\langle M\rangle$ of a square integrable martingale M always belongs to $\mathcal{V}^{p,+}$, by Proposition **??**, we can consider the processes $\langle M\rangle^V$ and $\langle M\rangle^{\perp V}$; in particular there exists a predictable process K with values in $[0,1]$ such that

$$\begin{cases} \langle M\rangle^V &= \int_0^\cdot \mathbb{1}_{\{K_s<1\}} d\langle M\rangle_s \\ \langle M\rangle^{\perp V} &= \int_0^\cdot \mathbb{1}_{\{K_s=1\}} d\langle M\rangle_s. \end{cases}$$

We can then set $M^V = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dM_s$ and $M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dM_s$ which are well-defined because K is predictable, and therefore $\mathbb{1}_{\{K_t<1\}}$ and $\mathbb{1}_{\{K_t=1\}}$ are also predictable. $M^V, M^{\perp V}$ are in $\mathcal{H}^2_0$ because their angular brackets are both bounded by $\langle M\rangle_T \in L^1$.

Since K takes values in $[0,1]$, we have

1. $M^V + M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dM_s + \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dM_s = M$;
2. $\langle M^V\rangle = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} d\langle M\rangle_s = \langle M\rangle^V$;
3. $\langle M^{\perp V}\rangle = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} d\langle M\rangle_s = \langle M\rangle^{\perp V}$;
4. $\langle M^V, M^{\perp V}\rangle = \int_0^\cdot \mathbb{1}_{\{K_s<1\}}\mathbb{1}_{\{K_s=1\}} d\langle M\rangle_s = 0$;

those properties imply all the conditions (**??**). □

Lemma C.2. *Let M_1, M_2 be in $\mathcal{H}^2_0$ and $|\langle M_1, M_2\rangle|$ denote the total variation of $\langle M_1, M_2\rangle$. Then*

$$\begin{cases} d|\langle M_1, M_2\rangle| &\ll d\langle M_1\rangle \\ d|\langle M_1, M_2\rangle| &\ll d\langle M_2\rangle. \end{cases}$$

Proof. We will prove here the first relation since the second one holds by symmetry. By Kunita-Watanabe theorem, we know that for any two square integrable martingales M_1, M_2 there exists a predictable process $H \in L^2(M_1)$ and a square integrable martingale N strongly orthogonal to M_1 such that

$$M_2 = \int_0^\cdot H dM_1 + N,$$

see for example Corollaries 1 and 2 in Chapter IV.3 of [?].
Therefore $\langle M_1, M_2\rangle = \int_0^\cdot Hd\langle M_1\rangle$ and $|\langle M_1, M_2\rangle| = \int_0^\cdot |H|d\langle M_1\rangle$ implying that $d|\langle M_1, M_2\rangle| \ll d\langle M_1\rangle$.

□

Proof of Proposition **??**.
Let M_1 and M_2 be in $\mathcal{H}^{2,V}$. By Lemma **??**, $d|\langle M_1, M_2\rangle| \ll d\langle M_1\rangle \ll dV$. So since $\langle M_1 + M_2\rangle = \langle M_1\rangle + 2\langle M_1, M_2\rangle + \langle M_2\rangle$, then $d\langle M_1 + M_2\rangle \ll dV$ which shows that $\mathcal{H}^{2,V}$ is a vector space.

If M_1 and M_2 are in $\mathcal{H}^{2,\perp V}$, then since $d|\langle M_1, M_2\rangle| \ll d\langle M_1\rangle$ we can write $|\langle M_1, M_2\rangle| = \int_0^\cdot \frac{d|\langle M_1,M_2\rangle|}{d\langle M_1\rangle} d\langle M_1\rangle$ which is almost surely singular with respect to dV since M_1 belongs to $\mathcal{H}^{2,\perp V}$. So, by the bilinearity of the angular bracket $\mathcal{H}^{2,\perp V}$ is also a vector space.

Finally if $M_1 \in \mathcal{H}^{2,V}$ and $M_2 \in \mathcal{H}^{2,\perp V}$ then $d|\langle M_1, M_2\rangle| \ll d\langle M_1\rangle \ll dV$ but we also have seen that if $d\langle M_2\rangle$ is singular to dV then so is $d|\langle M_1, M_2\rangle| \ll d\langle M_2\rangle$. For fixed ω, a measure being simultaneously dominated and singular with respect to to $dV(\omega)$ is necessarily the null measure, so $d|\langle M_1, M_2\rangle| = 0$ as a stochastic measure. Therefore M_1 and M_2 are strongly orthogonal, which implies in particular that M_1 and M_2 are orthogonal in $\mathcal{H}^2_0$.

So we have shown that $\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sublinear-spaces of $\mathcal{H}^2_0$ but we also know that $\mathcal{H}^2_0 = \mathcal{H}^{2,V} + \mathcal{H}^{2,\perp V}$ thanks to Proposition **??**, so

$$\mathcal{H}^2_0 = \mathcal{H}^{2,V} \oplus^\perp \mathcal{H}^{2,\perp V}.$$

This implies that $\mathcal{H}^{2,V} = (\mathcal{H}^{2,\perp V})^\perp$ and $\mathcal{H}^{2,\perp V} = (\mathcal{H}^{2,V})^\perp$ and therefore that these spaces are closed. So they are sub-Hilbert spaces. We also have shown that they were strongly orthogonal spaces, in the sense that any $M^1 \in \mathcal{H}^{2,V}$, $M^2 \in \mathcal{H}^{2,\perp V}$ are strongly orthogonal. □

We recall here a classical notion of martingale theory.

Definition C.3. *Let $p \in [1, \infty[$, a subset $\mathcal{H} \subset \mathcal{H}^p$ will be called a* ***stable subspace*** *if it is a closed sublinear space such that for any $M \in \mathcal{H}$, any event $A \in \mathcal{F}_0$ and any stopping time τ then $\mathbb{1}_A M^\tau \in \mathcal{H}$.*

Proposition C.4. *$\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are stable subspaces of $\mathcal{H}^2$.*

Proof. Since by Proposition **??**, $\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are closed sub-linear spaces of $\mathcal{H}^2$, then Proposition 4.3 in [?] states that the result will follow if we show that for any M in $\mathcal{H}^{2,V}$ (respectively in $\mathcal{H}^{2,\perp V}$) and H in $L^2(M)$ then $\int_0^\cdot HdM$ is in $\mathcal{H}^{2,V}$ (respectively in $\mathcal{H}^{2,\perp V}$). So let $M \in \mathcal{H}^{2,V}$ (respectively in $\mathcal{H}^{2,\perp V}$) and H in $L^2(M)$, then $\langle \int_0^\cdot HdM\rangle = \int_0^\cdot H^2 d\langle M\rangle$ therefore if $d\langle M\rangle$ is dominated by dV (respectively singular to dV), so is $d\langle \int_0^\cdot HdM\rangle$. □

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