Statistical ensembles of virialized halo matter density profiles

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Context and motivations

- Large $N$-body simulations are used in the cosmology used to solve for the Vlasov-Poisson equations for dark matter in an expanding universe, starting from cold, nearly homogeneous Gaussian initial conditions. They predict a cosmic web made out of filaments, voids, haloes.

- These simulations indicate that the spherically averaged density profiles can be described by a one-parameter family of profile ('NFW’, mid-nineties)

$$\rho(r) \propto \frac{1}{r} \frac{1}{\left(1 + \frac{r}{r_s}\right)^2}, \quad c = \frac{r_{\text{vir}}}{r_s} \sim c(M)$$
Context and motivations

Usual approach to profile fitting consider only haloes that are ’well behaved’ : e.g.

- The virial theorem $E_{\text{tot}} = \frac{1}{2} W$ must hold approximately.
- The halo must not have too much substructure
- It must have a well defined center.

The haloes so selected form an ensemble of virialized, spherically symmetric and of finite extent $r_{\text{vir}}$, turned into an ensemble of density profiles $\rho(r)$.

- Is it possible to point towards a statistically favored such function ?
- What are the quantities that are important beyond the mass and energy ? "What are the ’sufficient statistics’ "?
Outline:

1. Definition and discussion of the ensembles
2. Solution of the grand canonical and microcanonical ensembles
3. Regularization and connection to NFW haloes
A 'microstate’ is a spherically symmetric positive function $\rho(r)$, with $r$ from zero to some boundary $r_{\text{vir}}$, the virial radius. Characterization by:

- **Total mass**
  \[ M[\rho] = 4\pi \int_0^{r_{\text{vir}}} dr \ r^2 \rho(r) \]

- **Total energy defined as the total gravitational potential energy**
  \[ W[\rho] = -\frac{G}{2} \int_0^{\infty} dr \ \left( \frac{M(\leq r)}{r} \right)^2 \]
Ensembles

Let us define $p[\rho]:$ probability of observing state $\rho(r)$

- "Microcanonical", fix $M$ and $W$.

$$p[\rho] = \frac{1}{\Omega} \delta^D(M - M[\rho])\delta^D(W - W[\rho])$$

- "Grand canonical" $\sim$ max. entropy for mean $M$ and $W$.

$$p[\rho] = \frac{e^{-\lambda M[\rho] - \beta |W[\rho]|}}{Z(\lambda, \beta)}$$

We want the favored profiles

$$\bar{\rho}(r) = \int \mathcal{D}\rho \, p[\rho] \rho(r).$$

(See later for precise meaning of "$\mathcal{D}\rho$".)
A particle description sets $p(x_1, \cdots, x_N)$, (forgetting velocity degrees of freedom) with $x_i$ in some volume.

The canonical ensemble

$$p(x_1, \cdots, x_N) \propto e^{\beta \sum_{i \neq j} \frac{1}{|x_i - x_j|}}.$$

is equivalent to a collapse to a single point, chosen at random in the volume (Kiessling 89), for any $\beta > 0$. At $\beta = 0$ it is the sphere of uniform density.
Particle description II

Alternative, opposite approach to the previous slide. Assume the right limit is of the uncorrelated form

\[ p(x_1, \cdots, x_N) = \prod_{i=1}^{N} p(x_i). \]

With this assumption the mean profile is

\[ \bar{\rho}(x) \propto p(x). \]

Then the total entropy is \( N \) times the entropy of \( p(x) \). The canonical ensemble becomes

\[ p(x) = \bar{\rho}(x) \propto \exp(\beta \langle \Phi(x) \rangle) \]

the isothermal spheres obtained by Lynden-Bell (67) entropy maximisation of the phase space distribution \( f(x, v) \).
Solving the ensembles I

Grand canonical partition function:

\[ Z(\lambda, \beta) = \int_{\rho \geq 0} D\rho \ e^{-\lambda M[\rho] - \beta |W[\rho]|} \]

- \( \lambda M[\rho] + \beta W[\rho] \) quadratic in \( \rho \)
- Boundary conditions \( \rho(r) \geq 0 \).
- \( \rightarrow \) Not a Gaussian path integral, need some trick.
Solving the ensembles II: here is the trick

Introduce

\[ y(r) = 4\pi \int_r^{r_{\text{vir}}} ds \ s \ \rho(s). \]

Then for any spherically symmetric mass distribution, the total mass and energy are the first two moments of this function,

\[ M = \int_0^{r_{\text{vir}}} dr \ y(r), \quad W = -\frac{G}{2} \int_0^{r_{\text{vir}}} dr \ y^2(r). \]

Other relation

\[ \Phi(r) = -\frac{G}{r} \int_0^r ds \ y(s), \]

where \( \Phi = -GM/R \) for \( R \geq r_{\text{vir}}. \)
Constant Jacobian for this linear transformation. The partition function becomes

\[ Z(\lambda, \beta) = \int \mathcal{D}y \ e^{-\lambda \int dr \ y(r) - \beta \int dr \ y^2(r)}, \]

where the integral runs now over positive decreasing functions vanishing at \( r_{\text{vir}} \).

\[ y(r) \geq 0, \quad y'(r) \leq 0, \quad y(r_{\text{vir}}) = 0 \]
Solving the ensembles IV: discretization

We proceed with fine discretization of the radial distance (lattice),

- \( y_i = y \left( \frac{i}{N+1} r_{\text{vir}} \right) \quad i = 1, \ldots, N \).

- \( M \to \frac{1}{N} \sum_{i=1}^{N} y_i \), \( |W| \to \frac{1}{N} \sum_{i=1}^{N} y_i^2 \).

The partition function factorizes

\[
Z_N(\lambda, \beta) = \int dy^N \exp \left( -\lambda \sum y_i - \beta \sum y_i^2 \right)
\]

\[
\geq y_1 \geq \cdots \geq y_N \geq 0
\]

\[
= \frac{1}{N!} \left( \int_0^\infty dy e^{-\lambda y - \beta y^2} \right)^N
\]

in independent identical degrees of freedom, except for ordering. → It is all 'order statistics'.
Generating a halo in the grand canonical ensemble

1. Generate $N$ randoms from $p(y) \propto e^{-\lambda y - \beta y^2}$

2. Arrange them in decreasing order on the $N$ lattice points

3. Take the derivative of the resulting curve, divide by $r$. 

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\[ p(y) = \frac{1}{\sqrt{2\pi \beta}} e^{-\frac{1}{2\beta} (y - \lambda)^2} \]

\[ \rho(r) = \frac{1}{4\pi r^2} \]

\[ \frac{4\pi r^2 \rho(r)}{N_{vir}} \propto r \]

\[ N = 100 \]

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$y$ vs $p(y)$

$r / r_{vir}$ vs $\frac{4\pi r^2 \rho(r)}{N_{vir}}$
Large $N$ behavior

- Well defined limit for large $N$:

$$y(r) = F^{-1}\left(1 - \frac{r}{r_{\text{vir}}}\right), \text{ with } F(y) = \int_0^y dy' \rho(y')$$

For $\rho$

$$\bar{\rho}(r) \propto -\frac{y'(r)}{r} \propto \frac{1}{r} e^{\lambda y(r) + \beta y^2(r)},$$

but not sharp, $\rho(r)$ exponentially distributed at a given $r$.

- $\lambda$ and $\beta$ given by mass and energy through 1d non-linear equations:

$$-\frac{\partial \ln Z_1}{\partial \lambda} = \langle y \rangle = 1$$

$$-\frac{\partial \ln Z_1}{\partial \beta} = \langle y^2 \rangle = 2 \left(\frac{r_{\text{vir}}}{r_g}\right), \quad r_g = \frac{GM^2}{|W|}.$$
Special case $\beta = 0$

$\beta = 0$ leads to the singular isothermal sphere (with $r_{\text{vir}}/r_{g} = 1$):

$$p(y) = e^{-y}$$

$$F(y) = 1 - e^{-y}, \quad F^{-1}(1 - x) = \ln(x)$$

$$\rho(r) \propto \frac{1}{r^2}$$

Other values of $\beta$ involves the error function and its inverse.

$$y(r) = \frac{M}{r_{\text{vir}}} \left[ \text{erfc}^{-1} \left( \frac{r}{r_{\text{vir}} \text{erfc}(\gamma)} \right) - \gamma \right], \quad \gamma = \frac{\lambda}{2\sqrt{\beta}}.$$
\( \frac{r_{\text{vir}}}{r_g} \) sets \( \lambda \) and \( \beta \). \( \frac{r_{\text{vir}}}{r_g} \) cannot be arbitrarily small due to \( \langle y^2 \rangle - \langle y \rangle^2 \geq 0 \). We do not cover the interesting range with our ensemble so far.
Microcanonical ensemble

- Discretized ensemble:
  
  \[
  p(y_1, \cdots, y_N) \propto \delta^D \left(1 - \frac{1}{N} \sum_i y_i\right) \delta^D \left(2 \left(\frac{r_{\text{vir}}}{r_g}\right) - \frac{1}{N} \sum_i y_i^2\right),
  \]

  Plane (the standard simplex) intersects a sphere

  \[\infty \geq y_1 \geq \cdots \geq y_N \geq 0.\]

- Googled for help: ’A note about the uniform distribution on the intersection of a simplex and a sphere’ by S. Chatterjee solves the problem for large \(N\) (math.pr 1011.4043). Just add ordering.

- Transition at \(r_{\text{vir}}/r_g = 1\).
From Chatterjee’s note we infer:

- \( r_{\text{vir}}/r_g \leq 1 \): Same as grand canonical just solved.
- \( r_{\text{vir}}/r_g > 1 \): Same than \( \beta = 0 \) (SIS) plus energy condensate: the largest \( y \) drawn on that set grows in such a way with \( N \) that

\[
\frac{\text{frac. of energy}}{N} = \frac{1}{N} y^2 (1/N) \quad \text{remains finite}
\]

but

\[
\frac{\text{frac. of mass}}{N} = \frac{1}{N} y (1/N) \quad \text{still goes to zero}
\]

The other \( y \)'s behave just like for the SIS.
Results so far

Looks like reasonable results for gravitational physics, but not very insightful on dark matter haloes.
Obvious regularization

- The set of y’s always saturate the freedom we give to them (i.e. the range of p(y)):

\[ y(r_{\text{vir}}) = \min y = 0 \]
\[ y(0) = \max y = \infty \]

- Note that \( y(0) \) is the central gravitational potential \( \Phi(0) \) of the mass distribution.

\[ \Phi(r) = -G \frac{1}{r} \int_0^r ds \ y(s). \]

- Obvious modification: prevents the arbitrary growth of the potential well, by setting an upper limit to \( y \).
Now not one but two relevant dimensionless parameters: $r_{\text{vir}}/r_g$ and the ratio $\epsilon_\Phi$ of the gravitational potential at the center to that at the boundary of the halo,

$$\epsilon_\Phi = \frac{\Phi(0)}{\Phi(r_{\text{vir}})} = y(0) \left( \frac{r_{\text{vir}}}{M} \right), \quad \epsilon_\Phi \in (0, 1)$$

The entire analysis of the grand canonical ensemble carries over with no modification except the range of $p(y)$. The $y$-function is given by the inverse cumulative (quantile) function of the now doubly truncated 'Gaussian'.

To each acceptable values of $r_{\text{vir}}/r_g$ and $\epsilon_\Phi$ there are now corresponding $\lambda$ and $\beta$ parameters of the ensemble.
At any finite $\epsilon$, any mass and energy a inner slope of unity is
predicted and a characteristic radius appears, just like for NFW
haloes.

$$\rho(r) \propto -\frac{y'(r)}{r} \propto \frac{1}{r} \quad \text{at small } r.$$  

We can calculate the $\lambda$ and $\beta$ of the NFW haloes, and compare
the profiles.

$$\epsilon_{\Phi,NFW} = \frac{1}{c^2} \left[ (1 + c) \ln(1 + c) - c \right], \quad \frac{r_{\text{vir}}}{r_g} = \frac{1}{2c} \left( \frac{1 - 2\epsilon_{\Phi,NFW}}{\epsilon_{\Phi,NFW}^2} \right).$$
Now lower and upper bounds on the energy at fixed mass,

$$\langle x \rangle \geq \langle x^2 \rangle \geq \langle x \rangle^2 \quad \text{if} \ 0 \leq x \leq 1.$$
y-function and density profiles
Conclusions, and open questions

- Our class of exactly solvable ensembles point towards the $y$-function as the key degrees of freedom. The mass and energy and the first and second moment of this function. What is the meaning of the $y$-function?
- To each spherically symmetric matter distribution can always be assigned a $p(y)$. Our ensembles sets $p$ the ’Gaussian’ pdf.
- Exhibit condensation etc. Most obvious regularization remains exactly solvable and produce NFW-like haloes. Why this location in the $\epsilon_\Phi - r_{\text{vir}}/r_g$ plane?

Thank you!