

Effective attenuation in multiscale composite media

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- Wave equation in random media with long- or short-range correlations.
- Explain the apparent attenuation observed in seismic wave propagation.

Geophysics

- Experimental observations for body waves in the earth's crust: the seismic attenuation factor Q^{-1} defined through the transmission coefficient by

$$|T_L(\omega)|^2 \simeq \exp\left(-\frac{|\omega|}{c_o} Q^{-1}(\omega)L\right)$$

has a frequency dependence of the form

$$Q^{-1}(\omega) \approx |\omega|^\alpha$$

α is obtained through a fitting of measured data:

- $\alpha \simeq 0.4$ for low frequencies $< 10^{-3}\text{Hz}$ [Lekic 2009],
- $\alpha \in (-0.4, 0)$ for mid frequencies $(10^{-3}, 1)\text{Hz}$ [Choy 1986, Shito 2004],
- $\alpha \simeq -1$ for high frequencies $> 1\text{Hz}$ [Choy 1986, Cormier 2011].

- Two possible mechanisms: scattering and intrinsic attenuation (due to friction, viscosity, ...) [Knopoff 1964, Jackson 1970, ..., Ricard 2014 Is there seismic attenuation in the mantle?, ...].

Scalar wave equation in one-dimensional random medium

Scalar wave equation:

$$\frac{\partial^2 p^\varepsilon}{\partial z^2} - \frac{1}{c^2(z)} \frac{\partial^2 p^\varepsilon}{\partial t^2} = 0$$

A section of random medium is sandwiched in between two homogeneous half-spaces:

$$\frac{1}{c^2(z)} = \begin{cases} \frac{1}{c_o^2} \left(1 + \varepsilon \nu\left(\frac{z}{\varepsilon^2}\right)\right) & \text{for } z \in [0, L] \\ \frac{1}{c_o^2} & \text{for } z \in (-\infty, 0) \cup (L, \infty) \end{cases}$$

The background velocity is c_o .

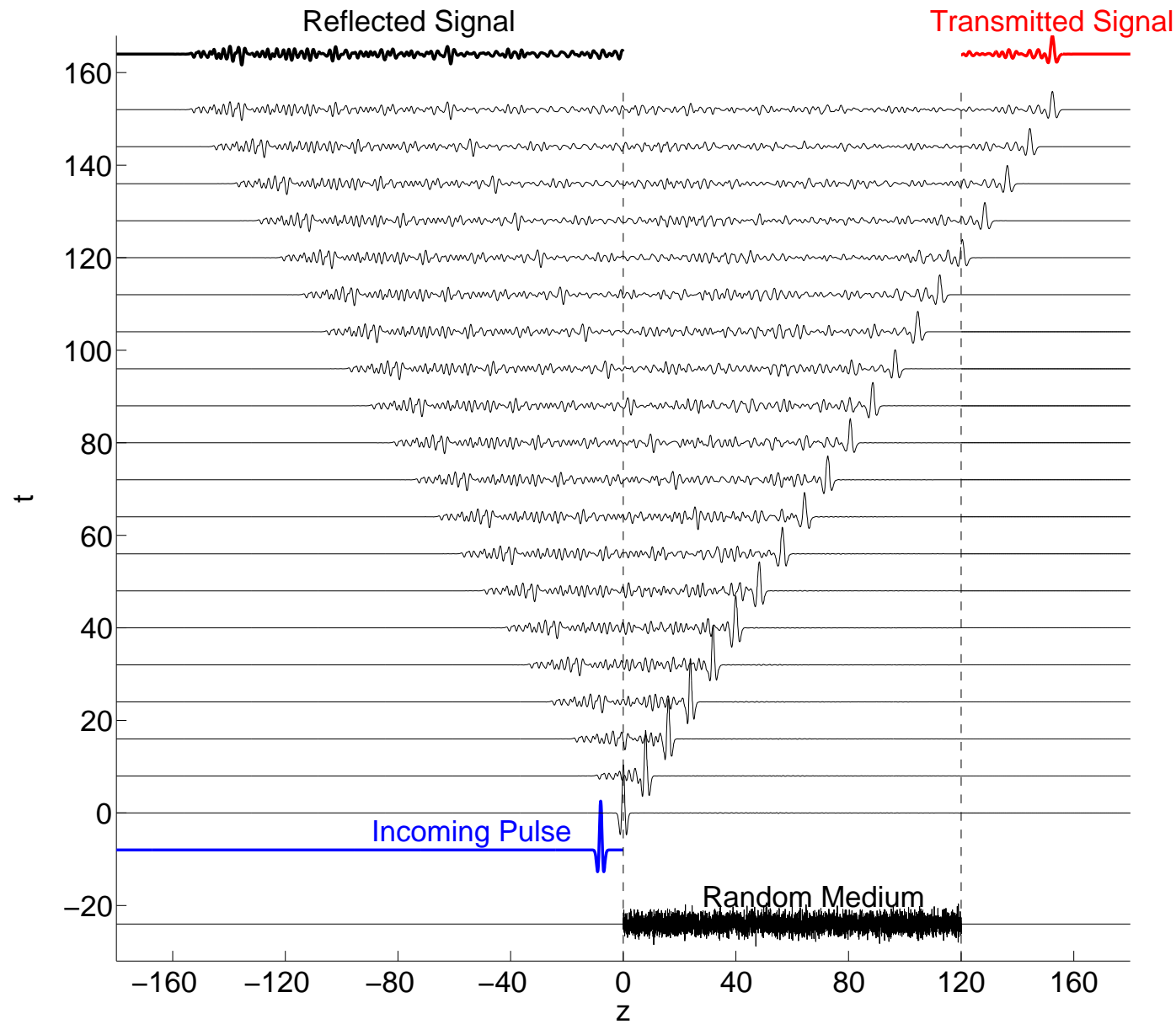
$\nu(z)$ is a zero-mean, stationary, random process with covariance function:

$$\phi(z) := \mathbb{E}[\nu(y)\nu(y+z)]$$

Pulse wave incoming from the left with pulse profile

$$p^\varepsilon(t, z) = p_0\left(\frac{t - z/c_o}{\varepsilon^2}\right), \quad t \ll 0$$

Numerical experiment in one-dimensional random medium



The mixing case

- $\phi(z) = \mathbb{E}[\nu(y)\nu(y+z)]$ decays fast enough at infinity (integrable) and is regular at zero (Lipschitz).

↔ The correlation length ℓ_c can be defined by

$$\ell_c = \frac{1}{\phi(0)} \int_0^\infty \phi(z) dz$$

↔ The covariance function can be expanded as

$$\phi(z) = \phi(0) \left(1 - d_c \frac{|z|}{\ell_c} + o\left(\frac{|z|}{\ell_c}\right) \right), \quad |z| \ll \ell_c$$

- Example: Binary medium. The process ν is stepwise constant.

$(l_j)_{j \geq 0}$: lengths of the elementary intervals.

$(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are independent and identically distributed (i.i.d.) with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

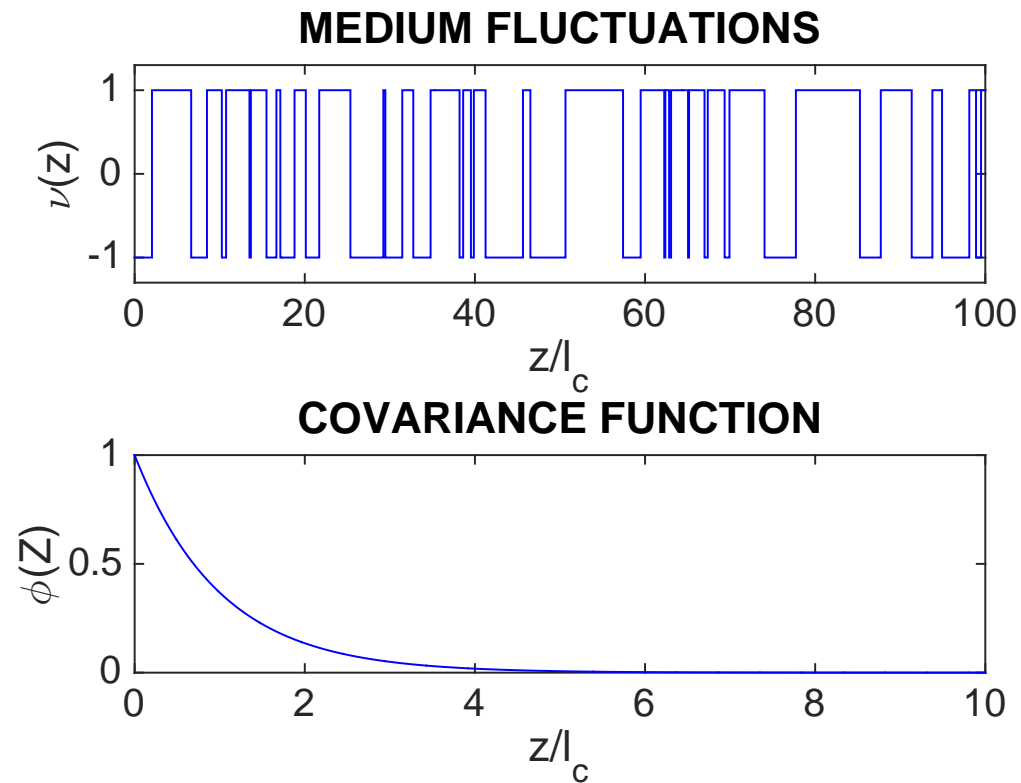
The lengths l_j are i.i.d. with the probability density function (pdf):

$$p_{l_1}(z) = \frac{1}{\ell_c} \exp\left(-\frac{z}{\ell_c}\right) \mathbf{1}_{[0, \infty)}(z)$$

↔ The covariance function is

$$\phi(z) = \sigma^2 \exp\left(-\frac{|z|}{\ell_c}\right)$$

Binary medium - mixing case



Realizations of a binary medium with exponentially distributed intervals.

Effective pulse propagation - mixing case (1/5)

Introduce the random travel time

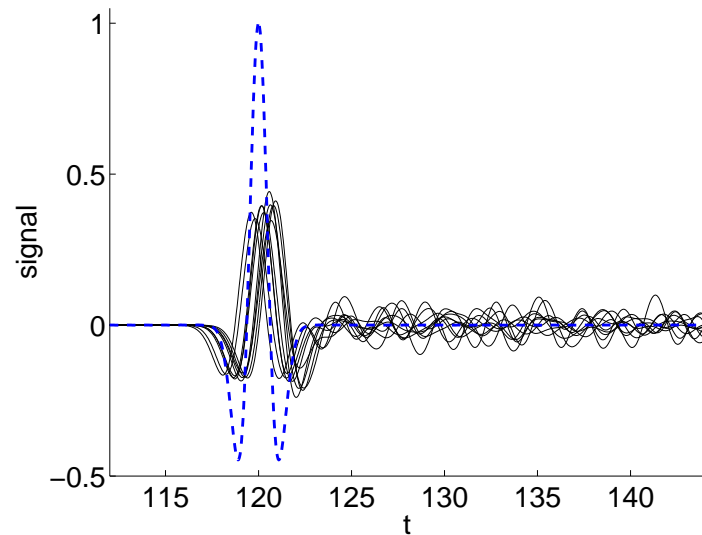
$$\tau^\varepsilon(L) = \frac{L}{c_o} + \varepsilon^2 \tau_c^\varepsilon(L), \quad \tau_c^\varepsilon(L) = \frac{1}{2c_o\varepsilon} \int_0^L \nu\left(\frac{z}{\varepsilon^2}\right) dz$$

In the limit $\varepsilon \rightarrow 0$, at $z = L$,

- 1) the *random time shift* $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one,
- 2) the pulse profile converges in probability to a *deterministic profile*:

$$p^\varepsilon(z = L, t = \tau^\varepsilon(L) + \varepsilon^2 \tau) \xrightarrow{\varepsilon \rightarrow 0} p_L(\tau)$$

Effective pulse propagation - mixing case (2/5)

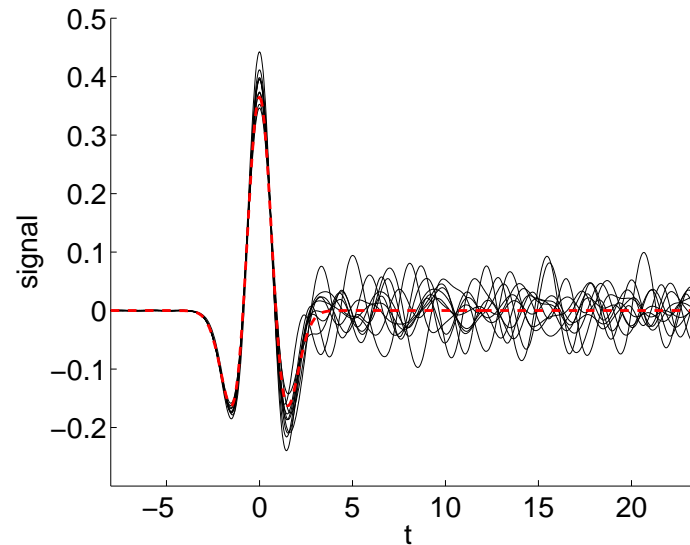


Time profiles of the transmitted wave at $z = L$

Without time shift

Black: ten numerical results
obtained with ten realizations
of the random medium

Blue: original pulse
 $p_0(t)$



With time shift

Red: theoretical (deterministic)
transmitted pulse shape

$p_L(t)$

Effective pulse propagation - mixing case (3/5)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ (\sim Brownian motion) with mean zero and variance:

$$\mathbb{E}[\tau_c(L)^2] = \frac{\hat{\phi}(0)\ell_c}{8c_o^2} L$$

2) in the limit $\varepsilon \rightarrow 0$, the pulse profile experiences a deterministic deformation:

$$p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega, \quad T_L(\omega) = \exp\left(-\frac{\omega^2[\hat{\phi}_c(\omega) + i\hat{\phi}_s(\omega)]}{8c_o^2} L\right)$$

$$\hat{\phi}_c(\omega) := 2 \int_0^\infty \phi(z) \cos\left(\frac{2\omega z}{c_o}\right) dz, \quad \hat{\phi}_s(\omega) := 2 \int_0^\infty \phi(z) \sin\left(\frac{2\omega z}{c_o}\right) dz$$

- Remark 1: Known as O'Doherty-Anstey theory in geophysics [70's]. Proved in mathematics by Papanicolaou and co [90's].

Cf: J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, Wave propagation and time reversal in randomly layered media, Springer, 2007.

- Remark 2: Convergence of p^ε towards p_L in the space of continuous functions with the sup norm (not in L^2).

Effective pulse propagation - mixing case (4/5)

- If $\frac{\omega \ell_c}{c_0} \ll 1$ (the wavelength $\gg \ell_c$ probes the tail of $\phi(z)$), then:

$$T_L(\omega) = \exp\left(-\frac{\hat{\phi}(0)}{8} \frac{\omega^2 \ell_c^2}{c_0^2} \frac{L}{\ell_c}\right)$$

↪ Effective second-order diffusion (attenuation), no effective dispersion.

Here $\alpha = 1$.

- Remark 3: What happens if $\int_0^\infty \phi(z) dz = \infty$?

- If $\frac{\omega \ell_c}{c_0} \gg 1$ (the wavelength $\ll \ell_c$ probes the small- z behavior of $\phi(z)$), then:

$$T_L(\omega) = \exp\left(-\frac{\phi(0)d_c}{16} \frac{L}{\ell_c} - i \frac{\phi(0)}{8} \frac{\omega \ell_c}{c_0} \frac{L}{\ell_c}\right)$$

↪ Effective constant attenuation, no effective dispersion, additional time shift.

Here $\alpha = -1$.

- Remark 4: What happens if $\phi(z)$ is not smooth at zero ?

Effective pulse propagation - mixing case (5/5)

- The mean field approach is quantitatively wrong:

$$p_L^{\text{MF}}(\tau) = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[p^\varepsilon \left(z = L, t = \frac{L}{c_o} + \varepsilon^2 \tau \right) \right] = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L^{\text{MF}}(\omega) e^{-i\omega\tau} d\omega,$$

$$T_L^{\text{MF}}(\omega) = T_L(\omega) \exp \left(- \frac{\hat{\phi}(0)}{8} \frac{\omega^2 \ell_c^2}{c_o^2} \frac{L}{\ell_c} \right)$$

↔ additional frequency-dependent attenuation that comes from the averaging with respect to the random time shift.

Long-range and short-range correlations

- Long-range correlations: ϕ is not integrable and has a power decay at infinity:

$$\phi(z) \stackrel{|z| \rightarrow \infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}$$

where $r_{H'} > 0$ and $H' \in (1/2, 1)$ (i.e., $2H' - 2 \in (-1, 0)$).

ℓ_c is the critical length scale beyond which the power law behavior is valid.

- Short-range correlations: ϕ is not smooth at zero:

$$\phi(z) \stackrel{|z| \rightarrow 0}{\simeq} \phi(0) \left(1 - d_H \left| \frac{z}{\ell_c} \right|^{2H} + O\left(\left| \frac{z}{\ell_c} \right|\right) \right)$$

where $d_H > 0$ and $H \in (0, 1/2)$ (i.e., $2H \in (0, 1)$).

ℓ_c is the critical length scale below which the expansion is valid.

Binary medium - long-range correlations

The process ν is stepwise constant.

$(l_j)_{j \geq 0}$: lengths of the elementary intervals.

$(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf ($H' \in (1/2, 1)$):

$$p_{l_1}(z) = \frac{3 - 2H'}{\ell_c} \frac{\ell_c^{4-2H'}}{z^{4-2H'}} \mathbf{1}_{[\ell_c, \infty)}(z)$$

Note: The average length of the intervals is $\frac{3-2H'}{2-2H'} \ell_c$ while the variance is infinite.

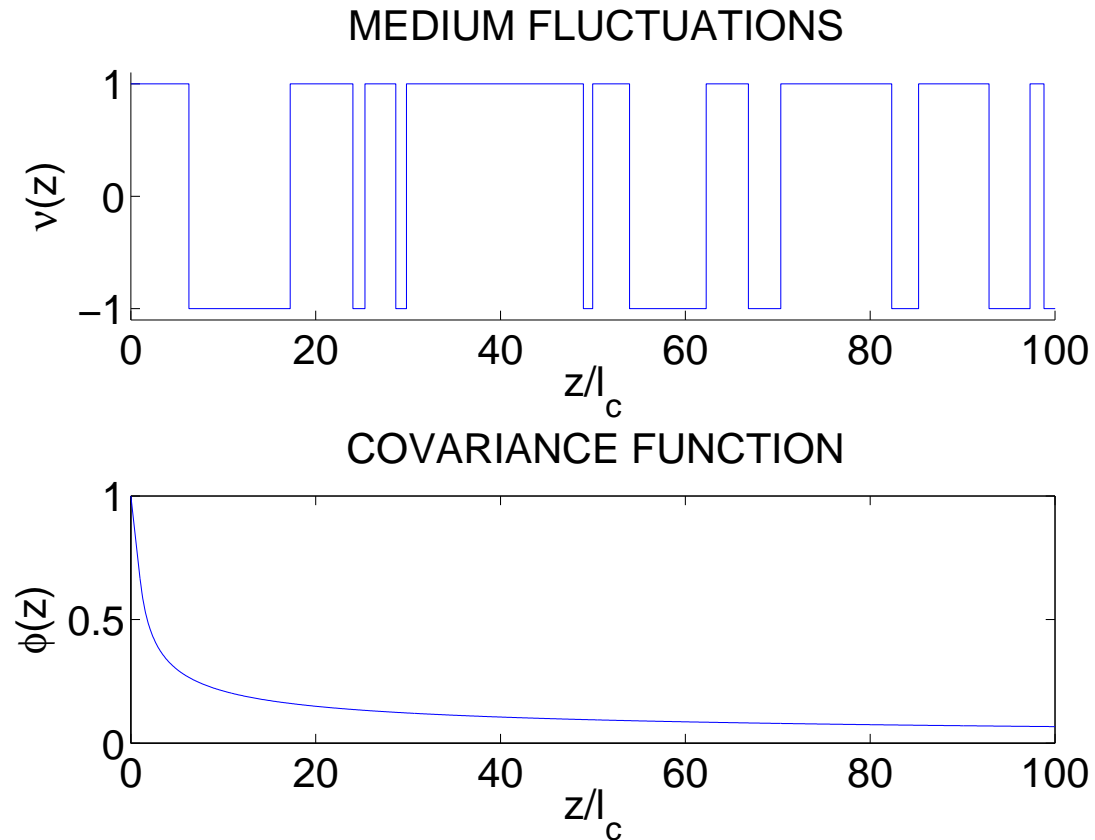
The covariance function is

$$\phi(z) = \sigma^2 \left[\frac{1}{3 - 2H'} \frac{\ell_c^{2-2H'}}{|z|^{2-2H'}} \mathbf{1}_{[\ell_c, \infty)}(|z|) + \left(1 - \frac{2 - 2H'}{3 - 2H'} \frac{|z|}{\ell_c}\right) \mathbf{1}_{[0, \ell_c)}(|z|) \right]$$

We have

$$\phi(z) \stackrel{|z| \rightarrow \infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}, \quad \text{with } r_{H'} = \frac{\sigma^2}{3 - 2H'}$$

Binary medium - long-range correlations



Realizations of a binary medium with the index $H' = 0.75$.

Generation of intervals longer than the average responsible for the long-range correlation property.

Binary medium - short-range correlations

The process ν is stepwise constant.

$(l_j)_{j \geq 0}$: lengths of the elementary intervals.

$(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf ($H \in (0, 1/2)$, $l_i \ll \ell_c$):

$$p_{l_1}(z) = \frac{1 - 2H}{\ell_c[(l_i/\ell_c)^{2H-1} - 1]} \frac{\ell_c^{2-2H}}{z^{2-2H}} \mathbf{1}_{[l_i, \ell_c]}(z)$$

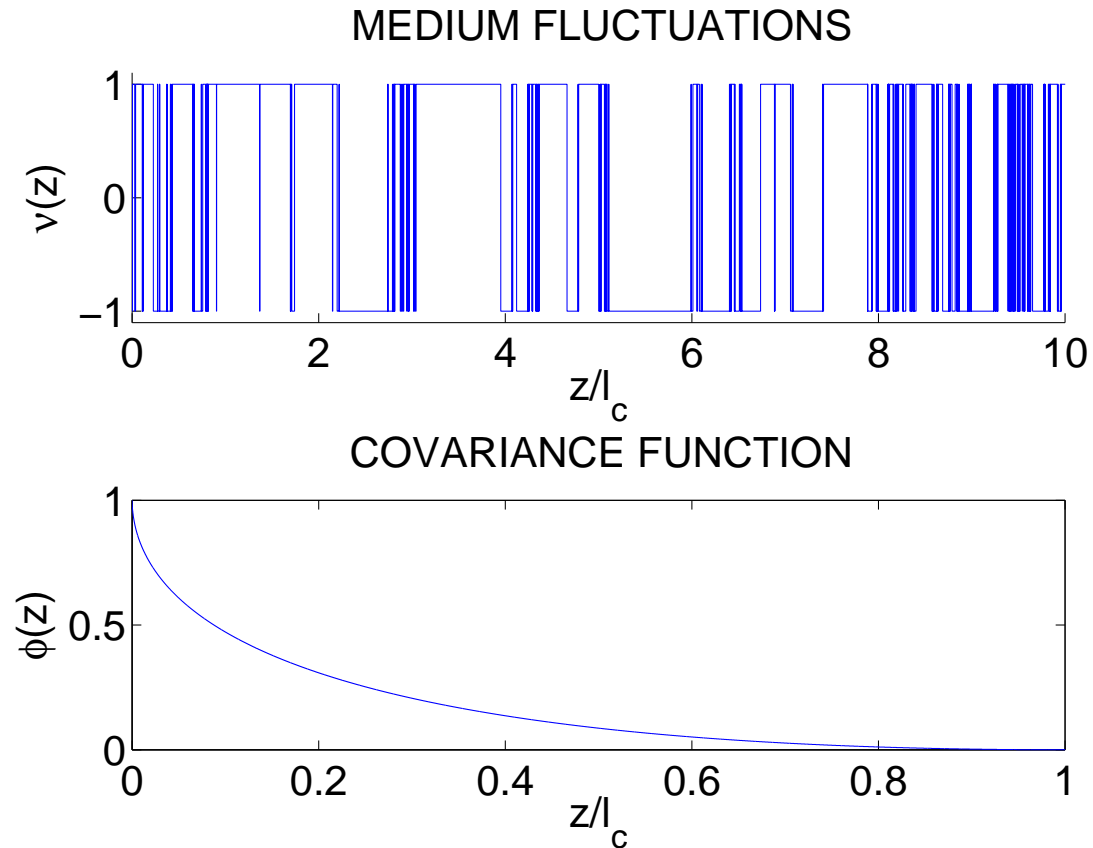
The covariance function is

$$\begin{aligned} \phi(z) = & \frac{\sigma^2}{1 - \delta^{2H}} \left(1 - \frac{1}{1 - 2H} \frac{|z|^{2H}}{\ell_c^{2H}} + \frac{2H}{1 - 2H} \frac{|z|}{\ell_c} \right) \mathbf{1}_{(l_i, \ell_c]}(|z|) \\ & + \sigma^2 \left(1 - \frac{2H}{1 - 2H} \frac{\delta^{2H} - \delta}{1 - \delta^{2H}} \frac{|z|}{\ell_i} \right) \mathbf{1}_{(0, l_i]}(|z|) \end{aligned}$$

with $\delta = l_i/\ell_c$. We have

$$\phi(z) \stackrel{l_i < |z| \ll \ell_c}{\simeq} \sigma^2 \left(1 - d_H \left| \frac{z}{\ell_c} \right|^{2H} + o\left(\left| \frac{z}{\ell_c} \right|\right) \right), \quad \text{with } d_H = \frac{1}{1 - 2H}$$

Binary medium - short-range correlations



Realizations of a binary medium with the index $H = 0.25$.

Accumulation of very small intervals responsible for the short-range correlation property.

Binary medium - short- and long-range correlations

The process ν is stepwise constant.

$(l_j)_{j \geq 0}$: lengths of the elementary intervals.

$(n_j)_{j \geq 0}$: values $\in \{-\sigma, +\sigma\}$ taken by the process over each elementary interval.

The values n_j are i.i.d. with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}$$

The lengths l_j are i.i.d. with the pdf:

$$\begin{aligned} p_{l_1}(z) &= \left(1 - a\delta^{1-2H}\right) \frac{1 - 2H}{\delta^{2H-1} - 1} \frac{\ell_c^{1-2H}}{z^{2-2H}} \mathbf{1}_{[l_i, \ell_c]}(z) \\ &\quad + a\delta^{1-2H} (3 - 2H') \frac{\ell_c^{3-2H'}}{z^{4-2H'}} \mathbf{1}_{[\ell_c, \infty)}(z), \end{aligned}$$

where $H \in (0, 1/2)$, $H' \in (1/2, 1)$, $a \in (0, 1)$, $\delta = l_i/\ell_c$, and $0 < l_i \ll \ell_c$.

The pdf is continuous at ℓ_c if we choose a as

$$\frac{1}{a} = \frac{3 - 2H'}{1 - 2H} - 2\delta^{1-2H} \frac{1 + H - H'}{1 - 2H},$$

but it is not required.

Binary medium - short- and long-range correlations

The covariance function has both H -short-range and H' -long-range properties:

- if $|z| \geq \ell_c$, then

$$\phi(z) \simeq \frac{a\sigma^2}{(2 - 2H') \left(\frac{1-2H}{2H} + a \frac{3-2H'}{2-2H'} \right)} \frac{\ell_c^{2-2H'}}{|z|^{2-2H'}}$$

which shows it has the H' -long-range property in the range $|z| \in [\ell_c, \infty)$.

- if $\ell_i \ll |z| \ll \ell_c$, then

$$\phi(z) \simeq \sigma^2 \left\{ 1 - \frac{1}{1 - 2H + a2H \frac{3-2H'}{2-2H'}} \frac{|z|^{2H}}{\ell_c^{2H}} \right\}$$

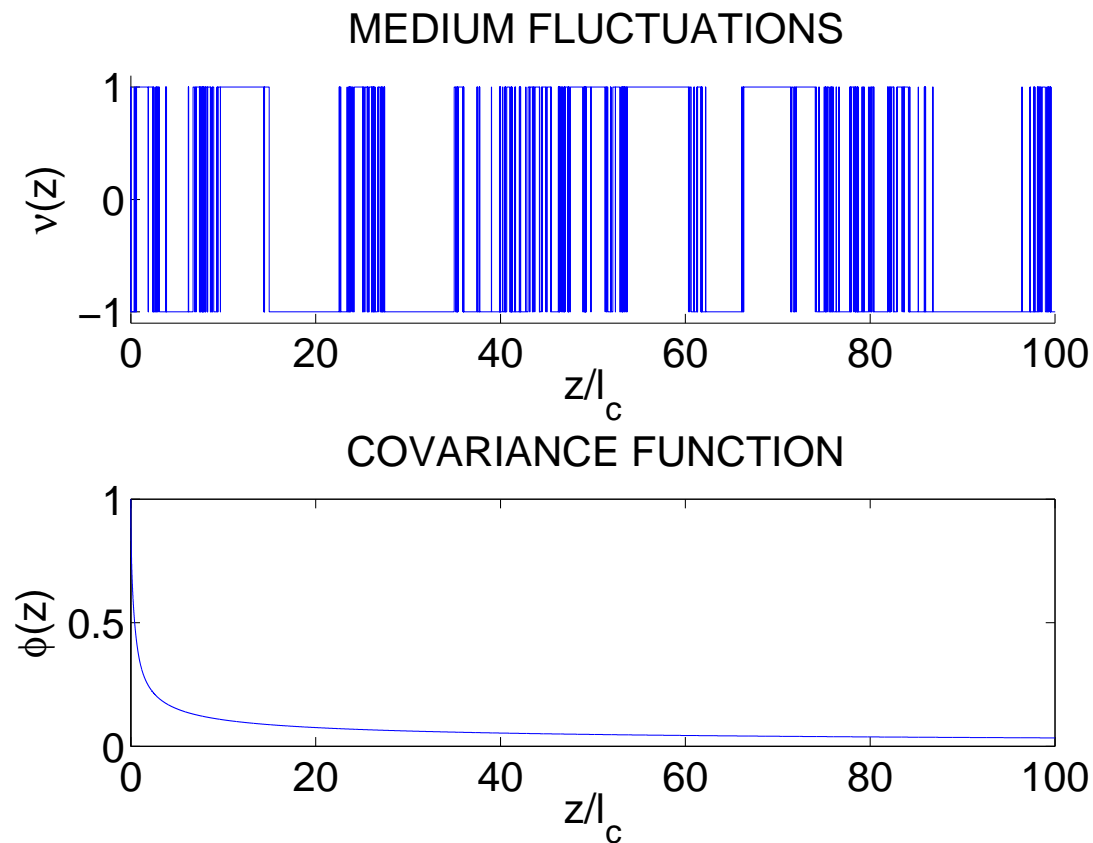
which shows it has the H -short-range property in the range $|z| \in (\ell_i, \ell_c)$.

- if $|z| \leq \ell_i$, then

$$\phi(z) \simeq \sigma^2 \left\{ 1 - \frac{1}{\left(\frac{1-2H}{2H} + a \frac{3-2H'}{2-2H'} \right) \delta^{1-2H}} \frac{|z|}{\ell_c} \right\}$$

which shows it has the regular property in the range $|z| \in [0, \ell_i]$.

Binary medium - short- and long-range correlations



Realizations of a binary medium long- and short-range properties with $H = 0.25$ and $H' = 0.75$.

Accumulation of very small intervals responsible for the short-range correlation property and generation of intervals longer than the average responsible for the long-range correlation property

Effective pulse propagation - long-range correlations (1/5)

- Covariance function:

$$\phi(z) \stackrel{|z| \rightarrow \infty}{\simeq} r_{H'} \left| \frac{z}{\ell_c} \right|^{2H'-2}, \quad H' \in (1/2, 1)$$

- Introduce the random travel time

$$\tau^\varepsilon(L) = \frac{L}{c_o} + \varepsilon^{3-2H'} \tau_c^\varepsilon(L), \quad \tau_c^\varepsilon(L) = \frac{1}{2c_o \varepsilon^{2-2H'}} \int_0^L \nu \left(\frac{z}{\varepsilon^2} \right) dz$$

- In the limit $\varepsilon \rightarrow 0$, at $z = L$,

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a zero-mean random variable with variance of order one,

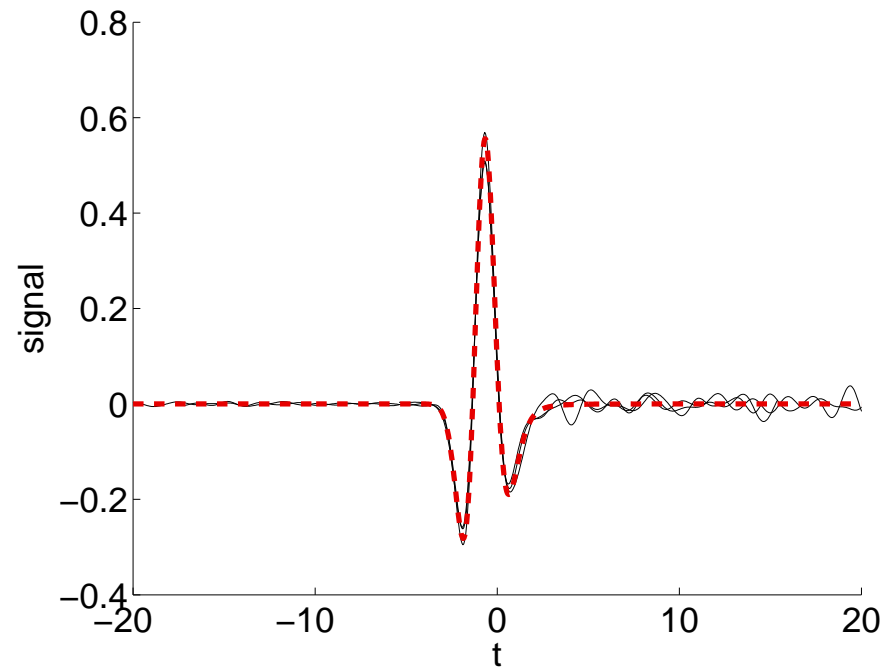
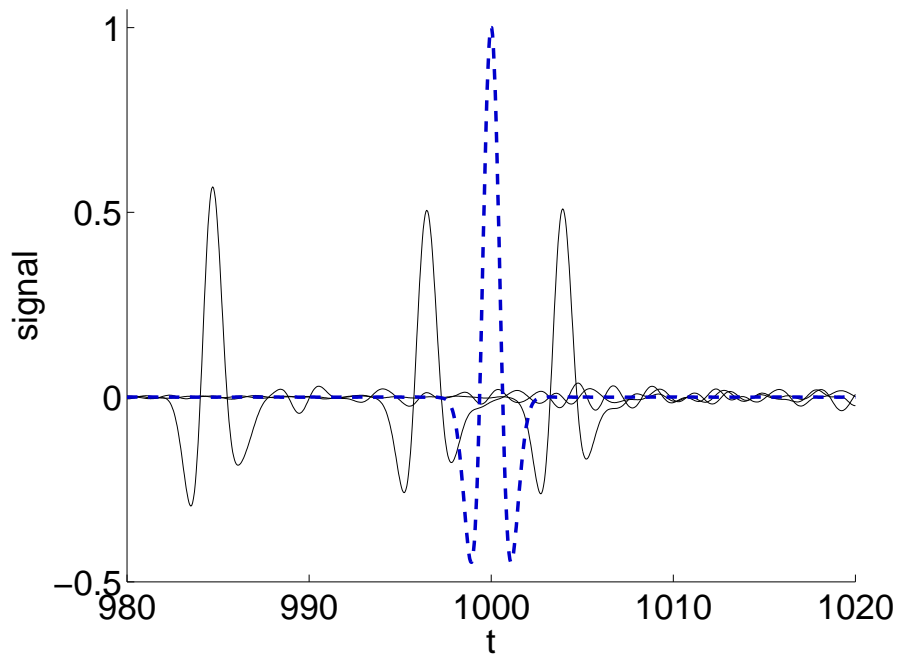
2) the pulse profile converges in probability to a deterministic profile:

$$p^\varepsilon(z = L, t = \tau^\varepsilon(L) + \varepsilon^2 \tau) \xrightarrow{\varepsilon \rightarrow 0} p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega$$

Note that the time shift is of order $\varepsilon^{3-2H'} \gg \varepsilon^2$ (here $H' > 1/2$, i.e. $3 - 2H' < 2$):

the random time shift is larger than the deterministic deformation.

Effective pulse propagation - long-range correlations (2/5)



Blue: original pulse profile $p_0(t)$

Red: theoretical transmitted profile $p_L(t)$

Black: results of three numerical simulations (with centering in the right picture).

Here $H' = 0.875$.

Effective pulse propagation - long-range correlations (3/5)

More precisely:

1) the random time shift $\tau_c^\varepsilon(L)$ converges in distribution to a Gaussian random variable $\tau_c(L)$ with mean zero and variance (\sim fractional Brownian motion):

$$\mathbb{E}[\tau_c(L)^2] = \frac{\ell_c^{2-2H'}}{c_o^2} \frac{r_{H'}}{4H'(2H'-1)} L^{2H'}$$

2) If $\frac{\omega \ell_c}{c_o} \ll 1$, then

$$T_L(\omega) = \exp \left(-\frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \cos \left(\left(H' - \frac{1}{2} \right) \pi \right) \left(\frac{|\omega| \ell_c}{c_o} \right)^{3-2H'} \frac{L}{\ell_c} \right. \\ \left. -i \frac{r_{H'}}{2} \frac{\Gamma(2H'-1)}{2^{2H'}} \sin \left(\left(H' - \frac{1}{2} \right) \pi \right) \left(\frac{|\omega| \ell_c}{c_o} \right)^{2-2H'} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c} \right)$$

\hookrightarrow Effective fractional diffusion (attenuation) $\sim |\omega|^{3-2H'}$, $3-2H' \in (1, 2)$.

\hookrightarrow Effective fractional dispersion $\sim |\omega|^{2-2H'} \omega$.

Here $\alpha = 2-2H' \in (0, 1)$.

Effective pulse propagation - long-range correlations (4/5)

Effective fractional wave equation in the “original” frame (up to random time correction):

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} = \frac{r_{H'} \ell_c^{2-2H'}}{2^{2H'} c_o^{4-2H'}} \int_0^\infty \frac{1}{s^{2-2H'}} \frac{\partial^3 p}{\partial t^3}(t-s) ds$$

- same form as wave equations used in sound propagation in lossy media;
- slightly different from the standard models [M. Caputo, Geophys. J. R. Astron. Soc. 1, 529 (1967), T. L. Szabo, J. Acoust. Soc. Am. 96, 491 (1994)];
- equivalent to the model proposed in [W. Chen and S. Holm, J. Acoust. Soc. Am. 114, 2570 (2003)];
- respects causality (dispersion relation satisfies Kramers-Kronig relation).

Effective pulse propagation - long-range correlations (5/5)

- The mean field approach is qualitatively wrong.

$$p^{\text{MF},\varepsilon}(L, \tau) := \mathbb{E} \left[p^\varepsilon \left(L, \frac{L}{c_o} + \varepsilon^2 \tau \right) \right]$$

↔ additional frequency-dependent decay that comes from the averaging with respect to the random time shift. This term is strong, it becomes of order one for a small propagation distance, of order $\varepsilon^{2-\frac{1}{H'}}$:

$$p^{\text{MF},\varepsilon} \left(\varepsilon^{2-\frac{1}{H'}} L, \tau \right) \xrightarrow{\varepsilon \rightarrow 0} p_L^{\text{MF}}(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L^{\text{MF}}(\omega) e^{-i\omega\tau} d\omega$$

where

$$T_L^{\text{MF}}(\tau) = \exp \left(- \frac{r_{H'}}{8H'(2H'-1)} \frac{\omega^2 \ell_c^2 L^{2H'}}{c_o^2 \ell_c^{2H'}} \right)$$

Thus, the mean field is described by an anomalous diffusion equation.

Effective pulse propagation - short-range correlations (1/2)

$$\phi(z) \stackrel{|z| \rightarrow 0}{\simeq} \phi(0) \left(1 - d_H \left| \frac{z}{\ell_c} \right|^{2H} + O\left(\left| \frac{z}{\ell_c} \right|\right) \right), \quad H \in (0, 1/2)$$

In the limit $\varepsilon \rightarrow 0$, at $z = L$, the pulse profile converges in probability to a deterministic profile:

$$p^\varepsilon \left(z = L, t = \frac{L}{c_o} + \varepsilon^2 \tau \right) \xrightarrow{\varepsilon \rightarrow 0} p_L(\tau) = \frac{1}{2\pi} \int \hat{p}_0(\omega) T_L(\omega) e^{-i\omega\tau} d\omega$$

Note that **the random time shift is vanishing**.

If $\frac{\omega \ell_c}{c_o} \gg 1$, then

$$T_L(\omega) = \exp \left(-\frac{\phi(0)d_H}{8} \frac{\Gamma(1+2H)}{2^{2H}} \sin(H\pi) \left(\frac{|\omega|\ell_c}{c_o} \right)^{1-2H} \frac{L}{\ell_c} \right. \\ \left. -i \frac{\phi(0)}{8} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c} + i \frac{\phi(0)d_H}{8} \frac{\Gamma(1+2H)}{2^{2H}} \cos(H\pi) \left(\frac{|\omega|\ell_c}{c_o} \right)^{-2H} \frac{\omega \ell_c}{c_o} \frac{L}{\ell_c} \right)$$

↪ **Deterministic time shift** (proportional to L).

↪ **Effective fractional diffusion (attenuation)** $\sim |\omega|^{1-2H}$, $1 - 2H \in (0, 1)$.

↪ **Effective fractional dispersion** $\sim |\omega|^{-2H} \omega$.

Here $\alpha = -2H \in (-1, 0)$.

Effective pulse propagation - short-range correlations (2/2)

- Remark 1: Effective fractional wave equation in the “original” frame:

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} = \frac{H d_H}{2^{1+2H} c_o^{2-2H} \ell_c^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial^2 p}{\partial t^2}(t-s) ds$$

- Remark 2: The mean field approach is quantitatively correct.

Effective pulse propagation - short- and long-range correlations

$$\frac{1}{L} \ln |T_L(\omega)| \approx \begin{cases} - \left(\frac{\omega \ell_c}{c_o} \right)^{3-2H'} \frac{\Gamma(2H' - 1) \cos((H' - 1/2)\pi)}{2^{2H'} (3 - 2H')(1 + \mathcal{C})} & , \quad \frac{\omega}{c_o} \ll \frac{1}{\ell_c} , \\ - \left(\frac{\omega \ell_c}{c_o} \right)^{1-2H} \frac{\Gamma(1 + 2H) \sin(H\pi)}{2^{2+2H} (1 - 2H)(1 + \mathcal{C}^{-1})} & , \quad \frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i} , \\ - \frac{H\delta^{2H-1}}{4(1 - 2H)(1 + \mathcal{C}^{-1})} & , \quad \frac{\omega}{c_o} \gg \frac{1}{\ell_i} , \end{cases}$$

with $\mathcal{C} = \sigma^2 \frac{(1-2H)(2-2H')(1-a\delta^{1-2H})(1-\delta^{2H})}{a2H(3-2H')(1-\delta^{1-2H})}$.

- Summary:

- For low frequencies $\frac{\omega}{c_o} \ll \frac{1}{\ell_c}$, the H' -long-range correlation property gives an exponent $\alpha = 2 - 2H' \in (0, 1)$.
- For mid frequencies $\frac{1}{\ell_c} \ll \frac{\omega}{c_o} \ll \frac{1}{\ell_i}$, the H -short-range correlation property gives an exponent $\alpha = -2H \in (-1, 0)$.
- For high frequencies $\frac{\omega}{c_o} \gg \frac{1}{\ell_i}$, the regular property gives an exponent $\alpha = -1$.

- Remember the experimental findings:

- $\alpha \simeq 0.4$ for low frequencies $< 10^{-3}\text{Hz}$ [Lekic 2009],
- $\alpha \in (-0.4, 0)$ for mid frequencies $(10^{-3}, 1)\text{Hz}$ [Choy 1986, Shito 2004],
- $\alpha \simeq -1$ for high frequencies $> 1\text{Hz}$ [Choy 1986, Cormier 2011].

Conclusions

- In a random medium with a covariance function that decays at infinity as $|z|^{2H'-2}$, $H' \in (1/2, 1)$, the attenuation factor at low frequency is $Q^{-1}(\omega) \sim |\omega|^\alpha$, with $\alpha = 2 - 2H' \in (0, 1)$.
- In a random medium with a covariance function that behaves at zero like $1 - d_H|z|^{2H}$, $H \in (0, 1/2)$, the attenuation factor at high frequency is $Q^{-1}(\omega) \sim |\omega|^\alpha$, with $\alpha = -2H \in (-1, 0)$.
- A special frequency-dependent phase is associated to the frequency-dependent attenuation and it ensures that causality and Kramers-Kronig relations are respected.
- Effective fractional wave equations can be written that have the form of equations studied in the literature in the context of wave propagation in lossy media.
- A simple binary medium with short- and long-range correlation properties can explain the frequency dependence of the apparent attenuation observed for seismic body waves.