Advanced rigid body dynamics: tensor invariants and problem of hamiltonization — I.

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1. Rolling of a body on a plane

Equations of motion
Consider the problem of a rigid body rolling on a horizontal plane under the condition that at the point of contact there is no slipping and spinning at the same time. The configuration space of the system under consideration is the product $\mathcal{M} = \mathbb{R}^2 \times SO(3)$, where the first factor describes the position of the point of contact on the plane and the second factor describes the orientation of the body. We introduce two coordinate systems (see Fig. 1):

- a fixed system OXYZ whose origin is located at some point of the plane and the axis OZ is perpendicular to the plane;
- a movable system Cxyz whose origin C coincides with the center of mass of the body and whose axes are directed along the principal axes of inertia of the body.

![Fig. 1. A body on a plane](image)
Let \( \alpha, \beta, \gamma \) be the projections of the unit vectors of a fixed space (i.e. the unit vectors of the axes \( OXYZ \)) onto the movable axes \( Cxyz \) and \( \mathbf{R}_p = (x, y) \) be the coordinates of the point of contact on the plane. Introducing the orthogonal matrix

\[
\mathbf{Q} = \begin{pmatrix}
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2 \\
\alpha_3 & \beta_3 & \gamma_3
\end{pmatrix} \in SO(3),
\]

we see that the pair \( (\mathbf{R}_p, \mathbf{Q}) \in \mathbb{R}^2 \otimes SO(3) \) uniquely determines the position of the body. Let \( \mathbf{\omega} = (\omega_1, \omega_2, \omega_3), \mathbf{v} = (v_1, v_2, v_3) \) be the projections of the angular velocity and the velocity of the body’s center of mass onto the movable axes \( Cxyz \). Since there is no slipping at the point of contact, the velocity of the body at this point vanishes, i.e.,

\[
\mathbf{v} + \mathbf{\omega} \times \mathbf{r} = 0, \quad (1)
\]

where \( \mathbf{r} \) is the vector from the center of mass of the body \( C \) to the point of contact \( P \). The absence of spinning implies that the projection of the angular velocity onto the normal to the plane vanishes:

\[
(\mathbf{\omega}, \gamma) = 0. \quad (2)
\]
We obtain a system of equations governing the rolling of a rigid body on a plane without slipping and spinning:

\[
\dot{I}\omega = I\omega \times \omega - m r \times (\omega \times \dot{r}) + M_P + \lambda_0 \gamma,
\]

\[
\lambda_0 = - \frac{(I^{-1} \gamma, I\omega \times \omega - m r \times (\omega \times \dot{r}) + M_P)}{(\gamma, I^{-1} \gamma)},
\]

\[
\dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \quad \dot{\gamma} = \gamma \times \omega,
\]

\[
\dot{x} = (\alpha, \dot{r}), \quad \dot{y} = (\beta, \dot{r}),
\]

where \(\tilde{I} = I + mr^2E - mr \otimes r\) is the tensor of inertia relative to the point of contact and \(M_P = M_C + F_C \times r\) is the moment of the external forces relative to the point of contact. In addition, these equations must be supplemented with algebraic relations connecting the normal with the vector \(r\) with the help of the Gauss projection:

\[
\gamma = - \frac{\partial f}{\partial r} \begin{vmatrix} \partial f \\ \partial r \end{vmatrix}^{-1} \frac{\partial f}{\partial r},
\]

where \(f(r) = 0\) is the equation of the body's surface.
In what follows we shall consider the case where the external forces are potential with the potential depending only on $\gamma$, i.e. $U = U(\gamma)$, such that $M_P = \gamma \times \frac{\partial U}{\partial \gamma}$. The equations governing the evolution of the vectors $\omega$, $\gamma$ separate in this case:

$$\dot{\mathbf{I}}\omega = \mathbf{I}\omega \times \omega - m\mathbf{r} \times (\omega \times \dot{\mathbf{r}}) + \gamma \times \frac{\partial U}{\partial \gamma} + \lambda_0 \gamma, \quad \dot{\gamma} = \gamma \times \omega.$$  

(5)

Thus, for example, during the rolling of the body in a field of gravity on a horizontal plane

$$U(\gamma) = -mg(\gamma, \mathbf{r}(\gamma)),$$

where $m$ is the mass of the body and $g$ is the free fall acceleration.

The reduced system (5) is analogous to the Euler – Poisson equations and their generalizations in rigid body dynamics [16]. From the known solutions $\omega(t), \gamma(t)$ of this system the law of change of the other variables $\alpha, \beta, x, y$ is obtained according to (3) by quadratures.

Therefore, the properties of the system (5) determine considerably the properties of the dynamics of the general system.
Rolling of a body on a sphere

The configuration space of the system is $\mathcal{M} = S^2 \times SO(3)$, where the first factor corresponds to a possible position of the point of contact $P$ and the second factor corresponds to the orientation of the body.

Fig. 2.
In the case where the moment of external forces \( M_p \) depends only on the normal \( \gamma \), the equations governing the evolution of \( \omega, \gamma \) separate:

\[
\tilde{I}\omega = \tilde{I}\omega \times \omega - m r \times (\omega \times r) + M_p + \lambda_k \gamma,
\]

where \( \lambda_k \) is determined higher, where \( \Gamma \) is a matrix with the elements \( \Gamma_{ij} = \delta_{ij} - k \frac{\partial r_i}{\partial \gamma_j} \).

A generalization of the moment of potential external forces in this case is

\[
M_p = \gamma \times \left( (\Gamma^{-1})^T \frac{\partial U}{\partial \gamma} \right).
\]

**First integrals**
The system of equations (6), as in the case of a plane, admits a pair of general interals:

\[
F_0 = \gamma^2 = 1, \quad F_1 = (\omega, \gamma) = 0,
\]

and, in addition, in the case of potential external forces at the level \( F_1 = 0 \), the energy is conserved

\[
E = \frac{1}{2} (\omega, \tilde{I}\omega) + U(\gamma).
\]
The Poincaré map and the hierarchy of dynamics

The Poincaré map
In many cases for a numerical analysis and illustration of the behavior of the trajectories of both the system (5) and the system (6) it is convenient to use the method of the Poincaré cross-sections (which in this case turns out to be two-dimensional), therefore, we consider here in more detail its construction for the systems under consideration. First of all, we restrict the systems to a four-dimensional manifold of the level of general integrals:

\[ \mathcal{M}^4 = \{ (\omega, \gamma) \mid \gamma^2 = 1, (\omega, \gamma) = 0 \}, \]  

and obtain a four-dimensional flow with the energy integral \( \tilde{E} = E \big|_{\mathcal{M}^4} \). In performing numerical analysis for parametrization of this manifold we shall use special variables \( (L, G, l, g) \) analogous to the Andoyer variables, which are systematically applied for these purposes in rigid body dynamics [16]. In this case:

\[ \omega_1 = \sqrt{G^2 - L^2 \sin l}, \quad \omega_2 = \sqrt{G^2 - L^2 \cos l}, \quad \omega_3 = L, \]
\[ \gamma_1 = \frac{L}{G} \cos g \sin l + \sin g \cos l, \quad \gamma_2 = \frac{L}{G} \cos g \cos l - \sin g \sin l, \quad \gamma_3 = -\sqrt{1 - \frac{L^2}{G^2}} \cos g. \]

where \( l, g \in [0, 2\pi) \) are the angular variables and \( L, G \) satisfy the obvious inequality

\[ -1 \leq \frac{L}{G} \leq 1. \]
Then we fix the level of the energy \( \tilde{E}(L, G, l, g) = h \), thereby we obtain a one-parameter family of three-dimensional flows on the manifolds

\[ \mathcal{M}^3_h = \{ x \in \mathcal{M}^4 \mid \tilde{E}(x) = h \}, \]

and as the secant for this flow we choose a manifold given by the relation

\[ g = g_0 = \text{const}. \]

Thus, integrating numerically the systems under consideration, we finally obtain a two-parameter family of point two-dimensional maps:

\[ \Phi_{h, g_0} : \mathcal{M}^2_{h, g_0} \to \mathcal{M}^2_{h, g_0}, \]

\[ \mathcal{M}^2_{h, g_0} = \{ x \in \mathcal{M}^4 \mid \tilde{E}(x) = h, g(x) = g_0 \}. \]
The hierarchy of dynamical behavior

For the system of interest a complete set of (tensor) invariants can additionally include an invariant measure and another first integral. Depending on combinations of these additional conservation laws, a hierarchy of possible types of dynamical behavior arises (for the case of usual rolling see [13, 23]). We describe it schematically, omitting details and denoting for brevity the invariant measure by $\mu$ and the additional integral by $F_2$. The corresponding Poincaré cross-sections are shown in Figs. 3–8.

$\exists F_2, \mu$. The system is integrable by the generalized Euler – Jacobi theorem (see, e.g., [5]); nonsingular invariant manifolds are tori, and all trajectories are regular and are either quasi-periodic or periodic coils of the invariant tori. In neighborhoods of the tori the system is conformally Hamiltonian. A typical view of the Poincaré cross-section in this case is shown in Fig. 3. It can be seen that the phase portrait is foliated into invariant curves and fixed points.

$\exists F_2, \nexists \mu$. The phase space is foliated into two-dimensional manifolds (in the well-known example two-dimensional tori). All trajectories are regular, but on the invariant manifolds (tori) there can exist limit cycles. The corresponding Poincaré map (see Fig. 4) does not differ materially in its appearance from the analogous map of the integrable Hamiltonian system. The only difference is that in addition to the family of tori filled with quasi-periodic coils there coexist tori on which there are limit cycles (in this case one can see limit cycles of period 4 and 6) (one of the examples was considered in [6]).
**Fig. 3.** The Poincaré cross-section for $h = 10$ and $g_0 = 0$ in the problem of rolling of a balanced dynamically asymmetric ball on a plane ($I_1 = 1$, $I_2 = 2$, $I_3 = 3$, $m = 1$, $R = 3$).

**Fig. 4.** The Poincaré cross-section for $h = 1$ and $g_0 = \pi$ in the problem of rolling of an unbalanced dynamically asymmetric ball on a plane in the absence of a gravitational field ($I_1 = 2.6$, $I_2 = 1.2$, $I_3 = 1.8$, $m = 1$, $R = 3$, $c = (0, 0, 2.9)$). The discontinuous curves correspond to individual trajectories on the torus with limit cycles.
∀F_2, ∃μ. The properties of the trajectories of the system are analogous to those of the trajectories in nonintegrable two-degree-of-freedom Hamiltonian systems (although the law of motion along the trajectories is different). There coexist invariant “KAM tori” and stochastic layers near unstable periodic solutions. A typical Poincaré cross-section in this case is shown in Fig. 5. It does not differ in its properties from the cross-section of the nonintegrable Hamiltonian system. For example, the presence of a (conservative) stochastic layer points to the absence of an additional first integral.

Fig. 5. The Poincaré cross-section for \( h = 100 \) and \( g_0 = \pi \) in the problem of rolling of an ellipsoid with special mass distribution on a plane in a gravitational field (\( I_1 = 40, I_2 = 64, I_3 = 100, m = 1, a_1 = 7, a_2 = 8, a_3 = 10, a_g = 9.8 \), see Section 26).

Fig. 6. The Poincaré cross-section for \( h = 4 \) and \( g_0 = \frac{\pi}{2} \) in the problem of rolling of an unbalanced dynamically asymmetric ball with a rotor on a plane (\( I_1 = 1, I_2 = 2, I_3 = 3, m = 1, R = 3, c = (2, 2, 3), S = (1, -0.35, 0) \).
On the level surface of the energy integral $\mathcal{M}^3_h$, a sufficiently general dissipative system arises. The behavior of the trajectories can either be regular with simple attractors (fixed points, limit cycles, and tori) or admit a typical “dissipative chaos” with quasi-attractors, Lorenz type strange attractors etc. There is an extensive literature on this subject; see, for example, [38, 52, 67]. Some possible Poincaré cross-sections in this case are presented in Figs. 6–8. In these figures, the largest cluster of points (nearly black regions) corresponds to simple attractors — fixed points, cycles and a limit curve; lighter colored clusters on large areas correspond to stochastic layers (Fig. 7), a strange attractor (on the left in Fig. 8) and a strange repeller (on the right in Fig. 8, it was obtained by integrating in the opposite direction in time). Furthermore, Fig. 7 shows invariant KAM curves which include conservative fixed points and cycles. The attractors and repellers on these cross-sections are transformed into each other by a transformation (involution) of the velocities that is discussed in the Appendix.
Fig. 7. The Poincaré cross-section for $h = 50$ and $g_0 = \frac{\pi}{2}$ in the problem of rolling of an unbalanced dynamically asymmetric ball on a plane in a gravitational field ($I_1 = 1$, $I_2 = 2$, $I_3 = 3$, $m = 1$, $R = 3$, $c = (1, 1.4, 0)$, $a_g = 9.8$).

Fig. 8. The Poincaré cross-section for $h = 50$ and $g_0 = \frac{\pi}{2}$ in the problem of rolling of an unbalanced dynamically asymmetric ball on a plane in a gravitational field ($I_1 = 1$, $I_2 = 2$, $I_3 = 3$, $m = 1$, $R = 3$, $c = (1, 1.5, 0.5)$, $a_g = 9.8$).
The results of the existence of $F_2$ and $\mu$ are presented in the form of tables showing the geometrical and dynamical properties of the bodies (such a description of nonholonomic systems was first presented in [13, 23]). This makes it possible not only to visualize the entire hierarchy of dynamics of a body rolling on a plane but also to predict new results for “white spots” in these tables. Most results of these tables are contained in [20]. In these tables, grey denotes the existence of this tensor invariant for corresponding geometrical and dynamical restrictions; those cases for which the date they were found is not indicated in the cells are first presented in this paper.

In addition to the first integrals and the invariant measure, the system can possess other tensor invariants [49], for example, symmetry fields, Poisson structure etc. In some cases these tensor invariants are possible only after an appropriate change of time (in particular, after a change of time the conformally Hamiltonian systems admit a Poisson structure). In what follows, it will be shown that for the systems (5) and (6) the existence of an invariant measure is equivalent to the possibility of representation in a conformally Hamiltonian form.
Conditions for the existence of an invariant measure and conformally hamiltonian representation

First of all, we recall that the system \( \dot{x} = v(x) \) possesses an invariant measure if there exists a strictly positive definite function \( \rho(x) \) satisfying the Liouville equation

\[
\text{div} \left( \rho(x)v(x) \right) = \sum_k \frac{\partial \rho(x)v_k(x)}{\partial x_k} = 0.
\]

We shall assume that \( \rho(x) \) is analytic (i.e. the invariant measure is also analytic).

As is well known, if two different invariant measures \( \rho_1(x) d^n x \) and \( \rho_2(x) d^n x \) are given for the system \( \dot{x} = v(x) \), then the ratio of their densities is a first integral:

\[
\frac{\rho_1(x)}{\rho_2(x)} = \text{const}.
\]

To prove this, it suffices to use the formula \( (\ln \rho_1)' = (\ln \rho_2)' = - \text{div} \, v(x) \), which follows from the Liouville equation \( \text{div} \, \rho \, v = 0 \).
Thus, arbitrariness in the choice of density of the invariant measure (defined for the fixed variables) is restricted only to multiplication by an arbitrary (analytical) function of the first integrals of the system.

Under a change of variables $\mathbf{x} = \mathbf{x}(\mathbf{y})$ the density of the invariant measure is multiplied by the Jacobian of the transformation:

$$
\tilde{\rho}(\mathbf{y}) = \rho(\mathbf{x}) \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}.
$$

General and the most robust conditions for the existence of an invariant measure for nonholonomic systems were derived in [47]. For Eqs. (5) and (6) one can obtain more robust conditions for the existence of an analytical measure and find cases where this measure does not obviously exist. We note that results analogous to those presented below are difficult to establish for the classical rolling problem. The cases of existence of invariant measures which are considered in [13] do not prevent their existence in more general situations.
Table 1. The hierarchy of dynamics of a body rolling without spinning on a plane

<table>
<thead>
<tr>
<th>tensor of inertia</th>
<th>dynamically asymmetric case $I_1 \neq I_2 \neq I_3 \neq I_1$</th>
<th>axisymmetrical symmetry $I_1 = I_2, U = U(\gamma_3)$</th>
<th>complete dynamical symmetry $I_1 = I_2 = I_3 = \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>surface of the body</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>geometrical and dynamical restrictions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>the center of mass coincides with the geometrical center</td>
<td>the center of mass does not coincide with the geometrical center</td>
<td>geometrical and dynamical axes coincide ( I_k = m \det B(\alpha + a_k^2) \times (\beta + a_k^2)^{-1} ) for ( k = 1, 2, 3 )</td>
<td>body of revolution</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>geometrical and dynamical axes coincide and contain the center of mass</td>
</tr>
<tr>
<td>( I \neq m B + \mu E )</td>
<td>( I = m B + \mu E )</td>
<td>( g_1(\gamma_3) )</td>
<td></td>
</tr>
<tr>
<td>measure</td>
<td>((\gamma, J^{-1} \gamma)^{-\frac{1}{2}})\ Borisov, Mamaev, 2005</td>
<td>does not exist</td>
<td>( g_1(\gamma_3)^{1/2} g_2(\gamma_3) ) Borisov, Mamaev, 2008</td>
</tr>
<tr>
<td>additional integral</td>
<td>((J \omega \times \gamma, J \omega \times \gamma)) Borisov, Mamaev, 2005 ((\omega \times \gamma, J \omega \times \gamma)) Borisov, Mamaev, 2008</td>
<td>does not exist in the general case</td>
<td>( g_1^{1/2}(\gamma_3) \omega_3 ) Borisov, Mamaev, 2008</td>
</tr>
<tr>
<td>integrable addition of a gyrostat</td>
<td>possible; Borisov, Mamaev, 2005</td>
<td>unknown</td>
<td>possible — along the axis of dynamical symmetry</td>
</tr>
<tr>
<td>generalizations and remarks</td>
<td>integrable addition of the Bruns field is possible; (-\frac{E}{4}(J \gamma, \gamma)) Borisov, Mamaev, 2005</td>
<td>unknown but the measure is preserved</td>
<td>unknown</td>
</tr>
</tbody>
</table>

\[ g_1(\gamma_3) = l_1 \gamma_3^2 + l_3 (1 - \gamma_3^2) + m(r, \gamma)^2, \quad g_2(\gamma_3) = l_1 + m r^2 \]

\( \tilde{I} = I + m r^2 E - m r \otimes r \) is the tensor of inertia relative to the point of contact

\[ J = \text{diag}(l_1 + m R^2, l_2 + m R^2, l_3 + m R^2), \]

\( R \) is the radius of the ball, \( a \) is the displacement of the center of mass,

\[ B = \text{diag}(a_1^2, a_2^2, a_3^2), \quad \chi_b = (\beta^2 + \beta \text{ Tr } B + \Delta_b)(r, \gamma)^2, \]

\[ \Delta_b = \det B(\alpha + \text{ Tr } B^{-1}), \quad f(\beta) = (\beta + a_1^2)(\beta + a_2^2)(\beta + a_3^2) \]

\(^1\) According to the results of numerical construction of the Poincaré cross-section for some fixed values of the parameters.
The conditions for existence of an invariant measure on a plane

Let us solve Eqs. (5) for the derivatives and represent them in symbolic form:

\[
\dot{\omega} = W(\omega, \gamma), \quad \dot{\gamma} = V(\omega, \gamma).
\]

at the level \( \gamma^2 = 1, (\omega, \gamma) = 0 \) we obtain the identity

\[
\sum_{i=1}^{3} \frac{\partial W_i}{\partial \omega_i} = (\xi(\gamma), \omega) = (\gamma \times \xi, \gamma \times \omega).
\]

The Liouville equation \( \text{div}(\rho \mathbf{v}) = 0 \) for the density of the invariant measure \( \rho(\omega, \gamma) \, d\omega \, d\gamma \) can be represented as

\[
\frac{1}{\rho} \left( \left( \frac{\partial \rho}{\partial \gamma}, \mathbf{v} \right) + \left( \frac{\partial \rho}{\partial \omega}, \mathbf{W} \right) \right) = -\sum_{i=1}^{3} \frac{\partial W_i}{\partial \omega_i},
\]

\[
\left( \frac{1}{\rho} \frac{\partial \rho}{\partial \gamma} + \gamma \times \xi(\gamma), \gamma \times \omega \right) + \left( \frac{1}{\rho} \frac{\partial \rho}{\partial \omega}, \mathbf{W} \right) = 0,
\]

where \( \xi(\gamma), W(\omega, \gamma) \) are given vector functions. Throughout our discussion we shall assume the density \( \rho > 0 \) to be an analytical function on \( \mathcal{M}^4 \) (8).
Proposition 1. The system (5) admits an invariant measure which is analytical in a neighborhood of the manifold $\mathcal{M}_0 = \{(\omega, \gamma) \mid \omega = 0\}$ if and only if

$$G(\gamma) + \left( \gamma, \frac{\partial G}{\partial \gamma} \right) = \sum_{i=1}^{3} \frac{\partial \xi_i(\gamma)}{\partial \gamma_i}, \quad G(\gamma) = (\gamma, \xi(\gamma)).$$

(11)

Conformally Hamiltonian representation
We define the Lagrange function of the system (5) in the usual manner:

$$L = T - U(\gamma_0),$$

$$T = \frac{1}{2}(\omega, \tilde{I}\omega) = \frac{1}{2}(G_{uu}\dot{u}^2 + 2G_{uv}\dot{u}\dot{v} + G_{vv}\dot{v}^2),$$

$$G_{uu} = (\Omega_u, \tilde{I}_o \Omega_u), \quad G_{uv} = (\Omega_u, \tilde{I}_o \Omega_v), \quad G_{vv} = (\Omega_v, \tilde{I}_o \Omega_v).$$

(12)

Let us introduce the generalized momenta of the system using the formulae

$$P_u = \frac{\partial L}{\partial \dot{u}} = G_{uu}\dot{u} + G_{uv}\dot{v}, \quad P_v = \frac{\partial L}{\partial \dot{v}} = G_{uv}\dot{u} + G_{vv}\dot{v}.$$

By the Chaplygin theorem (for more details) the following proposition holds:
Proposition 2. If the system admits an invariant measure, the equations of motion can be represented in conformally Hamiltonian form:

\[ \dot{u} = N \frac{\partial H}{\partial p_u}, \quad \dot{v} = N \frac{\partial H}{\partial p_v}, \quad \dot{p}_u = -N \frac{\partial H}{\partial u}, \quad \dot{p}_v = -N \frac{\partial H}{\partial v}, \]

\[ p_u = NP_u, \quad p_v = NP_v, \quad H = E|_{\dot{u}, \dot{v} \rightarrow p_u, p_v} = \frac{1}{2} (G^{-1} p, p) + U(u, v), \]

where \( p = (p_u, p_v) \) and the reducing multiplier is calculated from the formula

\[ N = \frac{\tilde{\rho}(u, v)}{\det G} = \rho(\gamma_o) \frac{(\gamma_o, \gamma_u \times \gamma_v)^2}{\det G}, \quad G = \begin{vmatrix} G_{uu} & G_{uv} \\ G_{uv} & G_{vv} \end{vmatrix}. \]
The body of revolution with axial dynamical symmetry \((l_1 = l_2 \neq l_3)\)

Consider a dynamically symmetric rigid body, that is \(l_1 = l_2\) whose surface is also axisymmetric, the dynamical and geometrical axes of symmetry coincide. The equation describing the surface of the body of revolution is represented as \(f(r_1^2 + r_2^2, r_3) = 0\). We make use of the following parametrization of the vector \(r(\gamma)\):

\[
\mathbf{r} = (f_1(\gamma_3)\gamma_1, f_1(\gamma_3)\gamma_2, f_2(\gamma_3)), \quad (15)
\]

here, the functions \(f_1(\gamma_3), f_2(\gamma_3)\) are determined by the specific type of the body’s surface but are not independent. Using the relation \((\dot{r}, \gamma) = 0\), we obtain that the functions \(f_1(\gamma_3), f_2(\gamma_3)\) satisfy the equation:

\[
\frac{df_2}{d\gamma_3} = f_1 - \frac{1 - \gamma_3^2}{\gamma_3} \frac{df_1}{d\gamma_3}.
\]

Fig. 9. The body of revolution.
In this case, assuming that the density of the invariant measure depends only on $\gamma_3$, Eq. (12) can be represented as:

$$
\rho'(\gamma_3) - \rho(\gamma_3) \left( \frac{g'_1(\gamma_3)}{2g_1(\gamma_3)} + \frac{g'_2(\gamma_3)}{g_2(\gamma_3)} \right) = 0
$$

(16)

$$
g_1 = l_1\gamma_3^2 + l_3(1 - \gamma_3^2) + m(r, \gamma)^2, \quad g_2 = l_1 + m r^2.
$$

Solving this equation, we find that the invariant measure has the form [20]:

$$
\sqrt{g_1 g_2} \ d\omega \ d\gamma.
$$

Conformally Hamiltonian representation

Using the relation (14), the equations of motion can be represented in conformally Hamiltonian form

$$
N^{-1} \dot{M} = M \times \frac{\partial H}{\partial M} + \gamma \times \frac{\partial H}{\partial \gamma}, \quad N^{-1} \dot{\gamma} = \gamma \times \frac{\partial H}{\partial \gamma}
$$

where the reducing multiplier $N = g_1^{-1/2}$ and the corresponding variables $M$ are explicitly expressed in terms of $\omega, \gamma$ using the formula

$$
M_1 = \frac{g_2(1 - \gamma_3^2)\omega_1 + (g_2 - g_1)\gamma_1\gamma_3\omega_3}{(1 - \gamma_3^2)\sqrt{g_1}},
$$

$$
M_2 = \frac{g_2(1 - \gamma_3^2)\omega_2 + (g_2 - g_1)\gamma_2\gamma_3\omega_3}{(1 - \gamma_3^2)\sqrt{g_1}}, \quad M_3 = \sqrt{g_1}\omega_3.
$$

(17)
The Hamiltonian in the orbit \( M_e \) can be represented as:

\[
H = \frac{g_2 M_3^2 + g_1 (M_1 \gamma_2 - M_2 \gamma_1)^2}{2g_2(1 - \gamma_3^2)}.
\]

**Additional integral**

There exists another additional integral which for an arbitrary body of revolution is expressed in terms of the elementary functions [20]:

\[
F_2 = \sqrt{g_1} \omega_3. \tag{18}
\]

**Gyrostatic generalization**

When a rotor with gyrostatic moment \( S = (0, 0, s) \) is added which is directed along the axis of symmetry, the measure does not change and the integral (18) reads:

\[
F_2 = \sqrt{g_1} \omega_3 - s \int \frac{\gamma_3}{\sqrt{g_1}} d\gamma_3.
\]
A body with a spherical tensor of inertia

Invariant measure

If the surface bounding the body is arbitrary but its tensor of inertia is spherical \( I = \mu E \), then Eq. (12) determining the invariant measure simplifies considerably:

\[
\frac{d}{dt} \ln \rho = \frac{2m(r, \dot{r})}{\mu + m r^2} + \frac{m(r, \gamma)(r, \dot{\gamma})}{\mu + m(r, \gamma)^2}.
\]  

(19)

Taking into account the geometrical relation \((\dot{r}, \gamma) = 0\), this equation can be easily integrated, and we obtain an invariant measure in the form:

\[
(\mu + m r^2)(\mu + m(r, \gamma)^2)^{1/2} d\omega d\gamma.
\]  

(20)

Conformally Hamiltonian representation

As was shown above, the system obviously turns out to be conformally Hamiltonian in this case. The reducing multiplier, the change of variables and the Hamiltonian have the form

\[
N = (\mu + m(r, \gamma)^2)^{-1/2},
\]

\[
M = N(\mu \omega + m r \times (\omega \times r) - m (r \times (\omega \times r), \gamma \gamma),
\]

\[
H = \frac{(\mu + m(r, \gamma)^2) M^2 + m(r, M)^2}{2(\mu + m r^2)}.
\]

REMARK 1. Geometrically, \( M = \tilde{N} l_{\gamma} \omega \), where \( \tilde{l}_\gamma \) is the projection of the vector \( \tilde{l} \) onto the plane perpendicular to the vector \( \gamma \).
An Ellipsoid with Special Mass Distribution

Consider an ellipsoid whose principal axes of inertia coincide with the principal geometrical axes. Then the equation of surface of the ellipsoid is represented as \((r, B^{-1}r) = 1\), where \(B = \text{diag}(a_1^2, a_2^2, a_3^2)\), \(a_1, a_2, a_3\) are the principal semi-axes of the ellipsoid. Inverting the Gauss map (4), we obtain the explicit expression:

\[ r = -\frac{B\gamma}{\sqrt{(B\gamma, \gamma)}}. \]

Invariant measure

Using the criterion for the existence of an invariant measure — Proposition 1, we find that in this case the relation (11) is identically satisfied under the condition

\[(a_2^2 - a_3^2)(ma_1^2l_1 - l_2l_3) + (a_3^2 - a_1^2)(ma_2^2l_2 - l_1l_3) + (a_1^2 - a_2^2)(ma_3^2l_3 - l_1l_2) = 0 \quad (21)\]

For fixed values of the semi-axes \(a_1, a_2\) and \(a_3\), the condition (21) is an equation of a one-sheet hyperboloid relative to the moments of inertia \(l_1, l_2\) and \(l_3\). We parameterize this surface as follows:

\[ l_k = m \det B \frac{\alpha + a_k^{-2}}{\beta + a_k^2}, \quad (k = 1, 2, 3), \quad (22) \]

where the parameters \(\alpha\) and \(\beta\) should be chosen so as to obtain physically admissible values of the moments of inertia. Using this parameterization, we obtain the invariant measure

\[ \frac{\left(\chi_b + (\alpha\beta - 1) \det B\right)^{1/2}((r, \gamma)^2(\Delta_b + \beta r^2) - \det B)}{(r, \gamma)^2} d\omega d\gamma, \]

where

\[ \chi_b = (\beta^2 + \beta \text{Tr}B + \Delta_b)(r, \gamma)^2, \quad \Delta_b = \det B(\alpha + \text{Tr}B^{-1}), \]
Conformally Hamiltonian representation

As was shown above, due to the existence of the invariant measure the equations of motion of the ellipsoid with special mass distribution turn out to be conformally Hamiltonian, and the reducing multiplier and the change of variables have the form:

\[ N = (\chi_b + (\alpha\beta - 1) \det B)^{-1/2} \]

\[ M_i = \frac{N}{m(r, \gamma)^2 f(\beta)} \left( (\beta + a_i^2) \gamma_i (B \gamma, \omega) (\chi_b - \beta a_i^2 (\beta + a_{jk})) ight. \]

\[ - m \omega_i (r, \gamma)^2 ((r^2 - a_{jk}) \chi_b - r^2 f(\beta) - \det B (\beta + a_{jk}) (\alpha\beta - 1)) \right), \tag{23} \]

where

\[ a_{jk} = a_j^2 + a_k^2, \quad i, j, k = 1, 2, 3, \quad f(\beta) = \prod_i (\beta + a_i^2). \]

Numerical experiments (Figs. 5) show (see also [3]) that in the general case the family (22) does not admit an additional integral.

**The particular case** \( I = mB + \mu E \)

Among the ellipsoids (21) there is an exceptional one-parameter family of the form

\[ I = mB + \mu E, \tag{24} \]

where \( \mu \) is an arbitrary parameter which is characterized by the fact that the reducing multiplier is constant: \( N(u, v) = \text{const} \). Hence the equations of motion are Hamiltonian (without change of time).
For the equations on the algebra \( e(3) \) the change of variables (23) considerably simplifies and can be represented in the standard form

\[
\mathbf{M} = \mathbf{I}\omega,
\]

which is a consequence of the identity \( \left( \frac{\partial T}{\partial \omega}, \gamma \right) = 0 \), which is satisfied for this particular case. The Hamiltonian in this case reads

\[
H = \frac{1}{2}(\mathbf{M}, \mathbf{I}^{-1}\mathbf{M}) = \frac{1}{2} \frac{\mu(r, \gamma)^2(\mathbf{M}, \mathbf{AM}) + m(\mathbf{M}, \mathbf{B}\gamma)^2}{(\mu + m\text{Tr}\mathbf{B})(r, \gamma)^2(\mu + mr^2) + m^2 \det\mathbf{B}},
\]

\[
\mathbf{A} = \text{diag}(\mu + m(a_2^2 + a_3^2), \mu + m(a_1^2 + a_3^2), \mu + m(a_1^2 + a_2^2)).
\]

We also note that in the problem of an ellipsoid rolling only under the no-slip condition this family (24) also admits an invariant measure [13, 76].

**An additional integral**

Under the condition (24) and for \( U(\gamma) = 0 \) the system of equations (5) also admits the additional integral

\[
F_2 = \frac{\det\mathbf{I}}{(\mu + mr^2)}(\mathbf{M}, (\mathbf{IBI}^{-1}\mathbf{M})
\]

(26)
A balanced dynamically asymmetric ball

In this section we consider the rolling of an inhomogeneous but balanced ball, i.e. we shall assume that the principal moments of inertia are different and the center of mass coincides with the geometrical center of the ball. This case is discussed in Koiller [20]. We obtain from Eq. (4)

\[ r = -R\gamma, \]

where \( R \) is the radius of the ball. Using this relation and the constraint equation \((\omega, \gamma) = 0\), we find

\[ \tilde{I}\dot{\omega} = I\omega, \quad J = I + mR^2E. \]

This leads to simplification of the equations of motion (5):

Then the equations of motion (5) can be represented as:

\[ J\dot{\omega} = J\omega \times \omega + \lambda_0\gamma + \gamma \times \frac{\partial U}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \omega, \quad (27) \]

where

\[ \lambda_0 = - \frac{(J\omega \times \omega + \gamma \times \frac{\partial U}{\partial \gamma}, J^{-1}\gamma)}{\gamma, J^{-1}\gamma}. \]

Invariant measure

The invariant measure of the system of equations (27) is equal to [45]:

\[ (\gamma, J^{-1}\gamma)^{1/2} d\omega d\gamma. \]
**Conformally Hamiltonian representation**

In this case the reducing multiplier, the change of variables and the Hamiltonian have the form:

\[
N = (\gamma, J^{-1}\gamma)^{-1/2},
\]
\[
M = N(J\omega - (J\omega, \gamma)\gamma),
\]
\[
H = \frac{1}{2} (\gamma, J^{-1}\gamma)(M, J^{-1}M).
\]

**Additional integrals**

In the case of the balanced ball \( U(\gamma) = 0 \) there exists an additional integral [20]:

\[
F_2 = (J\omega \times \gamma, J\omega \times \gamma).
\]

Recall that a complete system governing the evolution of rotation of a ball is obtained by adding the equations for direction cosines:

\[
\dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega
\]  \hspace{1cm} (28)

As in the well-known problem of rolling of the Chaplygin ball, the equations of motion (27), (28) admit two additional integrals linear in velocities

\[
(J\omega \times \gamma, \alpha) = \text{const}, \quad (J\omega \times \gamma, \beta) = \text{const}.
\]  \hspace{1cm} (29)

Thus, the vector \( J\omega \times \gamma \) is constant in the fixed axes.
The Poisson structure
The equations of motion (27) can be represented in Hamiltonian form after the time substitution [20] \( N dt = d\tau \), \( N = (\gamma, J^{-1}\gamma)^{-1/2} \). The new variables \( M, n \), the Hamiltonian and the Poisson bracket have the form:

\[
M = \rho \sqrt{\det JJ^1/2} \omega, \quad n = \frac{J^{-1/2}\gamma}{\rho},
\]

\[
H = \frac{1}{2} \frac{(n, Jn)M^2}{\det J},
\]

\[
\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0. \tag{30}
\]

The bracket (30) is the Li–Poisson bracket of the algebra \( e(3) \). The functions \( F_0 = (n, n) \), \( F_1 = (M, n) \) are its Casimir functions.

Gyrostatic generalization
When a rotor with gyrostatic moment \( S \) is added, the invariant measure \( \rho \) does not change and the integral becomes equal to [60]:

\[
F_2 = (J\omega + S, J\omega + S) - (J\omega + S, \gamma)^2.
\]
An unbalanced dynamically asymmetric ball

We now consider the generalization of the previous system to the case where the center of mass of a dynamically asymmetric ball \((I_i \neq I_j)\) does not coincide with the geometrical center. We obtain from Eq. (4)

\[ r = -R\gamma - a, \]

where \(a\) is the vector connecting the center of mass with the geometrical center (see Fig. 10). In the general case, this system admits no invariant measure with analytical density. The necessary criterion (11) is satisfied only in the particular cases considered above: the axial symmetry \((I_1 = I_2, a_1 = a_2 = 0)\), the complete dynamical symmetry \((I_1 = I_2 = I_3)\) and the balanced ball \((a = 0)\).

In [5] it is shown that the absence of an invariant measure is due to the existence of limiting cycles.
**Additional integral**

Nevertheless, in the case at hand there exists one additional integral [20]:

$$F_2 = (I\omega \times \gamma, I\omega \times \gamma) - 2mR(\gamma, a)(I\omega, \omega).$$

Thus, an asymmetric ball with a displaced center of mass possesses the necessary number of integrals for integrability. In addition, as shown in [Bolsinov, Borisov, and Mamaev], in the system we consider here the foliation of the phase space into invariant manifolds completely coincides with the corresponding foliation in the integrable Euler system governing the motion of a rigid body with a fixed point.
Fig. 11. Graph of the number of dependence of rotation of the system on the parameter $\delta$ for $\kappa = 3, \alpha = \frac{2}{3}, \mu = 0.95$. Figures (b) and (c) show the enlarged fragments near the values $\rho = 1$ и $\rho = \frac{1}{2}$. 
Table 2. The hierarchy of dynamics of a rubber body moving on a sphere

<table>
<thead>
<tr>
<th>Tensor of Inertia</th>
<th>Dynamically Asymmetric Case</th>
<th>Ellipsoid $(r, B^{-1}r) = 1$</th>
<th>Body with Flat Area</th>
<th>Arbitrary Body of Revolution</th>
<th>Unbalanced Ball</th>
<th>Body with Flat Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface of the Body</td>
<td>Ball</td>
<td>Ellipsoid $(r, B^{-1}r) = 1$</td>
<td>Body with Flat Area</td>
<td>Arbitrary Body of Revolution</td>
<td>Unbalanced Ball</td>
<td>Body with Flat Area</td>
</tr>
<tr>
<td>Geometrical and Dynamical Restrictions</td>
<td>The center of mass coincides with the geometrical center</td>
<td>The center of mass does not coincide with the geometrical center</td>
<td>I = $mB + \mu E$</td>
<td>The center of mass lies on a plane of contact ($z = 0$)</td>
<td>Geometrical and Dynamical Axes coincide and contain the center of mass</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>$R = -2a$</td>
<td>$R = -2a$</td>
<td>$R = a$</td>
<td>$R = a$</td>
<td>$R = a$</td>
<td></td>
</tr>
<tr>
<td>Measure</td>
<td>$(\gamma, J^{-1} \gamma)^{1/2}$, $\kappa = \frac{a}{a + R}$ Koiler, Ehlers, 2006</td>
<td>$\det I(\gamma, J^{-1} \gamma)$</td>
<td>$p = \rho ell$</td>
<td>$I_1 I_2 + m \times (I_1 r_1^2 + I_2 r_2^2)$</td>
<td>$g_1(\gamma_3) \frac{I_1 + m(r_1^2 + r_2^2 + x^2) + k \mu E}{\mu + m r^2} \times \frac{(r_1 \omega_1 + r_2 \omega_2) \times}{x} \frac{k \mu E}{\mu + m r^2}$</td>
<td></td>
</tr>
<tr>
<td>Integral</td>
<td>Unknown</td>
<td>Unknown</td>
<td>Does not exist in the general case</td>
<td>$-m(r_1 \omega_1 + r_2 \omega_2) \times (I_1 \omega_1 - I_1) \times (I_2 \omega_2 - I_2)$</td>
<td>$g_1(\gamma_3) \frac{I_1 + m(r_1^2 + r_2^2 + x^2) + k \mu E}{\mu + m r^2} \times \frac{(r_1 \omega_1 + r_2 \omega_2) \times}{x} \frac{k \mu E}{\mu + m r^2}$</td>
<td>Possible — along the axis of dynamical symmetry</td>
</tr>
<tr>
<td>Integrable Addition of a Gyrostat</td>
<td>Unknown but the measure is preserved</td>
<td>Unknown but the measure is preserved</td>
<td>Unknown but the measure is preserved</td>
<td>Unknown but the measure is preserved</td>
<td>Unknown but the measure is preserved</td>
<td>Possible — along the axis of dynamical symmetry</td>
</tr>
<tr>
<td>Generalizations and Remarks</td>
<td>Integrable addition of a field is possible $c(I^{-1} \gamma, \gamma)$</td>
<td>Integrable addition of a field is preserved</td>
<td>Integrable addition of a field is preserved</td>
<td>Integrable addition of a field is preserved</td>
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<td>Integrable addition of a field is preserved</td>
</tr>
</tbody>
</table>

$g_1(\gamma_3) = I_1 \gamma_3^2 + I_3 (1 - \gamma_3^2) + m(r, \gamma)^2$, $g_2(\gamma_3) = I_1 + m r^2$

$\tilde{I} = I + m r^2 E - m r \otimes r$ is the tensor of inertia relative to the point of contact $J = \text{diag}(I_1 + m R^2, I_2 + m R^2, I_3 + m R^2)$, $R$ is the radius of the ball, $a$ is the radius of the supporting sphere, $k = \frac{1}{a}$

$\rho_{ell} = \frac{\det \tilde{I}}{(\mu + m r^2) \det I} \times \left(1 + k \frac{\det B - (r, \gamma)^3 (TrB - r^2)}{(r, \gamma)^4} \right)^2$

$^1$According to the results of numerical construction of the Poincaré cross-section for some fixed values of the parameters.
Table 1. The hierarchy of dynamics of a body rolling without spinning on a plane

<table>
<thead>
<tr>
<th>tensor of inertia</th>
<th>dynamically asymmetric case  ( I_1 \neq I_2 \neq I_3 \neq I_1 )</th>
<th>axial dynamical symmetry  ( I_1 = I_2, U = U(\gamma_3) )</th>
<th>complete dynamical symmetry  ( I_1 = I_2 = I_3 = \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>surface of the body</td>
<td>model</td>
<td>model</td>
<td>model</td>
</tr>
<tr>
<td>geometrical and dynamical restrictions</td>
<td>the center of mass coincides with the geometrical center</td>
<td>geometrical and dynamical axes coincide</td>
<td>geometrical and dynamical axes coincide and contain the center of mass</td>
</tr>
<tr>
<td></td>
<td>( g_1(\gamma_3) = l_1 \gamma_3^2 + l_3(1 - \gamma_3^2) + m(r, \gamma)^2 ), ( g_2(\gamma_3) = l_1 + mr^2 )</td>
<td>( g_1(\gamma_3) = l_1 \gamma_3^2 + l_3(1 - \gamma_3^2) + m(r, \gamma)^2 ), ( g_2(\gamma_3) = l_1 + mr^2 )</td>
<td>( g_1(\gamma_3) = l_1 \gamma_3^2 + l_3(1 - \gamma_3^2) + m(r, \gamma)^2 ), ( g_2(\gamma_3) = l_1 + mr^2 )</td>
</tr>
<tr>
<td>measure</td>
<td>((J, \tilde{J}^{-1} J)^{-1/2} ) Borisov, Mamaev, 2005</td>
<td>does not exist</td>
<td>((\chi_b + (\alpha \beta - 1) \det B)^{1/2} \times \chi((r, \gamma)^2 (\Delta b + \beta r^2) - \Delta b)^{-1} ) Borisov, Mamaev, 2008</td>
</tr>
<tr>
<td>additional integral</td>
<td>((J \omega \times J, \tilde{J} \omega \times \gamma) ) Borisov, Mamaev, 2005</td>
<td>does not exist in the general case</td>
<td>( \frac{\det \tilde{I}}{\mu + m \gamma r^2} (\omega, B^{-1} \omega) ) Borisov, Mamaev, 2008</td>
</tr>
<tr>
<td>integrable addition of a gyrostat</td>
<td>possible; Borisov, Mamaev, 2005</td>
<td>unknown but the measure is preserved</td>
<td>possible — along the axis of dynamical symmetry</td>
</tr>
<tr>
<td>generalizations and remarks</td>
<td>integrable addition of the Bruns field is possible; (-\xi(J \gamma, \gamma)) Borisov, Mamaev, 2005</td>
<td></td>
<td>unknown</td>
</tr>
</tbody>
</table>

\( g_1(\gamma_3) = l_1 \gamma_3^2 + l_3(1 - \gamma_3^2) + m(r, \gamma)^2 \), \( g_2(\gamma_3) = l_1 + mr^2 \)

\( \tilde{I} = I + m r^2 E - m r \otimes r \) is the tensor of inertia relative to the point of contact

\( J = \text{diag}(l_1 + m R^2, l_2 + m R^2, l_3 + m R^2) \),

\( R \) is the radius of the ball, \( a \) is the displacement of the center of mass,

\( B = \text{diag}(a_1^2, a_2^2, a_3^2) \), \( \chi_b = (\beta^2 + \beta \text{ Tr } B + \Delta_b)(r, \gamma)^2 \),

\( \Delta_b = \det B(\alpha + \text{ Tr } B^{-1}) \), \( f(\beta) = (\beta + a_1^2)(\beta + a_2^2)(\beta + a_3^2) \)

\(^1\)According to the results of numerical construction of the Poincaré cross-section for some fixed values of the parameters.
The hierarchy of dynamics related to reversibility

It turns out that *discrete symmetries*, along with conservation laws for integrals and an invariant measure considered above, play an important role in explaining some phenomena in the systems discussed in this paper. Of particular importance are *time-reversing involutive transformations*, which are related to the conformally Hamiltonian behavior in neighborhoods of periodic trajectories.

We start with the most illustrative example. As shown above, the system (5) in the case of a triaxial ellipsoid possesses an invariant measure only if the special condition (21) is satisfied. On the other hand, when this condition is not satisfied in constructing the Poincaré map, the phase portrait is in many cases almost indistinguishable from the portrait of an area-preserving map (see Fig. 12 a, b, c). Moreover, it is impossible to (numerically) reveal any robust manifestations of dissipation (i.e. attractors and repellers), since the multipliers of all known fixed points and cycles satisfy the relation $\lambda_1 \lambda_2 = 1$.

**Question.** What determines the conservative properties (at least those which show themselves visually) of the system in the case where there is no invariant measure?

An analogous situation also occurs in the problem of rolling of a dynamically asymmetric ball whose center of mass lies on one of the principal dynamical axes, Fig. 15 a, b, c.
Fig. 12. A typical view of phase portraits in the problem of rolling of an ellipsoid on a plane with different numbers of involutions and magnitudes of the gravitational field. The dark regions in the figures correspond either to conservative stochastic layers or to attractors (each attractor corresponds to a repeller symmetric with respect to involution $r(0)$). The lighter colored regions correspond to transition trajectories from repellers to attractors.
Systems with Involution and Conservative Regions

Symmetries and involutions of the flow

We represent the equations of motion (5) as

$$\dot{x} = v(x),$$

where $x = (\gamma_1, \gamma_2, \gamma_3, \omega_1, \omega_2, \omega_3)$. Recall that the map of a phase space $\Sigma(x)$ is called:

— symmetry if

$$(\Sigma(x)) \cdot = v(\Sigma(x)),$$

— time-reversing involution if

$$(\Sigma(x)) \cdot = -v(\Sigma(x)), \quad \Sigma \circ \Sigma = \text{id},$$

where $\text{id}$ is the identity map and $\circ$ denotes a composition of maps.

In what follows, we shall illustrate most results with the problem of rolling of a balanced ellipsoid.

1) A completely symmetric ellipsoid — the principal axes of inertia coincide with the principal geometrical axes, and the symmetry group is $D_{2h}$.

2) A partially symmetric ellipsoid — only one of the principal axes of inertia coincides with the principal geometrical axis, and the symmetry group is $C_{2h}$.

3) A completely asymmetric ellipsoid — all principal geometrical and dynamical axes are different.
Examples. Regardless of the form of the surface, the system (5) admits a standard involution — reversion of velocities:

\[
R^{(0)}: \gamma_1 \to \gamma_1, \gamma_2 \to \gamma_2, \gamma_3 \to \gamma_3, \omega_1 \to -\omega_1, \omega_2 \to -\omega_2, \omega_3 \to -\omega_3.
\]

In the case of a completely symmetric ellipsoid the system (5) also admits the symmetry group \(D_{2h}\) including the following transformations (time does not change in this case):

*three turns through 180° \((i = 1, 2, 3, j, k \neq i)\):*

\[
\Pi^{(i)}: \gamma_i \to \gamma_i, \gamma_j \to -\gamma_j, \gamma_k \to -\gamma_k, \omega_i \to \omega_i, \omega_j \to -\omega_j, \omega_k \to -\omega_k,
\]

(33)

*three symmetry planes \((i = 1, 2, 3, j, k \neq i)\):*

\[
\Sigma^{(i)}: \gamma_i \to -\gamma_i, \gamma_j \to \gamma_j, \gamma_k \to \gamma_k, \omega_i \to \omega_i, \omega_j \to -\omega_j, \omega_k \to -\omega_k,
\]

(34)

*inversion:*

\[
\Sigma^{(0)}: \gamma_1 \to -\gamma_1, \gamma_2 \to -\gamma_2, \gamma_3 \to -\gamma_3, \omega_1 \to \omega_1, \omega_2 \to \omega_2, \omega_3 \to \omega_3.
\]

(35)

The symmetries of this system possess two distinctive properties:

1) all symmetries are involutive: \(\Sigma^{(i)} \circ \Sigma^{(i)} = \Pi^{(i)} \circ \Pi^{(i)} = \Sigma^{(0)} \circ \Sigma^{(0)} = \text{id}\);

2) all symmetries commute with involution \(R^{(0)}\).

Hence, it follows that

the composition of any of symmetries (33), (34), and (35) with involution \(R^{(0)}\) is a time-reversing involution (i.e. \(R\)-symmetry).

We write the symmetries and involutions for a completely symmetric ellipsoid in explicit form on the manifold \(\mathcal{M}^4 = \{\gamma^2 = 1, (\omega, \gamma) = 0\}\) using the Andoyer variables. We will need this in the sequel to construct involutions of the Poincaré cross-section. Denoting the compositions by

\[
\tilde{\Sigma}^{(i)} = R^{(i)} \circ \Sigma^{(i)}, \quad \tilde{\Pi}^{(i)} = R^{(i)} \circ \Pi^{(i)},
\]
Symmetries and involutions of the Poincaré map

As we saw above, the Poincaré cross-section provides one of the most important tools for studying the systems under consideration and, in particular, the system (5). Therefore, we discuss in more detail the symmetries and involutions of two-dimensional maps

\[ \Phi: \mathcal{M}^2 \to \mathcal{M}^2. \]  

(36)

We recall that the transformation \( \sigma: \mathcal{M}^2 \to \mathcal{M}^2 \) is called
— symmetry if
\[ \Phi \circ \sigma = \sigma \circ \Phi; \]
— map-reversing involution if
\[ \Phi \circ \sigma = \sigma \circ \Phi^{-1}, \quad \sigma \circ \sigma = \text{id}. \]
Proposition 3. The eigenvalues of $\lambda_1$ and $\lambda_2$ of linearization of the map $\Phi$ in a neighborhood of a symmetric fixed point satisfy the relation

$$\lambda_1 \lambda_2 = 1.$$ 

The multipliers of fixed points for area-preserving maps possess an analogous property.

Thus, in neighborhoods of symmetric fixed points the behavior of the trajectories of a reversible map is similar to that of the trajectories of an area-preserving map. Therefore, we shall call such points locally conservative.

If, in addition, the symmetric fixed point is elliptic, then, provided some general conditions (analogs of the twisting and non-resonance conditions) are satisfied, the map $\Phi$ possesses in a neighborhood of $x_*$ a one-parameter Cantor family of invariant curves (KAM tori) symmetric relative to involution \[65, 64\]. Consequently, such points are Lyapunov stable.

We show that not only symmetric fixed points but also many periodic solutions display conservative behavior. For brevity, we shall call the point $x$ periodic if it belongs to a periodic solution, i.e. $\Phi^n x = x$ for some $n$.

We recall \[31\] that any involution $\sigma$ generates a countable family of involutions reversing the map $\Phi$:

$$\sigma_n = \Phi^n \sigma, \quad \Phi \circ \sigma_n = \sigma_n \circ \Phi^{-1}, \quad n \in \mathbb{Z},$$

with $\sigma = \sigma_0$.

For each of these involutions we consider the set of its fixed points

$\text{Fix} \sigma_k = \{x \mid \sigma_k(x) = x\}$. 
Proposition 4. If \( k \neq m \), then the set \( \text{Fix} \sigma_k \cap \text{Fix} \sigma_m \) consists of points belonging to periodic solutions whose period divides the difference \( k - m \).
Conversely, if \( x \in \text{Fix} \sigma_k \) is a periodic point of \( \Phi \), then \( x \in \text{Fix} \sigma_m \) for some \( m \neq k \).

Such periodic points arising as an intersection of fixed-point sets of involutions (37) are called symmetric periodic points, and the corresponding orbits are said to be symmetric periodic solutions.
That is, $x_*$ is a symmetric fixed point of the reversible map $\Phi^n$. Hence, we conclude that in a neighborhood of $x_*$ the map $\Phi^n$, and hence the original map $\Phi$ in a neighborhood of a periodic trajectory generated by $x_*$, is close to the area-preserving map.

As in the case of fixed points, symmetric periodic solutions will be called locally conservative. Moreover, in a neighborhood of symmetric periodic solutions possessing the ellipticity property there exist invariant KAM curves (see [64] for more details).

The search for symmetric periodic solutions is greatly simplified if we use the well-known [65, 31] relations:

$$\text{Fix } \sigma_{2n} = \Phi^n(\text{Fix } \sigma_0), \quad \text{Fix } \sigma_{2n+1} = \Phi^n(\text{Fix } \sigma_1).$$

Consequently, to find symmetric periodic points it is necessary to iterate the lines of fixed points of involutions $\sigma$ and $\sigma_1 = \Phi \sigma$ and to find their intersections. This algorithm for finding symmetric periodic solutions in the case of Hamiltonian systems has been applied in [30].
The first ten direct and inverse iterations of the lines of fixed points of involutions $\text{Fix} \sigma_{2n} = \Phi^n(\text{Fix} \sigma_0)$, $n = -10, \ldots, 10$ in the case of a completely symmetric ellipsoid for $a_g = 0$. Figs. a–c show iterations for individual lines of fixed points of various involutions. Fig. d shows all lines simultaneously for all involutions.
Fig. 14. Illustration of the separatrices and lines of fixed points of involutions in the case of a partially symmetric ellipsoid for \( a_g = 0, \psi_0 = 1 \). Fig. (a) shows the separatrices to one of the hyperbolic fixed points against the background of the phase portrait; Fig. (b) depicts the lines \( \text{Fix} \vec{\pi}^{(3)}_{2n} = \Phi^n \text{Fix} \vec{\pi}^{(3)}, n = -10, \ldots, 10 \).
The Hierarchy of Dynamics in the Problem of Rolling of an Ellipsoid

A completely symmetric ellipsoid (the top row of portraits in Fig. 12). As mentioned previously, the Poincaré map in this case admits the following four involutions on the chosen cross-section

\[ r_0, \quad \tilde{\pi}^{(3)}, \quad \tilde{\sigma}^{(1)}, \quad \tilde{\sigma}^{(2)}. \]

The last three of these involutions have nontrivial fixed-point sets on \( S^2 \). These sets are five main circles on the sphere \( S^2 = \left\{ \left( I, \frac{L}{G} \right) \right\} \), which pass through all “main” resonances of the system. As can be seen from Fig. 13, the iterations of these lines of involutions fill the entire sphere. Hence, the entire phase space of the system is actually a conservative region.
A partially symmetric ellipsoid (the middle row of portraits in Fig. 12). Here we consider only the situation where the dynamical axes have been rotated relative to the geometrical axes about the axis $e_3$. For the Poincaré cross-section formed by the intersection with the plane, only a pair of involutions: $r^{(0)}$ and $\pi^{(3)}$ is preserved in this case. The first one has no fixed points, for the second one $\text{Fix} \, \pi^{(3)} = \left\{ \frac{L}{G} = 0 \right\}$. The first ten iterations $\Phi^n(\text{Fix} \, \pi^{(3)}) \ n = -10 \ldots 10$, are presented in Fig. 14. The phase portraits in Fig. 12d–f demonstrate that the conservative regions in neighborhoods of symmetric solutions adjoin simple (fixed points) attractors and repellers conjugate to them relative to $\pi^{(3)}$; the interval between them is filled with transition trajectories (from repellers to attractors).

A completely asymmetric ellipsoid (the bottom row of portraits in Fig. 12). This system, where the dynamical and geometrical axes do not coincide, has in the chosen cross-section only one time-reversing involution: $r^{(0)}$, which, as we saw above, has no fixed points. As a consequence, we can see that the behavior of the trajectories on the phase portrait is purely dissipative: there are attractors and repellers symmetric to them relative to $r^{(0)}$. In this case, as the gravitational field (or energy) changes, only sufficiently simple attractors are found on the Poincaré map: fixed points, periodic orbits and limit invariant curves. The question of the existence of complex attractors (strange attractors and quasi-attractors) in the problem of rolling of an asymmetric ellipsoid remains open.
The Hierarchy of Dynamics in the Problem of Rolling of an Unbalanced Dynamically Asymmetric Ball

The center of mass is located on the principal dynamical axis (the top row of portraits in Fig. 15). Here we consider only the situation where the center of mass is displaced along the axis $e_3$. In this case, the Poincaré map on the chosen cross-section admits four involutions

$$\tilde{r}^{(0)}, \tilde{\pi}^{(3)}, \tilde{\sigma}^{(1)}, \tilde{\sigma}^{(2)}.$$ 

The last three of them have nontrivial fixed-point sets on $S^2$. As can be seen in Fig. 15, the iterations of these lines of involution fill the entire sphere. Hence, the entire phase space of the system is a conservative region (as was the case for a completely symmetric ellipsoid).

The center of mass lies in the principal plane (the middle row of portraits in Fig. 15). Here we shall assume that the center of mass is displaced in the plane $Oe_2e_3$. In this case, only a pair of involutions ($\tilde{r}^{(0)}$ and $\tilde{\sigma}^{(1)}$) is preserved in the general position situation for the Poincaré cross-section considered. The first one has no fixed points, for the second $\text{Fix}\tilde{\sigma}^{(1)} = \{l = \pi\} \cup \{l = 0\}$.

The center of mass lies outside the principal planes (the bottom row of portraits in Fig. 15). In the general case this system possesses only one time-reversing involution, namely, $\tilde{r}^{(0)}$, which, as we know, has no fixed points. As a consequence, the behavior of the trajectories in the phase space is purely dissipative: there are attractors and repellers symmetric to them relative to $\tilde{r}^{(0)}$. In this case it turns out that for some values of parameters the system admits a strange attractor (Fig. 15h) due to A. O. Kazakov [42].
Fig. 15. A typical view of phase portraits in the problem of rolling of a dynamically asymmetric unbalanced ball on a plane for different numbers of involutions and magnitudes of the gravitational field. The dark and light regions are analogous to similar regions in Fig. 12 except for Fig. h. In Fig. h the darker region in the left part corresponds to a strange attractor.
The first ten direct and inverse iterations of the lines of fixed points of involutions $\text{Fix } \sigma_{2n} = \Phi^n(\text{Fix } \sigma_0)$, $n = -10, \ldots, 10$ in the case of a ball whose center of mass is located on the principal dynamical axis when $a_g = 98$ (Fig. 15b).
Fig. 17. The first ten direct and inverse iterations of the lines of fixed points of involutions \( \{ l = \pi \} \) for a ball whose center of mass lies in the principal plane when \( a_g = 9.8 \) (Fig. 15e).
General conclusions and a discussion

1. A (numerical) analysis of the phase portraits of the system at various values of the parameters suggests that when there is a sufficient number of involutions \( \sigma_1, \ldots, \sigma_k \) possessing a nontrivial fixed-point set (\( \text{Fix} \sigma_i \neq \emptyset \)), the conservative region of the system coincides with the entire phase space. Moreover, the behavior of the trajectories on the Poincaré map of the system actually does not differ from their behavior in the case of conservative systems.

2. As the number of nontrivial involutions (i.e. \( \text{Fix} \sigma_i \neq \emptyset \)) decreases, the conservative region of the system diminishes and regions appear which correspond to “dissipative behavior”. In the absence of involutions with \( \text{Fix} \sigma_i \neq \emptyset \), the entire phase space exhibits “dissipative behavior”.

Thus, the presence of various nontrivial involutions provides a partial explanation of why the system’s phase flow which does not admit an invariant measure is so similar to the flow of a typical Hamiltonian system.

3. According to the criterion (11), the systems considered above in this section do not admit an invariant measure with analytical density. On the other hand, as can be seen from Figs. 12a, b, c and 15a, b, c, manifestations of dissipation (for example, attractors like fixed point, cycles, limit curves or strange attractors) cannot be observed in this case either. Also, they cannot be revealed by more detailed numerical investigations in various (including sufficiently small) neighborhoods of elliptic fixed points and cycles. This raises two natural questions.

1. Can the points of intersection of the fixed-point sets of involutions (\( \text{Fix} \sigma_n \) for maps and \( \text{Fix} \Sigma_\tau \) for the flows) form everywhere a dense set in the phase space?

2. Are there any “more subtle” (than non-degenerate attracting points) obstructions to the existence of an invariant measure in a neighborhood of periodic solutions which are analogous to obstructions to the existence of single-valued integrals?
The simplest system having this physical meaning is a three-dimensional homogeneous ellipsoid on a plane, for which, in spite of the completely Hamiltonian structure of the phase portrait, we cannot obtain an invariant measure (except for the case of an ellipsoid of revolution). The classical case of nonholonomic rolling [13, 17] involves an analogous problem of motion of a homogeneous ellipsoid.

4. A fairly wide class of reversible systems without invariant measure is also related to the usual rolling without slipping [13, 23]. For such systems the fixed-point sets of the corresponding involutions $\Sigma_i$ satisfy the relation

$$\text{Fix } \Sigma_i \neq \frac{\dim M}{2}.$$ 

Effects occurring in such cases due to reversibility require additional study (see [14]).

5. Since the property of preservation of KAM tori in reversible systems was discovered, a substantial body of published research has emerged on the topic (see, e.g., the references in the review [63]).

As mentioned previously, the most fundamental properties of reversible systems consist in the existence of a large number of symmetric periodic orbits (exhibiting conservative behavior) and invariant KAM tori (invariant KAM curves in the case of maps). These invariant manifolds can form multi-parameter families whose number of parameters depends on the ratio of the dimensions of the phase space and the fixed-point set of involution (Fix $\Sigma$).

Furthermore, the numerical simulation (of maps) has revealed that in various regions of phase space these systems typically exhibit both conservative and dissipative behavior.
There are also a number of studies [59, 68] which show that from the viewpoint of formal expansions in a neighborhood of a symmetric singular point in a nonresonance case and in the case of some resonances, reversible systems are conjugate to Hamiltonian systems. As usual, due to the impossibility of proving the convergence of series obtained, these results can be used to gain insight into the dynamics of the system. For example, in [37] it is shown that in principle there may exist smooth reversible systems where there may exist sequences of dissipative periodic solutions which converge to both symmetric periodic solutions and invariant KAM tori. Of great interest is the study of bifurcations of periodic orbits for reversible systems related to the loss of symmetry [55, 37].

Such a mixed (Hamiltonian and dissipative) behavior was discovered in [42] where the so-called rubber rock-n-roller (a dynamically asymmetric ball with a displaced center) was considered as an example.
The dynamics of a spherical shell on a plane with a nonholonomic hinge inside

Consider a system consisting of two bodies (Fig. 18): an outer body and an inner body. The outer body is a dynamically symmetric spherical shell that moves without slipping on a plane. The inner body is an arbitrary body which is fixed inside the shell by means of two sharp-edged wheels. The sharp-edged wheels ensure that the components of angular velocities are equal in direction fixed in the inner body [70]. That is, $(\omega - \Omega, e) = 0$, where $\omega$ and $\Omega$ are the angular velocity components of the inner body and the shell in the moving coordinate system $C_{xyz}$ rigidly attached to the inner body, $e = (0, 0, 1)$ [43].

The equations of motion in the moving coordinate system $C_{xyz}$ are:

\[
\ddot{\Omega} = \dot{\Omega} \times \omega + mR_0^2(\gamma(\Omega, \gamma \times \omega) + \gamma \times \omega(\gamma, \Omega)) - \lambda_0 e
\]

\[
\dot{\omega} = \omega \times \omega + \lambda_0 e
\]

\[
\dot{\gamma} = \gamma \times \omega,
\]

(39)

Fig. 18. A dynamically symmetric spherical shell with a nonholonomic hinge inside on a plane
where \( \tilde{I} = \mu + mR^2(\gamma^2 - \gamma \otimes \gamma) \) is the inertia tensor of the shell relative to the contact point, \( I = \text{diag}(I_1, I_2, I_3) \) is the inertia tensor of the inner body and \( m \) is the mass of the entire system,

\[
\lambda_0 = -\frac{\mu(\mu + mR^2\gamma^2)(\omega_1\omega_2(I_1 - I_2) + I_3(\Omega_2\omega_1 - \Omega_1\omega_2))}{\mu(\mu + mR^2\gamma^2 + I_3) + mR^2I_3\gamma_3^2}.
\]

invariant measure:

\[
(\mu(\mu + mR_0^2\gamma^2 + I_3) + mR^2I_3\gamma_3^2)^{1/2} d\Omega d\omega d\gamma.
\]

first integrals

\[
F_0 = \gamma^2 = 1, \quad F_1 = \omega_3 - \Omega_3 = 0.
\]

\[
H = \frac{1}{2}((\Omega, \tilde{I}\Omega) + (\omega, I\omega)).
\]

\[
F_2 = M^2, \quad F_3 = (M, \gamma), F_4 = I_1(I_1 - I_3)\omega_1^2 + I_2(I_2 - I_3)\omega_2^2.
\]

where \( M = \tilde{I}\Omega + I\omega \) is angular momentum of the system relative to the point of contact.
$l_1 = l_2 \neq l_3$ symmetry field: $u = M_1 \frac{\partial}{\partial M_2} - M_2 \frac{\partial}{\partial M_1} + \omega_1 \frac{\partial}{\partial \omega_2} - \omega_2 \frac{\partial}{\partial \omega_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_1}$

It corresponds to the invariance of the system with respect to rotations about the axis of dynamical symmetry. Using this field we can reduce the order of system. For that we should choose the integrals of vector fields $u$ as reduced variables to present the equations in the simplest from:

$$
\begin{align*}
K_1 &= \gamma_1 \omega_1 + \gamma_2 \omega_2,
K_2 &= \gamma_1 \omega_2 - \gamma_2 \omega_1,
K_3 &= \gamma_1 M_2 - \gamma_2 M_1, 
F_3, M_3, \gamma_3.
\end{align*}
$$

$$
\dot{x} = \rho^{-1} J_a \nabla \left( \frac{2(\mu + mR^2)H - F_2}{2l_1} \right) = \rho^{-1} J_b \nabla \left( \frac{F_4}{2l_1^2(l_1 - l_3)} \right), \quad x = (K_1, K_2, K_3, M_3, \gamma_3)
$$

$$
J_a = \begin{pmatrix}
0 & 0 & \rho \gamma_3 K_1 & \rho K_2 & 0 \\
0 & 0 & \rho \gamma_3 K_2 & -\rho K_1 & 0 \\
-\rho \gamma_3 K_1 & -\rho \gamma_3 K_2 & 0 & J_{a}^{34} & \rho \left(1 - \gamma_3^2\right) \\
-\rho K_2 & \rho K_1 & -J_{a}^{34} & 0 & 0 \\
0 & 0 & -\rho \left(1 - \gamma_3^2\right) & 0 & 0
\end{pmatrix}
$$

$$
J_{a}^{34} = \frac{mR^2 \left(1 - \gamma_3^2\right) \left((\mu + l_3 + mR^2) \left(l_1 K_1 - K_4\right) + l_3 \gamma_3 M_3\right)}{\rho}
$$
\[ J_b = \begin{pmatrix}
0 & J_b^{12} & \rho l_1 (\gamma_3 K_4 - M_3) & \rho l_1 K_3 & 0 \\
-J_b^{12} & 0 & \rho l_1 \gamma_3 K_3 & \rho l_1 (\gamma_3 M_3 - K_4) & -\rho l_1 \left(1 - \gamma_3^2\right) \\
-\rho l_1 (\gamma_3 K_4 - M_3) & -\rho l_1 \gamma_3 K_3 & 0 & 0 & 0 \\
-\rho l_1 K_3 & -\rho l_1 (\gamma_3 M_3 - K_4) & 0 & 0 & 0 \\
0 & \rho l_1 \left(1 - \gamma_3^2\right) & 0 & 0 & 0
\end{pmatrix} \]

\[ J_b^{12} = \frac{l_3 \left(1 - \gamma_3^2\right) (m R^2 \gamma_3 (l_1 K_1 - K_4) - \mu M_3)}{\rho} \]
References


