An Optimal \( hp \)
Discontinuous Galerkin Method
for Computational Aeroacoustics

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Introduction
Why Develop Methods based on DGM to Compute Euler’s linearized equations?

Introduction of complex geometries is difficult with the FDM. FEM failed to solve Euler’s linearized equations.

DGM Advantages and Disadvantages:

Complex Geometries
Variational Formulation
Very Flexible

CPU and RAM Expensive
### Physical Modeling and Mathematical formulation

**Euler’s linearized equations**

\[ \mathbf{\varphi} = \begin{pmatrix} u_1 \\ v_1 \\ a_0 \rho_1/\rho_0 \end{pmatrix} \]

**Symmetric Friedrich System**

\[ \partial_t \mathbf{\varphi} + A_i \partial_i \mathbf{\varphi} + B \mathbf{\varphi} = 0 \]

**Matrix \( A_i \partial_i \) is symmetric**

**Variational formulation**

\[ \left\{ \varphi_h \in W^k(\omega_h) \mid \forall \psi_h \in W^k(\omega_h) ; \mathcal{L}(\varphi_h, \psi_h) = 0 \right\} \]

**\( A_i \mathbf{n}_i \) is diagonalizable**

\[ A_i \mathbf{n}_i = [A_i \mathbf{n}_i]^+ + [A_i \cdot \mathbf{n}_i]^- \]

**Fully Upwind Scheme**

\[ \mathcal{L}(\varphi_h, \psi_h) = \int_{\Omega} \psi_h \cdot \partial_t \mathbf{\varphi}_h + \int_{\Omega} \psi_h \cdot A_i \partial_i \mathbf{\varphi}_h + \int_{\Omega} \psi_h \cdot B \mathbf{\varphi}_h \]

\[ \quad + \int_{\partial \omega_h/\partial \Omega} \psi_h \cdot [A_i \mathbf{n}_i]^- (\mathbf{\varphi}_h^o - \mathbf{\varphi}_h^i) + \int_{\partial \omega_h \cap \partial \Omega} \psi_h \cdot (\mathcal{M} \mathbf{\varphi}_h - \mathbf{g}) - \int_{\Omega} \psi_h \cdot \mathbf{g} \]
Part One

HPC DGM

HPC = High Performance Computing
Simulations for 3D geometries

First Idea for 3D Simulation:
- HPC
- Formal Calculation
- Vectorization
- Massively Parallel Computation
Falcon CAA HPC Computation

TetMesh (INRIA/Simulog)  
500 000 Tetras  
64 domains
Falcon CAA HPC Computation

HP cluster - 64 nodes DEC Alpha 1.25 GHz - ~12 hours CPU
Part Three
Optimal hp Adaptation Principles
DGM Triangles for 2D geometries

Precision(36 \( P1 \)) \( \leq \) Precision(\( P6 \))

<table>
<thead>
<tr>
<th>( m(P_i) )</th>
<th>FEM deg</th>
<th>DGM deg</th>
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<tbody>
<tr>
<td>P1</td>
<td>36</td>
<td>28</td>
</tr>
<tr>
<td>P6</td>
<td>1</td>
<td>28</td>
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</table>

High Order Elements reduce the size of problems
### Mesh Refinement / Element Order

<table>
<thead>
<tr>
<th>element</th>
<th>$n(P_i)$</th>
<th>$m(P_i)$</th>
<th>$\alpha(P_i)$</th>
<th>$h_{min}(\leq 5%)$</th>
<th>$h_{min}(\leq 10%)$</th>
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<tbody>
<tr>
<td>P1</td>
<td>3</td>
<td>1</td>
<td>1.0</td>
<td>$\lambda/14$</td>
<td>$\lambda/12$</td>
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<tr>
<td>P2</td>
<td>6</td>
<td>4</td>
<td>0.5</td>
<td>$\lambda/4$</td>
<td>$\lambda/4$</td>
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<tr>
<td>P3</td>
<td>10</td>
<td>9</td>
<td>0.37</td>
<td>$\lambda/3$</td>
<td>$\lambda/2$</td>
</tr>
<tr>
<td>P4</td>
<td>15</td>
<td>16</td>
<td>0.31</td>
<td>$\lambda/2$</td>
<td>$\lambda/3$</td>
</tr>
<tr>
<td>P5</td>
<td>21</td>
<td>25</td>
<td>0.28</td>
<td></td>
<td>$\lambda$</td>
</tr>
<tr>
<td>P6</td>
<td>28</td>
<td>36</td>
<td>0.26</td>
<td></td>
<td>$\lambda$</td>
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</table>

$$\alpha(P_i) = \frac{1}{m(P_i)} \cdot \frac{n(P_i)}{n(P_1)}$$
Remeshing Tool

Original mesh
np: 67 108  nt: 134 212

Adapted mesh
np: 9 796  nt: 19 588
Optimal \( \text{hp} \) DGM with remeshing tool

non adapted
17024 triangles

adapted isotropic
6417 triangles

adapted anisotropic
3467 triangles
Element Orders Definition
Optimal hp DGM CAA

\[ f = 1 \text{ kHz} \]

<table>
<thead>
<tr>
<th></th>
<th>CPU</th>
<th>RAM</th>
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<tbody>
<tr>
<td>1 kHz</td>
<td>564.15'</td>
<td>100 MO</td>
</tr>
<tr>
<td>2 kHz</td>
<td>246.05'</td>
<td>50 MO</td>
</tr>
<tr>
<td>3 kHz</td>
<td>125.51'</td>
<td>25 MO</td>
</tr>
</tbody>
</table>
Conclusion
DGM

DGM is able to solve most CAA problems (and many others)
DGM is very expensive (especially for lower order elements)

HPC DGM

HPC is good for DGM

hp DGM

hp DGM mixes element orders and results a much less expensive cost
With hp DGM, CFD and CAA computations are handled on same mesh
Introduction of the doppler effect when determining local orders

HPC hp DGM

HPC is good for hp DGM
optimize HPC for hp DGM