Homogenization of elastic structures leading to second gradient models

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Waves in periodic media and metamaterials
Cargese, November 2016
The goal: second gradient materials

Considered geometries: thin walled structures

Reduction to a discrete problem

Main results

Explicit computation of the limit energy

Conclusion
The goal
Designing second gradient materials

- Second gradient materials (or strain gradient materials): their elastic energy depends on the second gradient of the displacement

\[ E(u) = \int_{\Omega} W(\nabla \nabla u) = \int_{\Omega} \tilde{W}(\nabla (e(u))) \]

- Very simple from the variationnal point of view.
  Equilibrium equations, natural boundary conditions.

- Controversial from the physical point of view.
  Compatibility with second principle of thermodynamics? with continuum mechanics theory?: the point is that boundary actions which go beyond a simple surface density of forces are hardly “understood”.

- Well known in the case of beams, plates, shells but the general belief is that low dimension is essential. Boundary actions are density of torques.


- Almost all the few known results give energies involving only the part \( \nabla (\nabla^{skew} u) \) of \( \nabla (\nabla u) \). (“incomplete” second gradient models)
  In which case boundary actions are density of torques.
The goal
Why are complete second gradient material so special?

Assume an energy $E(u) = \int_{\Omega} (\partial_1 \partial_1 u_1)^2$, no applied bulk forces. We just change the boundary conditions on the left hand side

from $u = 0, \partial_1 u_1 = 0$
The goal
Why are complete second gradient material so special?

Assume an energy $E(u) = \int_{\Omega} (\partial_1 \partial_1 u_1)^2$, no applied bulk forces. We just change the boundary conditions on the left hand side

to $u = 0, \partial_1 u_1 = 1$
The goal

The context

Many researchers use the model which has the advantage to regularize singularities (interfaces, plasticity, fractures, . . .)

Nobody has directly measured second gradient effects nor design materials with specific second gradient properties nor experimented the associated special boundary conditions. Do these materials actually exist?

A closure theorem (Camar-Eddine and P.S. 2003) states that any quadratic l.s.c. and objective functional can be obtained from classical linear elasticity through an homogenization process.

A very indirect method. The result is quite different when homogenizing conductivity problems (Camar-Eddine et P.S. 2002).
We want to describe the homogenized elastic behavior of this structure.
Thin walled structures

Assumptions

- Homogeneous linear elastic material (no micro buckling effect).
- Plane strain elasticity conditions: the problem reduces to a linear elastic problem set in a thickened periodic planar graph.
- The size $\ell$ of the period of the graph is small compared to the characteristic size $L$ of the domain $\Omega$ (standard asymptotic homogenization assumption) $\varepsilon = \ell/L << 1$.
- The thickness $e$ of the walls is small compared with $\ell$: $\delta = e/\ell << 1$. In the sequel $\delta = \varepsilon^\alpha$ with $\alpha > 1$. (actually $\alpha = 1$ is the most interesting case).
- Lamé coefficients $(\mu, \lambda)$ of the material tend to infinity like $\delta^{-1}\varepsilon^{-2}$:
  
  $$\mu = \frac{\mu_0}{\delta \varepsilon^2}, \quad \lambda = \frac{\lambda_0}{\delta \varepsilon^2}$$

- The material is fixed on some part of the domain to ensure some coercivity.
Thin walled structures

Geometry

- a prototype cell \( Y \) made of \( K \) nodes at points \( y_s, s \in \{1, \ldots, K\} \),
- two vectors \( t_1, t_2 \) for translating the cell.

Notation: \( y_{I,s} = y_s + it_1 + jt_2, l = (i,j) \in \mathbb{N}^2. I_\varepsilon := \{ (i,j) : \forall (p,s), \varepsilon y_{l+p,s} \in \Omega \} \).

- five matrices \( a^p \) defining the edges between the nodes of cells \( Y_l \) and \( Y_{l+p}, p \in \mathcal{P} := \{(0,0), (0,1), (1,0), (1,1), (1,-1)\} \). An edge links \( y_{l,s} \) and \( y_{l+p,s'} \) if \( a^p_{s,s'} > 0 \).

Notation: \( p = p_1 t_1 + p_2 t_2, \ell_{p,s,s'} := \|y_{l+p,s'} - y_{l,s}\|, \tau_{p,s,s'} := (y_{l+p,s'} - y_{l,s})/\ell_{p,s,s'} \).

\[ A := \{ (p,s,s') ; p \in \mathcal{P}, 1 \leq s \leq K, 1 \leq s' \leq K, a^p_{s,s'} > 0 \} \]

\[ G := \bigcup_{(l,p,s,s') \in I_\varepsilon \times A} [y_{l,s}, y_{l+p,s'}] \]

- Considered 2D domain: \( G_\varepsilon := \{ x \in \mathbb{R}^2 ; d(x, G) < \delta \} \).
Examples of periodic trusses

Example 1: the most standard truss

Geometry: \( K = 1 \) node, \( t_1 = (1, 0) \), \( t_2 = (-0.5, 0.866) \), \( a^{(0,0)} = (-) \), \( a^{(1,0)} = (1) \), \( a^{(0,1)} = (1) \), \( a^{(1,1)} = (1) \), \( a^{(1,-1)} = (0) \).

**Figure** – Regular triangle truss
Examples of periodic trusses

Example 2: A degenerated truss

We just modify $t_2 = (0, 1)$, $a^{(1,1)} = (0)$ in the previous example.

**Figure** – The regular square truss and its free shear deformation
Examples of periodic trusses

Example 2: A degenerated truss

We just modify $t_2 = (0, 1), \ a^{(1,1)} = (0)$ in the previous example.

**Figure** – The regular square truss and its free shear deformation
Examples of periodic trusses

Example 3: another structure with free shear

Geometry: \( K = 2 \) nodes, \( t_1 = (1, 0) \), \( t_2 = (0, 2) \), \( a^{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \),

\( a^{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), \( a^{(0,1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), \( a^{(1,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), \( a^{(1,-1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).
Examples of periodic trusses

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\[ \varepsilon t_2 \]

\[ \varepsilon t_1 \]
Examples of periodic trusses

Example 4: the truss we have printed

Geometry: $K = 6$ nodes, $t_1 = (4, 0)$, $t_2 = (-2, 4)$, $\ldots$

\[ \text{FIGURE} – \text{Pantographic truss} \]
Examples of periodic trusses

Example 4: the truss we have printed

Geometry: $K = 6$ nodes, $t_1 = (4, 0)$, $t_2 = (-2, 4)$, ...
Examples of periodic trusses

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Reduction to a discrete problem

Energies

Initial energy:

\[ E_{\varepsilon}(u) := \int_{\Omega_{\varepsilon}} \left( \frac{\mu_0}{\delta \varepsilon^2} \| e(u) \|^2 + \frac{\lambda_0}{2\delta \varepsilon^2} \text{tr}(e(u))^2 \right) \, dx \]

\((\mu_0 > 0, \lambda_0 + \mu_0 > 0, Y_0 := \frac{4\mu_0(\mu_0 + \lambda_0)}{2\mu_0 + \lambda_0}, v := v_0 := \frac{\lambda_0}{2\mu_0 + \lambda_0}). \text{ We fix} \)

\[(a^p)_{s,s'} = \frac{2Y_0}{\ell_{p,s,s'}}.\]

and to any discrete vector field \(U\), we associate the discrete energy

\[ E_{\varepsilon}(U) := \frac{1}{\varepsilon^2} \sum_{(l,p,s,s') \in I_{\varepsilon} \times A} \frac{a^p_{s,s'}}{2} \left( ((U_{l+p,s'} - U_{l,s}) \cdot \tau_{p,s,s'})^2 \right) \]
Reduction to a discrete problem

Convergences

- We say that a sequence of families of vectors \( Z^\varepsilon \) converges to \( z \) (\( Z^\varepsilon \rightharpoonup z \)), when

\[
\varepsilon^2 \sum_{l \in I_\varepsilon} \frac{1}{K} \sum_{s=1}^{K} Z^\varepsilon_{l,s} \delta_{\varepsilon y_{l,s}} \rightharpoonup z(x) \, dx
\]

- We say that a sequence of families of vectors \( Z^\varepsilon \) double-scale converges to \( z \) (\( Z^\varepsilon \rightharpoonup \rightharpoonup z \)), when

\[
\forall s \in \{1, \ldots, K\}, \quad \varepsilon^2 \sum_{l \in I_\varepsilon} Z^\varepsilon_{l,s} \delta_{\varepsilon y_{l,s}} \rightharpoonup z(x, s) \, dx
\]

- To any field \( u \in L^2(\Omega_\varepsilon; \mathbb{R}^2) \) we associate the family

\[
\bar{u}_{l,s} := \int_{B(\varepsilon y_{l,s}, \varepsilon \delta)} u(x) \, dx
\]

We say that the sequence of functions \( (u^\varepsilon) \) converges to \( u \) when \( \bar{u}^\varepsilon \rightharpoonup u \).
Reduction to a discrete problem

Result

Theorem

The $\Gamma$-limits of $(\mathcal{E}_\varepsilon)$ and $(E_\varepsilon)$ are identical: for any $u$, we have

(i) $\inf\{\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u^\varepsilon) : u^\varepsilon \rightharpoonup u\} \geq \inf\{\liminf_{\varepsilon \to 0} E_\varepsilon(U^\varepsilon) ; U^\varepsilon \rightharpoonup u\}$

(ii) $\inf\{\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u^\varepsilon) : u^\varepsilon \rightharpoonup u\} \leq \inf\{\limsup_{\varepsilon \to 0} E_\varepsilon(U^\varepsilon) ; U^\varepsilon \rightharpoonup u\}$

From now on we focus on $(E_\varepsilon)$
Study of the discrete problem

Double-scale limits

We set

\[ m_l := \sum_{s=1}^{K} U_{l,s}, \quad v_{l,s} := \frac{1}{\varepsilon} (U_{l,s} - m_l) \quad \text{and} \quad \chi_{p,l} := \frac{1}{\varepsilon} (m_{l+p} - m_l). \]

So that

\[ E_\varepsilon(U) = \bar{E}_\varepsilon(v, \chi) := \sum_{(l,p,s,s') \in I_\varepsilon \times \mathcal{A}} \frac{a_{s,s'}}{2} \left( (v_{l+p,s'} - v_{l,s} + \chi_{l,p}) \cdot \tau_{p,s,s'} \right)^2 \]

We assume that the sequences

\[ \varepsilon^2 \sum_{l \in I_\varepsilon} \|m_l\|^2, \quad \varepsilon^2 \sum_{l \in I_\varepsilon} \|v_{l,s}\|^2 \quad \text{and} \quad \varepsilon^2 \sum_{l \in I_\varepsilon} \|\chi_{l,p}\|^2 \]

are bounded. Thus there exist \( u, v \) and \( \chi \) in \( L^2 \) such that, up to a sub-sequence,

\[ m^\varepsilon \rightharpoonup u, \quad v^\varepsilon \rightharpoonup v \quad \text{and} \quad \chi_p^\varepsilon \rightharpoonup \chi_p. \]
We also have

\[ \mathcal{U}^\varepsilon \rightharpoonup u, \quad \sum_{s=1}^{K} \nu(x, s) = 0 \quad \text{and} \quad \chi_p = \nabla u \cdot p \]

To check that \( \chi_p = \nabla u \cdot p \), it is enough to notice that, for any smooth test field \( \varphi \),

\[
\int_{\Omega} \chi_p(x) \cdot \varphi(x) = \lim_{\varepsilon \to 0} \varepsilon^2 \sum_{l \in I_\varepsilon} \varepsilon^{-1} (m^\varepsilon_{l+p} - m^\varepsilon_l) \cdot \varphi(\varepsilon y_l) = \lim_{\varepsilon \to 0} \varepsilon^2 \sum_{l \in I_\varepsilon} m^\varepsilon_l \cdot \varepsilon^{-1} (\varphi(\varepsilon y_{l+p}) - \varphi(\varepsilon y_l))
\]

\[= \lim_{\varepsilon \to 0} \varepsilon^2 \sum_{l \in I_\varepsilon} m^\varepsilon_l \cdot (-\nabla \varphi(\varepsilon y_l) \cdot p) + O(\varepsilon) = -\int_{\Omega} u(x) \cdot (\nabla \varphi(x) \cdot p) = \int_{\Omega} (\nabla u(x) \cdot p) \cdot \varphi(x). \]
Study of the discrete problem

First homogenization result

\[ \varepsilon^2 \bar{E}_\varepsilon(v^\varepsilon, \chi^\varepsilon) := \varepsilon^2 \sum_{I \in I_\varepsilon} \sum_{(p, s, s') \in \mathcal{A}} \frac{a_{s, s'}^p}{2} ( (v_{l+p, s'} - v_{l, s} + \chi_{l, p}) \cdot \tau_{p, s, s'} )^2 \]

\[ \bar{E}(v, \eta u) := \int_\Omega \sum_{(p, s, s') \in \mathcal{A}} \left( \frac{a_{s, s'}^p}{2} ( (v_{s'}(x) - v_s(x) + (\eta u)_{p, s'}(x)) \cdot \tau_{p, s, s'} )^2 \right) dx \]

with \((\eta u)_{p, s}(x) := \nabla u(x) \cdot p\).

**Theorem**

The sequence \((\varepsilon^2 E_\varepsilon)\) Γ-converges to the function \(E := \inf_v \bar{E}(v, \eta u) :\)

(i) For all sequence \(U^\varepsilon\) such that \(U^\varepsilon \rightharpoonup u\), \(\liminf \varepsilon^2 E_\varepsilon(U^\varepsilon) \geq E(u)\).

(ii) \(\forall u \in L^2(\Omega), \exists U^\varepsilon\) such that \(U^\varepsilon \rightharpoonup u\) and \(\limsup \varepsilon^2 E_\varepsilon(U^\varepsilon) \leq E(u)\).
Study of the discrete problem

Main homogenization result

Consider a sequence such that \( \bar{E}_\varepsilon (\nu^\varepsilon, \chi^\varepsilon) < M \).

- Previous theorem implies \( \bar{E}(\nu, \eta_u) = 0 \) and so

\[
(v_s'(x) - v_s(x) + \nabla u \cdot p) \cdot \tau_{p,s,s'} = 0.
\]

- \( \omega^\varepsilon_{l,p,s,s'} := \varepsilon^{-2} (U^\varepsilon_{l+p,s'} - U^\varepsilon_{l,s}) \cdot \tau_{p,s,s'} \rightarrow \omega_{p,s,s'} \).

- Set \( \mathcal{D}_A := \{ (w_{s'} - w_s + \nabla \lambda \cdot p) \cdot \tau_{p,s,s'} : w_s, \lambda \in L^2(\mathbb{R}^2, \mathbb{R}^2) \} \)

\( \phi \in \mathcal{D}_A^\perp \) means

\[
\sum_{(p,s,s') \in \mathcal{A}} (\nabla \phi_{p,s,s'} \cdot p) \tau_{p,s,s'} = 0
\]

and

\[
\sum_{(p,s,s') \in \mathcal{P} \times \{1, \ldots, K\}^2} \tau_{p,s,s'} \phi_{p,s,s'} - \tau_{p,s',s} \phi_{p,s',s} = 0.
\]
Study of the discrete problem

Main homogenization result

For such functions we have

\[ \int \sum_{\Omega(p,s,s')} \omega_{p,s,s'}(x) \phi_{p,s,s'}(x) = \lim \varepsilon^2 \sum_{(l,p,s,s')} \varepsilon^{-2}(U_{l+p,s'} - U_{l,s}) \cdot (\phi_{p,s,s'}(\varepsilon y_l) \tau_{p,s,s'}) \]

\[ = \lim \varepsilon^2 \sum_{(l,p,s,s')} (\varepsilon^{-1}(v_{l+p,s'}^{\varepsilon} - v_{l,s'}^{\varepsilon}) + (\bar{v}_{l,s'}^{\varepsilon} - \bar{v}_{l,s}^{\varepsilon}) + \varepsilon^{-2}(m_{l+p}^{\varepsilon} - m_{l}^{\varepsilon})) \cdot (\phi_{p,s,s'}(\varepsilon y_l) \tau_{p,s,s'}) \]

Assuming, moreover, that they are smooth we get for the first addend:

\[ \lim \varepsilon^2 \sum_{(l,p,s,s')} \varepsilon^{-1}(v_{l+p,s'}^{\varepsilon} - v_{l,s'}^{\varepsilon}) \cdot (\phi_{p,s,s'}(\varepsilon y_l) \tau_{p,s,s'}) \]

\[ = \lim \varepsilon^2 \sum_{(l,p,s,s')} v_{l,s'}^{\varepsilon} \cdot (\varepsilon^{-1}(\phi_{p,s,s'}(\varepsilon y_{l-p}) - \phi_{p,s,s'}(\varepsilon y_l)) \tau_{p,s,s'}) \]

\[ = \lim \varepsilon^2 \sum_{(l,p,s,s')} v_{l,s'}^{\varepsilon} \cdot ((-\nabla \phi_{p,s,s'}(\varepsilon y_l) \cdot p) \tau_{p,s,s'}) + O(\varepsilon) \]

\[ = -\int \sum_{(p,s,s')} v_{s'}(x) \cdot ((\nabla \phi_{p,s,s'}(x) \cdot p) \tau_{p,s,s'}) \]

\[ = \left< \sum_{(p,s,s')} (\nabla v_{s'}(x) \cdot p) \cdot (\phi_{p,s,s'}(x) \tau_{p,s,s'}) \right>, \]
Study of the discrete problem

Main homogenization result

and for the second addend

\[
\lim \epsilon^2 \sum_{(l,p,s,s')} (\epsilon^{-2}(m_i^\epsilon + p - m_i^\epsilon)) \cdot (\phi_{p,s,s'}(\epsilon y_l)\tau_{p,s,s'}) \\
= \lim \epsilon^2 \sum_{(l,p,s,s')} \epsilon^{-2} m_i^\epsilon \cdot ((\phi_{p,s,s'}(\epsilon y_{l-p}) - \phi_{p,s,s'}(\epsilon y_l))\tau_{p,s,s'}) \\
= \lim \epsilon^2 \sum_{(l,p,s,s')} m_i^\epsilon \cdot ((-\epsilon^{-1}\nabla \phi_{p,s,s'}(\epsilon y_l) \cdot p + \frac{1}{2}\nabla \nabla \phi_{p,s,s'}(\epsilon y_l) \cdot p \cdot p)\tau_{p,s,s'} + O(\epsilon)) \\
= \int_\Omega \sum_{(p,s,s')} u(x) \cdot ((\frac{1}{2}\nabla \nabla \phi_{p,s,s'}(x) \cdot p \cdot p)\tau_{p,s,s'}) \\
= \left\langle \sum_{(p,s,s')} \frac{1}{2}(\nabla \nabla u(x) \cdot p \cdot p) \cdot (\phi_{p,s,s'}(x)\tau_{p,s,s'}) \right\rangle.
\]

Collecting these results we obtain that the distribution

\[
\omega_{p,s,s'}(x) - (\nabla v_k'(x) \cdot p + \frac{1}{2}\nabla \nabla u \cdot p \cdot p) \cdot \tau_{p,s,s'}
\]

is orthogonal to all smooth functions in \(D_A^\perp\).
Study of the discrete problem
Main homogenization result

There exist some fields $w_s$ and $\lambda$ in $L^2(\mathbb{R}^2)$ such that, for any $(p, s, s') \in \mathcal{A}$,

$$
\omega_{p,s,s'}(x) - \left( \nabla v_{s'}(x) \cdot p + \frac{1}{2} \nabla \nabla u \cdot p \cdot p \right) \cdot \tau_{p,s,s'} = \left( \nabla \lambda(x) \cdot p + w_{s'}(x) - w_s(x) \right) \cdot \tau_{p,s,s'}
$$

$$
\omega_{p,s,s'}(x) = \left( \nabla (v_{s'} + \lambda)(x) \cdot p + \frac{1}{2} \nabla \nabla u \cdot p \cdot p + w_{s'}(x) - w_s(x) \right) \cdot \tau_{p,s,s'}.
$$

$$
\liminf E_\varepsilon(U) = \liminf \varepsilon^2 \sum_{(l,p,s,s')} (\omega_{l,p,s,s'}^\varepsilon)^2 \geq \int_{\Omega} (\omega_{p,s,s'}(x))^2 dx = \bar{E}(w, \xi_{u,v+\lambda})
$$

with

$$
(\xi_{u,v})_{p,s} = \nabla v_s \cdot p + \frac{1}{2} \nabla \nabla u \cdot p \cdot p.
$$
Study of the discrete problem

Main homogenization result

\[ \mathcal{E}(u) := \inf_{w,v} \left\{ \bar{E}(w, \xi u, v); \bar{E}(v, \eta u) = 0 \right\} \]

Theorem

\( E_\varepsilon \) Γ-converges to \( \mathcal{E} \):

(i) For all sequence \( (U_\varepsilon) \) such that \( U_\varepsilon \rightharpoonup u \), \( \lim \inf E_\varepsilon(U_\varepsilon) \geq \mathcal{E}(u) \).

(ii) \( \forall u \), there exists a sequence \( U_\varepsilon \rightharpoonup u \) and \( \lim \sup E_\varepsilon(U_\varepsilon) \leq \mathcal{E}(u) \).

The approximating sequence is built by setting

\[ U_{i,s}^\varepsilon := u(\varepsilon y_i) + \varepsilon v_s(\varepsilon y_i) + \varepsilon^2 w_s(\varepsilon y_i) \]

where \( (v, w) \) are such that \( \mathcal{E}(u) = \bar{E}(w, \xi u, v) \) and \( \bar{E}(v, \eta u) = 0 \).
Study of the discrete problem
Explicit computation of the limit energy

We can easily rewrite the integrand of \( \bar{E} \):

\[
\sum_{p,s,s'} \frac{\partial_{s,s'}^p}{2} ((v_{s'} - v_s + \eta_{p,s'}) \cdot \tau_{p,s,s'})^2
\]

under the form

\[
\frac{1}{2} v \cdot A \cdot v + v \cdot B \cdot \eta + \frac{1}{2} \eta \cdot C \cdot \eta.
\]

(Identifying \( v \) with the 2K-dimensional vector \((v_1, v_2, \ldots, v_K)\) and \( \eta \) with the 10 \( \times \) K vector \((\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{5,K})\))

A solution of the minimization problem \( \inf_{v} \bar{E}(v, \eta) \) is \( \bar{v} := -A^+ \cdot B \cdot \eta \) and the minimal value is \( \frac{1}{2} \eta \cdot D \cdot \eta \) where \( D := C - B^t \cdot A^+ \cdot B \).

Let \( L \) (identified to a 10\( K \times 4 \) matrix) be the the linear operator such that, for any 2 \( \times \) 2 matrix \( M \) and any \((p, s), (L \cdot M)_{p,s} = M \cdot p.\)

\[
E(u) = \int_{\Omega} \frac{1}{2} \nabla u(x) \cdot L^t \cdot D \cdot L \cdot \nabla u(x) \, dx
\]

\[
R = L^t \cdot (C - B^t \cdot A^+ \cdot B) \cdot L.
\]
Study of the discrete problem

Explicit computation of the limit energy

The minimal value of \( \inf_w \tilde{E}(w, \xi_{u,v}) \) is again \( \int_\Omega \frac{1}{2} \xi_{u,v} \cdot D \cdot \xi_{u,v} \).

Remind that \( v \) has to satisfy \( \tilde{E}(v, \eta_u) = 0 \). Therefore \( v = -A^+ \cdot B \cdot L \cdot \nabla u + \tilde{v} \) with \( \tilde{v} \) in the kernel of \( A \). (\( \tilde{v} \rightarrow \mu \) in A. Abdulle, \( \rightarrow \hat{u}_1 \) in V. Vinoles)

\[
(\xi_{u,v})_{\rho,s}(x) = -(A^+ \cdot B \cdot L \cdot \nabla \nabla u(x) \cdot p)_s + \nabla \tilde{v}_s(x) \cdot p + \frac{1}{2} \nabla \nabla u(x) \cdot p \cdot p
\]

which we can rewrite under the form \( \xi_{u,v}(x) = \bar{L} \cdot \nabla \nabla u(x) + \tilde{L} \cdot \nabla \tilde{v}(x) \).

Setting \( G := \bar{L}^t \cdot D \cdot \bar{L} \), \( H := \bar{L}^t \cdot D \cdot \tilde{L} \), \( J := \tilde{L}^t \cdot D \cdot \tilde{L} \)

\[
\frac{1}{2} \xi_{u,v} \cdot D \cdot \xi_{u,v} = \frac{1}{2} \nabla \nabla u \cdot G \cdot \nabla \nabla u + \nabla \nabla u \cdot H \cdot \nabla \tilde{v} + \frac{1}{2} \nabla \tilde{v} \cdot J \cdot \nabla \tilde{v}.
\]

- Either stop there : the model is a a second gradient model coupled with an extra kinematic variable.

- Or (when possible) use \( \tilde{v} \) such that \( \nabla \tilde{v} = -J^+ \cdot H^t \cdot \nabla \nabla u \). In which case the model is a pure second gradient one :

\[
\mathcal{E}(u) = \inf_v \int_\Omega \frac{1}{2} \nabla \nabla u \cdot R_c \cdot \nabla \nabla u \, dx \quad \text{where} \quad R_c := G - H \cdot J^+ \cdot H^t.
\]
Results for the different examples

The regular triangular truss

\[ R \text{ is given in the basis corresponding to } (e_{11}, e_{22}, e_{21}, \partial_2 u_1, \partial_1 u_2) \text{ instead of } (\partial_1 u_1, \partial_2 u_1, \partial_1 u_2, \partial_2 u_2). \] We get

\[ \tilde{R} = a \sqrt{3} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \]

(\(\mu = \lambda = a \sqrt{3}\) and its Poisson ratio is \(\nu = 1/3\))

As \(E\) is non degenerated, only rigid motions are admissible in \(E\).
Results for the different examples

The square truss

For this example, our procedures lead to

\[ \tilde{R} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R = 0. \]

As expected the truss shows a degeneracy with respect to shear. As \( R = 0 \) this truss does not present any second gradient effect.

A free global degree of mobility is not a sufficient condition for observing second gradient effects.
Results for the different examples

The second structure with free shear

We get
\[ \tilde{R} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
and
\[ \tilde{R} = \frac{a}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

The kernel of \( \tilde{R} \) has dimension 1 and direction \( e_{12} \). The matrix \( \tilde{R} \) is thus written in the basis corresponding to \( (\partial_1 e_{12}, \partial_2 e_{12}) \). The limit energy reads

\[ \mathcal{E}(u) = \int_{\Omega} \frac{a}{8} (\partial_1 e_{12}(u))^2 \, dx, \quad \text{if} \ e_{11}(u) = e_{22}(u) = 0 \]
Examples of periodic trusses

The truss we have printed

\[ \mathbf{R} = a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

We get \( \tilde{\mathbf{R}} = \frac{12}{17} a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). The kernel of \( \tilde{\mathbf{R}} \) has dimension 2 and directions \((e_{11}, e_{12})\). The limit model is a coupled one, but when restricting to displacement fields of the type \( u(x) = u_1(x_1) \) the energy reduces to

\[ \mathcal{E}(u) = \int_{\Omega} 0.0656 a (\partial_{11}^2 u_1)^2 \]

where \( \partial_{11}^2 \) denotes the second derivative with respect to \( x_1 \).
Conclusion

- Homogenization of our structure reduces to the homogenization of a periodic truss.
- Homogenization of trusses reduces to simple algebraic formulas.
- Homogenization may lead to second gradient effects. This happens when
  - the structure has some global degrees of mobility,
  - this needs some weak zones simulating hinges,
  - the geometry is designed in such a way that the deformation of a cell is transmitted to its neighbors.
- Results must be extended to welded grids (taking into account the flexion energy of the bars).
- First experimental result: the walls of the sample we have printed are too thick for our asymptotic result to apply. Second gradient terms are hidden by the other terms . . .
- Future: optimize the structure to get an experimental evidence?