# Superlensing using hyperbolic metamaterials

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**Outline:**

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3. A toy problem for an elliptic/hyperbolic equation
4. Another toy problem for a forward-backward device
5. More interesting devices
6. Design stability ?
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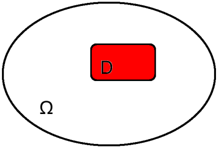
## 1. Introduction




Consider $\Omega$ smooth bounded domain in $\mathbb{R}^d$ and $D \subset\subset \Omega$

Let $k \geq 0$. Metamaterials generally concern PDE's of the type

$$\begin{cases} \mathrm{div}(s_\delta a \nabla u_\delta) + k^2 s_\delta \sigma u_\delta = f & \text{in } \Omega \\ a\nabla u_\delta \cdot \nu - iku_\delta = 0 & \text{on } \partial\Omega \end{cases}$$

where the coefficients $a(x), \sigma(x)$ satisfy for some $a_0, \sigma_0 > 0$

$$\begin{array}{ccccccc} 1/a_0|\xi|^2 & \leq & a(x)\xi\cdot\xi & \leq & a_0|\xi|^2, & & \xi \in \mathbb{R}^d \\ 1/\sigma_0 & \leq & \sigma(x) & \leq & \sigma_0 & & \end{array}$$

and where for $\delta > 0$

$$s_\delta(x) = \begin{cases} 1 & x \in \Omega \setminus \overline{D} \\ -1 - i\delta & x \in D \end{cases}$$

The parameter $\delta$ models absorption of EM energy in the medium

When $\delta > 0$, the PDE (with proper BC's) has a unique solution

- Under what conditions do the $u_\delta$'s remain bounded or do they converge to some limiting $u_0$ ?

- If this is the case, does $u_0$ solve the limiting equation

$$\begin{cases} \mathrm{div}(s_0(x)a(x)\nabla u_0(x)) + k^2 s_0(x)\sigma(x)u_0(x) \; = \; f \qquad \text{in } \Omega \\ a\nabla u_0 \cdot \nu - iku_0 \; = \; 0 \qquad \text{on } \partial\Omega \end{cases} \qquad ?$$

- What are the particular properties (localization, blow up,...) of the $u_\delta$'s or of $u_0$ ?

Metamaterials have become an active research area :

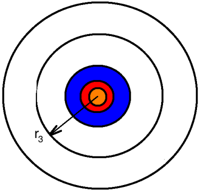
- Construction of metamaterials : Bouchitté-Bourel-Felbacq, Pendry, Shelby et al, Milton-McPhedran,. . .
- Well-posedness of transmission problems with sign-changing coefficients : Costabel-Stephan, Bonnet BenDiah-Ciarlet-Chesnel, Hoai-Minh Nguyen, . . .
- Cloaking : Nicorovici-Milton-McPhedran, Ammari-Ciraolo-Kang-Lee-Milton, Kohn-Lu-Schweizer-Weinstein, Hoai-Minh Nguyen, . . .

We are particularly interested in hyperbolic metamaterials, where for $\delta = 0$ the coefficient matrix in the inclusion has positive and negative eigenvalues

Hyperbolic metamaterials should be easier to build than materials with negative indices

Interesting effects have been observed experimentally, particularly superlensing, focusing, and enhancement of nonlinear response
[Poddubny et al, Nature Photonics, Vol 7. Dec. 2013]

## 2. Superlensing using NIM's




Consider $0 < r_0 < r_1 < r_2 < r_3 < R \qquad m, \alpha > 1$

$$mr_0 = r_2 \qquad mr_1 = r_3 \qquad r_3 = r_2^{\alpha}/r_1^{\alpha-1}$$

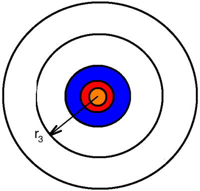
Let $F \;: B_{r_2} \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \overline{B_{r_2}}$ defined by $F(x) = r_2^{\alpha} x/|x|^{\alpha}$

Consider a conductivity map of the form

$$A(x) = \begin{cases} a(x) & \text{in } B_{r_0} \\ m^{d-2} I & \text{in } B_{r_1} \setminus B_{r_0} \\ F_*^{-1} I & \text{in } B_{r_2} \setminus B_{r_1} \\ I & \text{otherwise} \end{cases} \quad \text{and} \quad s_\delta(x) = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1} \\ 1 & \text{otherwise} \end{cases}$$

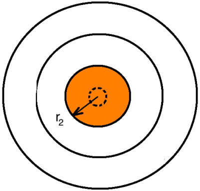
Let $u_\delta$ be the solution in $H^1_0(B_R)$ to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega = B_R$$




and let $\mathcal{U} \in H^1_0(B_R)$ be the solution to $\operatorname{div}(\mathcal{A}\nabla \mathcal{U}) = f$

with $$\mathcal{A} = \begin{cases} a(x/m) & \text{in } B_{r_2} \\ I & \text{otherwise} \end{cases}$$




**Thm** [HM Nguyen 15] Assume that $Spt f \cap B_{r_3} = \emptyset$, then

$$u_\delta \rightharpoonup \mathcal{U} \quad \text{in } H^1(\Omega \setminus B_{r_3}) \text{ as } \delta \to 0$$

**Remarks :**

- A similar construction is valid in 3D and for the Helmholtz equation
- This construction uses reflecting complementary media, i.e. such that

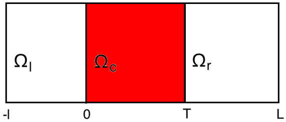
$$\begin{aligned} F_*(x)A(x) &= \frac{DF(y)A(y)DF^T(y)}{|\det DT(y)|} \qquad \text{with } y = F^{-1}(x) \\ &\equiv A(x), \;\; x \in B_{r_3} \setminus \overline{B_{r_2}} \\ F(x) &= x, \;\; x \in \partial B_{r_2} \end{aligned}$$

  so that $u_\delta$ and $u_\delta \circ F^{-1}$ satisfy the same equation and the same Cauchy data across $\partial B_{r_2}$

- Superlensing is achieved: for an observer outside $B_{r_3}$, the medium $a(x)$ that occupies $B_{r_0}$ is perceived as

$$m^{2-d} a(x/m) \;\; \text{in} \;\; B_{r_2} = B_{m r_0}$$

# 3. A toy problem for an elliptic/hyperbolic equation




Consider

$$\begin{aligned} \Omega &= \Omega_l \cup \Omega_c \cup \Omega_r \\ &= (-l,0)\times(0,2\pi) \cup (0,T)\times(0,2\pi) \cup (T,L)\times(0,2\pi) \end{aligned}$$

and $\Gamma = \partial\Omega, \quad \Gamma_l = \{0\}\times(0,2\pi), \quad \Gamma_r = \{T\}\times(0,2\pi)$

Let $a$ be a uniformly elliptic matrix-valued conductivity and

$$a_\delta = \begin{pmatrix} 1 - i\delta & 0 \\ 0 & -1 - i\delta \end{pmatrix}$$

Consider the conductivity

$$A_\delta(x) = \begin{cases} a(x) & \text{in } \Omega_l \cup \Omega_r \\ a_\delta(x) & \text{in } \Omega_c \end{cases}$$

Let $f \in L^2(\Omega)$, with $\quad Spt(f) \cap \Omega_c = \emptyset$

and consider the equation

$$\begin{cases} \operatorname{div}(A_\delta \nabla u_\delta) = f & \text{in } \Omega \\ u_\delta \in H_0^1(\Omega) \end{cases}$$

Assume that $u_\delta$ is uniformly bounded in $H^1(\Omega)$ and that $u_\delta \rightharpoonup u_0$ weakly in $H^1(\Omega)$

Then $u_0 \in H_0^1(\Omega)$ is a solution to

$$\begin{cases} \operatorname{div}(a \nabla u_0) & = & f & \text{in } \Omega \setminus \Omega_c \\ (\partial_{11}^2 - \partial_{22}^2) u_0 & = & 0 & \text{in } \Omega_c \end{cases}$$

with the transmission conditions

$$\begin{cases} u_0|_{\Omega_c} & = & u_0|_{\Omega_l} \\ \partial_1 u_0|_{\Omega_c} & = & a\partial_1 u_0|_{\Omega_l} \end{cases} \text{on } \Gamma_l \qquad \begin{cases} u_0|_{\Omega_c} & = & u_0|_{\Omega_r} \\ \partial_1 u_0|_{\Omega_c} & = & a\partial_1 u_0|_{\Omega_r} \end{cases} \text{on } \Gamma_r$$

This is an ill-posed probem, except for special choices of $T$

Consider the effective domain

$$\hat{\Omega} \quad = \quad \Omega_l \cup \left( \begin{pmatrix} -T \\ 0 \end{pmatrix} + \Omega_r \right)$$

and define

$$\hat{a}(x_1, x_2), \hat{f}(x_1, x_2) \;=\; \begin{cases} a(x_1, x_2), f(x_1, x_2) & \text{in } \Omega_l \\ a(x_1 + T, x_2), f(x_1 + T, x_2) & \text{in } \hat{\Omega} \setminus \Omega_l \end{cases}$$

Assume that $\hat{a}$ is smooth, so that the solution to

$$\begin{cases} \text{div}(\hat{a} \nabla \hat{u}) \;=\; \hat{f} & \text{in } \hat{\Omega} \\ \hat{u} \in H_0^1(\hat{\Omega}) \end{cases}$$

is in $H^2(\hat{\Omega})$

## Proposition : superlensing with tuned HHMs

Assume that $T$ is a multiple of $2\pi$ and that $\quad Spt(f) \cap \Omega_c = \emptyset$

Then the solutions $u_\delta$ to

$$\left\{ \begin{array}{lll} \mathrm{div}(A_\delta \nabla u_\delta) & = & f \qquad \text{in } \Omega \\ u_\delta \in H_0^1(\Omega) & & \end{array} \right.$$

are uniformly bounded and converge strongly in $H^1$ to $u_0$, the unique solution to

$$\left\{ \begin{array}{lll} \mathrm{div}(A_0 \nabla u_0) & = & f \qquad \text{in } \Omega \\ u_0 \in H_0^1(\Omega) & & \end{array} \right.$$

Moreover, $u_0$ satisfies

$$u_0(x_1, x_2) \quad = \quad \left\{ \begin{array}{ll} \hat{u}(x_1, x_2) & \text{in } \Omega_l \\ \hat{u}(x_1 - T, x_2) & \text{in } \Omega_r \end{array} \right.$$

In other words, $u_0$ can be computed in $\Omega_l \cup \Omega_r$ as if the part $\Omega_c$ had disappeared

## Proof: 1. Construction of $u_0$

The smoothness assumption on $\hat{a}$ implies that $\hat{u} \in H^2(\hat{\Omega})$ and

$$||\hat{u}||_{H^2(\hat{\Omega})} \quad \leq \quad C\,||f||_{L^2(\hat{\Omega})}$$

Interpreting $x_1$ as a time variable in $\Omega_c$, standard results for the wave equation show that there is a unique solution $v \in \mathcal{C}^0([0,T], H_0^1(0,2\pi)) \cap \mathcal{C}^1([0,T], L^2(0,2\pi))$ to

$$(\partial_{11}^2 - \partial_{22}^2)v \quad = \quad 0 \qquad \text{in } \Omega_c = (0,T) \times (0,2\pi)$$

with the boundary condition $v = 0$ on $\partial\Omega \cap \partial\Omega_c$, and with the initial conditions

$$\begin{cases} v(0,x_2) & = & \hat{u}(0,x_2) \\ \partial_1 v(0,x_2) & = & \partial_1 \hat{u}(0,x_2) \end{cases}$$

Moreover, $v$ satisfies

$$||\nabla v||_{L^2(\Omega_c)} \quad \leq \quad C\int_0^{2\pi} |\hat{u}(0,x_2)|^2 + |\partial_1 \hat{u}(0,x_2)|^2 \;\leq\; C||f||_{L^2(\Omega)}$$

As $v(x) = \sum_{n \geq 1} \sin(nx_2) \left( a_n \cos(nx_1) + b_n \sin(nx_1) \right)$

and since $T \in 2\pi\mathbf{N}$ we have

$$\left\{ \begin{array}{rcr} v(0, x_2) & = & v(T, x_2) \\ \partial_1 v(0, x_2) & = & \partial_1 v(T, x_2) \end{array} \right.$$

i.e., $v$ satisfies the same Cauchy data at $x_1 = 0$ and at $x_1 = T$

One can then define the $H^1_0$ function

$$u_0(x_1, x_2) = \left\{ \begin{array}{ll} \hat{u}(x_1, x_2) & \text{in } \Omega_l \\ v(x_1, x_2) & \text{in } \Omega_c \\ \hat{u}(x_1 - T, x_2) & \text{in } \Omega_r \end{array} \right.$$

which satisfies $\operatorname{div}(A_0 \nabla u_0) = f$ in $\Omega$, and

$$||u_0||_{H^1(\Omega)} \leq C\, ||f||_{L^2(\Omega)}$$

## 2. Uniqueness of $u_0$

Assume that $w_0$ is another solution

In $\Omega_c$, $w_0$ can be expanded as
$w_0(x) = \sum_{n\geq 1} \sin(nx_2)\,(\alpha_n \cos(nx_1) + \beta_n \sin(nx_1))$

$$\text{so that} \qquad \left\{ \begin{array}{rcl} w_0(0,x_2) & = & w_0(T,x_2) \\ \partial_1 w_0(0,x_2) & = & \partial_1 w_0(T,x_2) \end{array} \right.$$

one can then define

$$\hat{w}(x_1,x_2) \quad = \quad \left\{ \begin{array}{ll} w_0(x_1,x_2) & \text{in } \Omega_l \\ w_0(x_1+T,x_2) & \text{in } \hat{\Omega} \setminus \Omega_l \end{array} \right.$$

which is in $H^1_0(\hat{\Omega})$, and which solves $\quad \mathrm{div}(\hat{A}\nabla\hat{w}) \;=\; \hat{f} \quad$ in $\hat{\Omega}$

Uniqueness for this problem implies that $\hat{w} \equiv \hat{u}$, from which it follows that $w_0 \equiv u_0$

## 3. Convergence

Set $\quad v_\delta = u_\delta - u_0 \quad$ in $\Omega$. Then

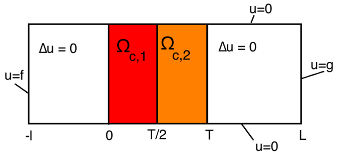
$$\begin{aligned} \mathrm{div}(A_\delta \nabla v_\delta) &= \mathrm{div}(A_\delta \nabla u_\delta) - \mathrm{div}(A_0 \nabla u_0) + \mathrm{div}(A_0 \nabla u_0) - \mathrm{div}(A_\delta \nabla u_0) \\ &= \mathrm{div}(i\delta 1_{\Omega_c} \nabla u_0) \end{aligned}$$

Multiplying by $\overline{v_d}$, integrating, taking the imaginary and real parts yields

$$\begin{aligned} ||\nabla v_\delta||^2_{L^2(\Omega_c)} &\leq \left| \int_\Omega \nabla u_0 \nabla v_\delta \right| \\ \int_\Omega |\nabla v_\delta|^2 &\leq C \left| \int_\Omega \nabla u_0 \nabla v_\delta \right| \end{aligned}$$

from which it follows that $u_\delta$ is uniformly bounded in $H^1$, and that $v_\delta$ converges strongly to $0$

# 4. Another toy problem for a forward-backward device




$$
\begin{aligned}
\Omega &= (-l,0)\times(0,2\pi)\ \cup\ (0,T/2)\times(0,2\pi)\ \cup\ (T/2,T)\times(0,2\pi)\ \cup\ (T,L)\times(0,2\pi)\\
&= \Omega_l\ \cup\ \Omega_{c,1}\ \cup\ \Omega_{c,2}\ \cup\ \Omega_r
\end{aligned}
$$

We consider a conductivity of the form

$$
A_\delta(x) = \begin{cases} a(x) & x\in\Omega_l\cup\Omega_r \\ \begin{pmatrix} 1+iO(\delta) & 0 \\ 0 & -1+iO(\delta)\end{pmatrix} & x\in\Omega_{c,1} \\ \begin{pmatrix} -1+iO(\delta) & 0 \\ 0 & 1+iO(\delta)\end{pmatrix} & x\in\Omega_{c,2} \end{cases}
$$

## Prop: superlensing via the complementary property

Assume that $Sptf \cap \Omega_c = \emptyset$

Then for some $C > 0$, independent of $\delta$ and $f$, we have

$$||u_\delta||_{H^1(\Omega)} \quad \leq \quad C||f||_{L^2(\Omega)}$$

Further, as $\delta \to 0$, $u_\delta \to u_0$ in $H^1(\Omega)$, where $u_0 \in H_0^1(\Omega)$ is the unique solution to

$$\mathrm{div}(A_0 \nabla u_0) \quad = \quad f \ \text{ in } \Omega$$

Additionnally, let $\hat{u}$ denote the $H_0^1(\hat{\Omega})$-solution to

$$\mathrm{div}(\hat{A} \nabla \hat{u}) \quad = \quad \hat{f} \ \text{ in } \hat{\Omega}$$

then $$u_0(x_1, x_2) \ = \begin{cases} \hat{u}(x_1, x_2) & \text{in } \Omega_l \\ \hat{u}(x_1 - T, x_2) & \text{in } \Omega_r \end{cases}$$

Again, the part $\Omega_c$ has disappeared in the limit. There is no hypothesis on $T$ here

The proof consists in the same 3 steps as in the previous proof

In the first step construct a solution to the wave equation

$$(\partial_{11}^2 - \partial_{22}^2)v = 0 \quad \text{in } \Omega_{c,1}$$

with Cauchy data

$$v(0,x_2) = \hat{u}(0,x_2) \qquad \partial_1 v(0,x_2) = \partial_1 \hat{u}(0,x_2)$$

The reflection of $v$ across $x_1 = T/2$

$$v_r(x_1,x_2) = v(T-x_1,x_2)$$

is also a solution to the wave equation and satisfies the transmission conditions

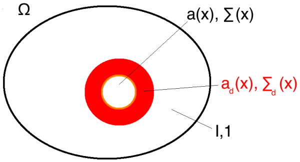
$$v(T/2,x_2) = v_r(T/2,x_2) \qquad \partial_1 v(T/2,x_2) = -\partial_1 v_r(T/2,x_2)$$

It follows that $\hat{u}(0,x_2) = v_r(T,x_2) \qquad \partial_1 \hat{u}(0,x_2) = -\partial_1 v_r(T,x_2)$

so that one can define

$$u_0 = \begin{cases} \hat{u} & \text{in } \Omega_l \\ v & \text{in } \Omega_{c,1} \\ v_r & \text{in } \Omega_{c,2} \\ \hat{u} & \text{in } \Omega_r \end{cases}$$

## 5. More interesting devices :




The metamaterials lie in $B_{r_2} \setminus B_{r_1}$

- Tuned superlensing in 2D, static and finite frequency case

- Tuned superlensing using HMM's in 3d, finite frequency case

- Superlensing via the complementary property in 2d and 3d, static and finite frequency case

## Tuned superlensing using HMM's in 3d, finite frequency case

Let $\Omega$ be a smooth bounded connected domain in $\mathbb{R}^3$

Assume that $0 < r_1 < r_2, \quad r_2 - r_1 \in 4\pi \mathbf{N}, \quad B_{r_2} \subset \Omega$

Assume that $k > 0$, $f \in L^2(\Omega \setminus B_{r_2})$

Medium description

$$A_\delta, \Sigma_\delta \quad = \quad \begin{cases} a \quad \text{(uniformly elliptic, smooth)} & \sigma \quad \text{(bounded)} & \text{in } B_{r_1} \\ \dfrac{1}{r^2} e_r \otimes e_r - (e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi) - i\delta & \dfrac{1}{4k^2 r^2} + i\delta & \text{in } B_{r_2} \setminus B_{r_1} \\ I & 1 & \text{in } \Omega \setminus B_{r_2} \end{cases}$$

**Thm :** The solutions $u_\delta$ to

$$\begin{cases} \mathrm{div}(A_\delta \nabla u_\delta) + k^2 \Sigma_\delta u_\delta & = & f & \text{in } \Omega \\ \partial_\nu u_\delta - iku_\delta & = & 0 & \text{on } \partial\Omega \end{cases}$$

are uniformly bounded in $H^1(\Omega)$ and converge strongly to $u_0$ the unique solution to the above system with $\delta = 0$

Moreover, $u_0 = \hat{u}$ in $\Omega \setminus B_{r_2}$, where $\hat{u}$ is the unique solution to

$$\begin{cases} \mathrm{div}(\hat{A} \nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} & = & f & \text{in } \Omega \\ \\ \partial_\nu \hat{u} - ik\hat{u} & = & 0 & \text{on } \partial\Omega \end{cases}$$

where $\quad \hat{A}, \hat{\Sigma} \; = \begin{cases} I & 1 & \text{in } \Omega \setminus B_{r_2} \\ \dfrac{r_1}{r_2} a(\dfrac{r_1}{r_2} x) & \dfrac{r_1^3}{r_2^3} \sigma(\dfrac{r_1}{r_2} x) & \text{in } B_{r_2} \end{cases}$

**Remarks :**

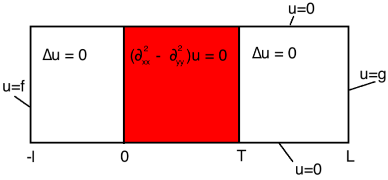
- The object in $B_{r_1}$ is magnified in the limit $\delta \to 0$ by a factor $\dfrac{r_2}{r_1}$
- The equation for $u_\delta$ in $B_{r_2} \setminus B_{r_1}$ takes the form

$$\partial^2_{rr} u - \Delta_{S^1} u + \frac{1}{4} u \quad = \quad 0$$

Expansion in spherical harmonics shows that the Cauchy data are transported from $\partial B_{r_2}$ to $\partial B_{r_1}$

## 6. Design stability

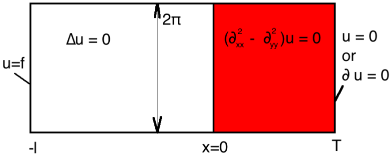
Consider again the toy problem




What happens when $T$ is not a multiple of $2\pi$ ?

If we can define a limiting solution $u_0$ when $\delta = 0$, then the same proof as for the case $T = 2\pi$ shows that $u_\delta \to u_0$ (in particular the $u_\delta$ are uniformly bounded)

By linearity, we can restrict the study to




Seek

$$u_0(x,y) = \begin{cases} \sum_{n\geq 1} sin(ny)\left(a_n e^{nx} + b_n e^{-nx}\right) & -l < x < 0 \\ \sum_{n\geq 1} sin(ny)\left(\alpha_n e^{inx} + \beta_n e^{-inx}\right) & 0 < x < T \end{cases}$$

Expressing the transmission and boundary conditions fixes the values of the $a_n, b_n, \alpha_n, \beta_n$'s provided the determinant of the associated linear system does not vanish

## The case of a homogeneous Dirichlet BC at $x = T$

$$\det \quad = \quad 2i\left[e^{nl}\left(\cos(nT) + \sin(nT)\right) - e^{-nl}\left(\cos(nT) - \sin(nT)\right)\right]$$

- If $T/\pi$ is irrational and diophantine of class $r$

$$\forall\, (p,q) \in \mathbf{Z} \times \mathbf{Z}^* \quad |l - \frac{p}{q}\pi| \quad > \quad \frac{\varepsilon}{q^r}$$

for some $\varepsilon > 0$, then

$$\begin{aligned} |e^{nl}\left(\cos(nT) + \sin(nT)\right)| \quad &= \quad e^{nl}|\left(\cos(nT) + \sin(nT)\right) \\ &\quad - \left(\cos(\frac{3\pi}{4} + 2\pi p) + \sin(\frac{3\pi}{4} + 2\pi p)\right)| \\ &\qquad \text{with} \quad 2\pi p < nl < 2\pi(p+1) \\ &\geq \quad e^{nl}\frac{\sqrt{2}}{2}n|l - \frac{3+8p}{4n}\pi| \\ &\geq \quad e^{nl}\frac{\sqrt{2}}{2}\frac{\varepsilon n}{(4n)^r} \; \geq \; c \; > \; 0 \end{aligned}$$

Thus, there is a unique solution $u_0$ to the elliptic/hyperbolic PDE

• If $T = \dfrac{4p+3}{4q}\pi,\ p, q \in \mathbf{Z}, q \neq 0$, then $\cos(nT) + \sin(nT)$ vanishes for an infinite number of $n$'s

The determinant is $O(e^{-nl})$ and there is no solution $u_0$

**Remarks :**

See also [Bourgin-Duffin 1939, F. John 1941]

Diophantine numbers in $(0, 1)$ form a set of Lebesgue measure 1

The behavior of HMM's strongly depends on the geometry of the inclusions

## 7. Conclusion

- The superlensing properties of NIM's are related to the unique continuation principle; those of HMM's to the uniqueness of solutions to the Cauchy problem for the wave equation

- HMM's can be constructed by homogenization of 'metals', for instance by homogenization of laminates. Are there other constructions ?
- Concerning superlensing using HMM's, there are many open questions :

  Other geometries (in 2D, in 3D) ?

  Extension to the Maxwell system or to the system of elasticity ?