Pontryagin’s Principle for State-Constrained Control Problems
Governed by Parabolic Equations with Unbounded Controls.

J. P. Raymond, H. Zidani
Université Paul Sabatier, UMR CNRS MIP, 31062 Toulouse cedex France

Abstract
This paper deals with optimal control problems governed by semilinear parabolic equations
with pointwise state constraints and unbounded controls. Under some strong stability assumption,
we obtain necessary optimality conditions in the form of a Pontryagin’s minimum principle in
qualified form. A Pontryagin’s principle in nonqualified form is also proved without any stability
condition.

Keywords. Optimal control, nonlinear boundary controls, semilinear parabolic equations, state
constraints, Pontryagin’s minimum principle, unbounded controls.

1 Introduction.
This article concerns control problems for the following parabolic system:
\[
\frac{\partial y}{\partial t} + Ay + f(x, t, y) = 0 \quad \text{in } Q, \quad \frac{\partial y}{\partial n_A} + g(s, t, y, v) = 0 \quad \text{on } \Sigma, \quad y(0) = w \quad \text{in } \Omega,
\]
where $\Omega \subset \mathbb{R}^N$, $Q = \Omega \times ]0, T[\,$, $\Sigma = \Gamma \times ]0, T[\,$, $T > 0$, $v$ is a boundary control, $w$ is a control of the
initial condition, $A$ is a second order elliptic operator. The control variables, $v$, $w$, and the state
variable $y$ must satisfy constraints of the form
\[
v \in V_{ad} \subset L^p(\Sigma), \quad w \in W_{ad} \subset C(\overline{\Omega}),
\]
\[
\phi(y) \in \mathcal{C}
\]
where $\phi$ is a continuous mapping from $C(\overline{Q})$ into $C(\overline{D})$, $\mathcal{C} \subset C(\overline{D})$ is a closed convex subset with
nonempty interior, and $\overline{D}$ is a nonempty compact subset of $\overline{Q}$. The control problem is
\[
(P) \quad \inf \{ J(y, v, w) \mid y \in W(0, T) \cap C(\overline{Q}), (v, w) \in V_{ad} \times W_{ad}, (y, v, w) \text{satisfies } (1), (2) \},
\]
where the cost functional is defined by
\[
J(y, v, w) = \int_Q F(x, t, y(x, t)) \, dx \, dt + \int_\Sigma G(s, t, y(s, t), v(s, t)) \, ds \, dt + \int_\Omega L(x, y(x, T), w(x)) \, dx.
\]
We are mainly interested in optimality conditions for such problems, in the form of Pontryagin’s
principles. The existence of optimal solutions for $(P)$ is a priori supposed.
In recent years there has been growing interest in optimality conditions for state-constrained control problems governed by partial differential equations (or variational inequalities). This is reflected by an important number of papers on this subject. For convex control problems we refer to [1], [4], [5], [7], [31]. In the case of nonconvex control problems the method of Lagrange multipliers provides optimality conditions, both for bounded and for unbounded controls [3], [11], [37], [32], [33].

When no qualification condition is assumed, optimality conditions are obtained in nonqualified form (optimality conditions of Fritz John type). To get optimality conditions in qualified form with a Lagrange multiplier theorem, a qualification condition such as the Zowe-Kurcyusz regularity condition is needed. (In many problems, this regularity condition corresponds to a Slater type qualification condition, see [37], [32].)

Another method proceeds by penalizing the state constraints and then characterizing optimal solutions of the original problem as ε-solutions of the penalized problems. The characterization of ε-solutions is carried out thanks to the Ekeland’s variational principle. By this method optimality conditions are obtained in the form of Pontryagin principles, which are in general more precise than optimality conditions deduced from Lagrange multiplier theorems. Moreover assumptions on the data of the problems (differentiability assumptions, convexity requirement,...) are less restrictive than those necessary for Lagrange multiplier theorems. To the best of our knowledge, except in [23], this method has so far been used only for problems with bounded controls (bounded in time [28], [29], [30] or in space and time [12], [21], [22], [24]).

There is a fundamental reason for this limitation. When we apply Ekeland’s principle to obtain Pontryagin’s principle, we need a complete metric space, let us say (V_{ad}, d_{E}) (the space of controls V_{ad}, endowed with the so-called Ekeland’s metrics d_{E}, in order to recover a Pontryagin principle) and a penalized functional F_{ε} (v) = J_{ε}(y_{v}, v) (v is the control variable, y_{v} is the solution of the state equation corresponding to v), which must be lower semicontinuous on (V_{ad}, d_{E}). To prove this lower semicontinuity property we need some assumptions on J_{ε} and we have to prove that the mapping T : v \mapsto y_{v} is continuous from (V_{ad}, d_{E}) into a Banach space Z (which depends on the problem considered). The continuity of T depends on regularity results for the state equation. In the problems studied in the articles mentioned above, V_{ad} is a subset of some Lebesgue space L^σ with 1 ≤ σ ≤ ∞, and (thanks to regularity results for partial differential equations) it can be proved that the mapping T is continuous from L^σ into Z for every σ > \bar{σ}, where \bar{σ} depends on the state equation.

If V_{ad} is bounded in L^∞ and if a sequence of controls converges for the Ekeland metric, it can easily be proved that this sequence still converges for the topology of L^σ for any σ > \bar{σ}. Therefore in this case, the mapping T is continuous and Ekeland’s principle can be applied.

If V_{ad} is not bounded in L^∞, convergence in the Ekeland metric does not imply convergence in the Lebesgue space norm, moreover (V_{ad}, d_{E}) is not necessarily complete. This is the reason why, up to now, in the presence of pointwise state constraints, Pontryagin’s principles have only been proved for bounded controls (at least for nonconvex problems, indeed for convex problems the optimality conditions deduced from Lagrange multiplier theorems correspond to Pontryagin principles).
Let us stress that the growth conditions on the integrands and the nonlinear terms in the state equations, postulated in ([30], Chapter 4, Hypothesis 2, p. 130) correspond to bounded controls. The same remark is valid for [12].

In [23], Fattorini and Sritharan prove a Pontryagin principle in nonqualified form for control problems of Navier-Stokes equations in which the controls are not necessarily bounded. Their idea is to work with bounded perturbations (see [23], p. 227). Here we consider a control set of the form

\[ V_{ad} = \{ v \in L^2(\Sigma) \mid v(s, t) \in K_V(s, t) \text{ a.e. in } \Sigma \} \]

where \( K_V \) is a measurable multimapping with nonempty and closed values in \( P(\mathbb{R}) \) (see Section 2).

We do not think that the method developed in [23] can be applied to such a control set. Moreover the method developed in [23] deals with Pontryagin principles in nonqualified form and requires some convexity condition on the cost functional (see Hypothesis 2.9 in [23]).

The purpose of this paper is to extend the method based on Ekeland’s principle to problems with unbounded controls. In order to explain the main ideas of this extension let us recall the starting point of the method described above. If \( \bar{v} \) is an \( \varepsilon^2 \)-solution of the problem

\( (P_\varepsilon) \inf \{ F_\varepsilon(v) \mid v \in V_{ad} \}, \)

where \( F_\varepsilon \) and \( (V_{ad}, d_E) \) satisfy the assumptions of the Ekeland’s principle, then there exists another \( \varepsilon^2 \)-solution \( v_\varepsilon \) such that

\[ d_E(v_\varepsilon, \bar{v}) \leq \varepsilon \quad \text{and} \quad F_\varepsilon(v_\varepsilon) - F_\varepsilon(v) \leq \varepsilon d(v, v_\varepsilon) \quad \text{for every } v \in V_{ad}. \]

In order to exploit this optimality condition, \( v \) is replaced by some perturbation of \( v_\varepsilon \). The methods developed in [8], [12], [20], [22], [24], [29], [30] differ both in their choices of \( F_\varepsilon \) and in their choices of the perturbations.

Pontryagin principles in qualified form are only obtained in [8], [12], by choosing for \( F_\varepsilon \) a regularization of an exact penalized functional. A Pontryagin principle is then obtained under a strong stability condition. Pontryagin principles in nonqualified form are obtained in [8] under a weak stability condition by a method of spike perturbations. With an other choice for the penalized functional and other kinds of perturbations, Pontryagin principles in nonqualified form are obtained in [22], [24], [29], [30].

In [22] Fattorini and Murphy use a method of multispike perturbations. The type of perturbations used in [29], [38], [24], [12], [13], [30] can be viewed as a generalization of multispike perturbations, which we call diffuse perturbations. In contrast to spike or multispike perturbations, which are precisely localized, a diffuse perturbation is not localized around some points but it is implicitly defined by some relations (see [12], [29], [38], [40], [24]). The existence of diffuse perturbations satisfying relations a priori defined is proved in [27], [28], [24] and in [12] in a constructive manner. To our knowledge this kind of perturbations has been introduced for the first time by Yao [38] and Li [26]. Connections with Lyapunov’s convexity theorem or with Uhl’s theorem are clarified in [40], p. 1315.
and [28]. We here prove that all the relations needed to define a diffuse perturbation can be obtained as a consequence of the Lyapunov convexity theorem.

Preliminary results related to the topic were announced in [34]. The metric space used in [34] is different from the one defined in Section 3.2. This is the reason why a convexity condition (assumption A7) is needed in [34] to ensure some semicontinuity property. Thus, the methods of the present paper improve upon those of [34].

The paper is organized as follows. In the next section we formulate the control problem governed by a semilinear parabolic equation and we state the main results: the weak and strong Pontryagin’s principles. In Section 3 we give some regularity results for solutions of the state and adjoint equations. In Section 4, we derive some technical results used in Section 5 to prove the main result stated in Section 2.

2 Assumptions and Main results.

Throughout the sequel Ω denotes a bounded open subset of \( \mathbb{R}^N \) \((N \geq 2)\) of class \( C^{2,\beta} \) for some \( 0 < \beta \leq 1 \) (that is, the boundary \( \Gamma \) of \( \Omega \) is an \((N-1)\)-dimensional manifold of class \( C^{2,\beta} \) such that \( \Omega \) lies locally on one side of \( \Gamma \). A function is of class \( C^{1,\beta} \) if it is of class \( C^2 \) and if its second order derivatives are Hölder continuous of exponent \( \beta \)). We denote by \( q, \sigma \) positive numbers satisfying

\[
q > N/2 + 1, \quad \sigma > N + 1 \quad \text{and} \quad q\sigma + q > qN + 2\sigma.
\]

The differential operator \( A \) in equation (1) is defined by

\[
Ay(x) = - \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_jy(x)),
\]

with coefficients \( a_{ij} \) belonging to \( C^{1,\beta}(\overline{\Omega}) \) and satisfying the conditions

\[
a_{ij}(x) = a_{ji}(x) \quad \text{for every } i,j \in \{1,\ldots,N\}, \quad m_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \tag{4}
\]

for all \( x \in \overline{\Omega} \) and all \( \xi \in \mathbb{R}^N \), with \( 0 < m_0 \). (\( D_i \) denotes the partial derivative with respect to \( x_i \)). In equation (1), \( \frac{\partial y}{\partial n_A} \) is the conormal derivative of \( y \) with respect to \( A \), that is

\[
\frac{\partial y}{\partial n_A}(s,t) = \sum_{i,j} a_{ij}(s)D_jy(s,t)n_i(s),
\]

where \( n = (n_1,\ldots,n_N) \) is the unit normal to \( \Gamma \) outward \( \Omega \).

For every \( 1 \leq \tau \leq \infty \), the norms in the spaces \( L^\tau(\Omega), L^\tau(\Gamma), L^\tau(Q), L^\tau(\Sigma) \) will be denoted by \( \| \cdot \|_{\tau,\Omega}, \| \cdot \|_{\tau,\Gamma}, \| \cdot \|_{\tau,Q}, \| \cdot \|_{\tau,\Sigma} \). The Hilbert space \( W(0,T;H^1(\Omega),(H^1(\Omega))') = \{ y \in L^2(0,T;H^1(\Omega)) \mid \frac{\partial y}{\partial t} \in L^2(0,T;(H^1(\Omega))') \} \), endowed with its usual norm, will be denoted by \( W(0,T) \). We also set \( \overline{\Omega}_0 = \overline{\Omega} \times \{0\} \) and \( \overline{\Omega}_T = \overline{\Omega} \times \{T\} \).
2.1 Assumptions.

(A1) - For every \( y \in \mathbb{R} \), \( f(\cdot, y) \) is measurable on \( Q \). For almost every \((x, t) \in Q\), \( f(x, t, \cdot) \) is of class \( C^1 \) on \( \mathbb{R} \). The following estimates hold:

\[
|f(x, t, 0)| \leq M_1(x, t), \quad C_0 \leq f_y(x, t, y) \leq M_1(x, t)\eta(|y|),
\]

where \( M_1 \) belongs to \( L^1(Q) \), \( \eta \) is a nondecreasing function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) and \( C_0 \in \mathbb{R} \). (We have denoted by \( f_y \) the partial derivative of \( f \) with respect to \( y \), and in the sequel we adopt the same kind of notation for other functions.)

(A2) - For every \((y, v) \in \mathbb{R}^2\), \( g(\cdot, y, v) \) is measurable on \( \Sigma \). For almost every \((s, t) \in \Sigma \) and every \( v \in \mathbb{R} \), \( g(s, t, \cdot, v) \) is of class \( C^1 \) on \( \mathbb{R} \). For almost every \((s, t) \in \Sigma\), \( g(s, t, \cdot) \) and \( g_y(s, t, \cdot) \) are continuous on \( \mathbb{R} \times \mathbb{R} \). The following estimates hold:

\[
|g(s, t, 0, v)| \leq M_2(s, t) + m_1|v|, \quad C_0 \leq g_y(s, t, y, v) \leq (M_2(s, t) + m_1|v|)\eta(|y|),
\]

where \( M_2 \) belongs to \( L^\sigma(\Sigma) \), \( m_1 > 0 \), \( C_0 \) and \( \eta \) are as in \((A1)\).

(A3) - For every \((y, w) \in \mathbb{R}^2\), \( L(\cdot, y, w) \) is measurable on \( \Omega \). For almost every \( x \in \Omega \), \( L(x, \cdot) \) is of class \( C^1 \) on \( \mathbb{R} \times \mathbb{R} \). The following estimate holds:

\[
|L(x, y, w)| + |L'_w(x, y, w)| + |L'_y(x, y, w)| \leq M_3(x)\eta(|w|)\eta(|y|),
\]

where \( M_3 \in L^1(\Omega) \), and \( \eta \) is as in \((A1)\).

(A4) - For every \( y \in \mathbb{R} \), \( F(\cdot, y) \) is measurable on \( Q \). For almost every \((x, t) \in Q\), \( F(x, t, \cdot) \) is of class \( C^1 \) on \( \mathbb{R} \). The following estimate holds:

\[
|F(x, t, y)| + |F'_y(x, t, y)| \leq M_4(x, t)\eta(|y|),
\]

where \( M_4 \in L^1(Q) \), and \( \eta \) is as in \((A1)\).

(A5) - For every \((y, v) \in \mathbb{R}^2\), \( G(\cdot, y, v) \) is measurable on \( \Sigma \). For almost every \((s, t) \in \Sigma \) and every \( v \in \mathbb{R} \), \( G(s, t, \cdot, v) \) is of class \( C^1 \) on \( \mathbb{R} \). For almost every \((s, t) \in \Sigma\), \( G(s, t, \cdot) \) and \( G'_y(s, t, \cdot) \) are continuous on \( \mathbb{R} \times \mathbb{R} \). The following estimate holds:

\[
|G(s, t, y, v)| + |G'_y(s, t, y, v)| \leq (M_5(s, t) + m_1|v|)\eta(|y|),
\]

where \( M_5 \in L^1(\Sigma) \), \( m_1 \) and \( \eta \) are as in \((A2)\).

(A6) - The set of constraints on \( v \) is defined by

\[
V_{ad} = \{v \in L^\sigma(\Sigma) \mid v(s, t) \in K_V(s, t) \text{ for a.e. } (s, t) \in \Sigma\},
\]

where \( K_V \) is a measurable multimapping with nonempty and closed values in \( \mathcal{P}(\mathbb{R}) \) (that is, the set of all subsets of \( \mathbb{R} \)). The constraint on the initial condition is \( w \in W_{ad} \), where \( W_{ad} \) is a closed convex
subset of $C(\overline{\Omega})$.

(A7) - In the state constraint (2), $\phi$ is a mapping of class $C^1$ from $C(\overline{Q})$ into $C(\overline{D})$, $\overline{D}$ is a nonempty compact subset of $\overline{Q}$, $C \subset C(D)$ is a closed convex subset with nonempty interior in $C(D)$.

In Section 3.1 we recall an existence and uniqueness result in $W(0,T) \cap C(\overline{Q})$ for equation (1), already proved in [35]. Therefore, the state constraint (2) makes sense because the weak solution of (1) is continuous on $\overline{Q}$. Let us give some examples of state constraints described by (2).

Example 2.1. If we choose $D = \Omega \times \{T\}$ (where $y|_D$ is the restriction of $y$ to $D$), we have a problem with a terminal state constraint. We may consider $\phi(y) = y|_D$ and $C = \{z \in C(D) | ||z - y_T||_{C(D)} \leq \epsilon\}$ where $\epsilon > 0$ and $y_T \in C(D)$ are given.

Example 2.2. We consider $\phi(y) = \psi(\cdot, y(\cdot))|_D$, where $\psi \in C(\overline{Q} \times \mathbb{R})$ is such that $\psi'_y$ (the partial derivative of $\psi$ with respect to $y$) belongs to $C(\overline{Q} \times \mathbb{R})$, $C = \{z \in C(D) | z \leq 0\}$ and $\overline{D}$ is any nonempty compact subset of $\overline{Q}$.

2.2 Strong stability assumption.

For $\gamma \geq 0$, we define the subset $C_\gamma$ of $C(D)$ by

$$C_\gamma = \{\varphi \in C(D) | \inf_{z \in C} ||\varphi - z||_{C(D)} \leq \gamma\},$$

and we consider the perturbed state constraint

$$\phi(y) \in C_\gamma. \tag{5}$$

We denote by $(P_\gamma)$ the following problem :

$$(P_\gamma) \quad \inf \{J(y, v, w) | y \in W(0, T) \cap C(\overline{Q}), (v, w) \in V_{ad} \times W_{ad}, (y, v, w) \text{ satisfies (1), (5)}\}.$$ 

Observe that $(P)$ is identical to $(P_0)$. Following [8], [12], [13], we say that $(P_\gamma)$ is strongly stable on the right if there exist $\tilde{\epsilon} > 0$ and $\tilde{r} > 0$ such that, for every $\gamma' \in [\gamma, \gamma + \tilde{\epsilon}]$, we have

$$\inf(P_\gamma) - \inf(P_{\gamma'}) \leq \tilde{r}(\gamma - \gamma').$$

With the additional assumption (A8), a Pontryagin principle for $(P)$ may be obtained in qualified form. Some remarks on (A8) are made after Theorem 2.1.

(A8) $(P)$ is strongly stable on the right.

2.3 Statement of the main result.

We define the boundary Hamiltonian function by:

$$H_\Sigma(s, t, y, v, p, \nu) = \nu G(s, t, y, v) - pg(s, t, y, v)$$

for every $(s, t, y, v, p, \nu) \in \Gamma \times [0, T] \times \mathbb{R}^4$. The main result of this paper is the Pontryagin principle for $(P)$, stated in the following theorem.
Theorem 2.1 If (A1) – (A7) are fulfilled and if \((\bar{y}, \bar{v}, \bar{w})\) is a solution of \((P)\), then there exists \(\bar{p} \in L^1(0, T; W^{1,1}(\Omega))\), and there exist \(\bar{v} \in \mathbb{R}, \bar{\mu} \in \mathcal{M}(\mathcal{D})\) (the space of Radon measures on \(\mathcal{D}\)) and a measurable subset \(\overline{\Sigma} \subset \Sigma\) such that

\[
(\bar{v}, \bar{\mu}) \neq 0, \quad \bar{v} \geq 0, \quad (\bar{\mu}, z - \phi(\bar{y}))_{\mathcal{M}(\mathcal{D}) \times C(\mathcal{D})} \leq 0 \quad \forall z \in \mathcal{C},
\]

\[
-\frac{\partial \bar{p}}{\partial t} + A\bar{p} + f'_y(x, t, \bar{y})\bar{p} = \bar{v}F'_y(x, t, \bar{y}) + [\phi'(\bar{y})^*\bar{\mu}]_Q \quad \text{in } Q,
\]

\[
\frac{\partial \bar{p}}{\partial n_A} + g'_y(s, t, \bar{y}, \bar{v})\bar{p} = \bar{v}G'_y(s, t, \bar{y}, \bar{v}) + [\phi'(\bar{y})^*\bar{\mu}]_\Sigma \quad \text{on } \Sigma,
\]

\[
\bar{p}(T) = \bar{v}L'_y(x, \bar{y}(T), \bar{w}) + [\phi'(\bar{y})^*\bar{\mu}]|_{\mathcal{P}_T} \quad \text{in } \Omega,
\]

\[
\mathcal{L}^N(\overline{\Sigma}) = \mathcal{L}^N(\Sigma),
\]

\[
H_\Sigma(s, t, \bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t), \bar{v}) = \min_{\nu \in K} H_\Sigma(s, t, \bar{y}(s, t), \nu, \bar{p}(s, t), \bar{v}) \quad \text{for all } (s, t) \in \overline{\Sigma},
\]

\[
\int_\Omega \bar{v}L_w(x, \bar{y}(T), \bar{w}) (\bar{w} - w) dx + \langle \bar{p}(0) + [\phi'(\bar{y})^*\bar{\mu}]_{\mathcal{P}_0}, \bar{w} - w \rangle_{\mathcal{M}(\mathcal{D}) \times C(\mathcal{D})} \leq 0 \quad \text{for all } w \in W_{ad},
\]

where \([\phi'(\bar{y})^*\bar{\mu}]_Q\) is the restriction of \([\phi'(\bar{y})^*\bar{\mu}]\) to \(Q\), \([\phi'(\bar{y})^*\bar{\mu}]_\Sigma\) is the restriction of \([\phi'(\bar{y})^*\bar{\mu}]\) to \(\Sigma\), \([\phi'(\bar{y})^*\bar{\mu}]|_{\mathcal{P}_T}\) is the restriction of \([\phi'(\bar{y})^*\bar{\mu}]\) to \(\mathcal{P}_T\) and \([\phi'(\bar{y})^*\bar{\mu}]|_{\mathcal{P}_0}\) is the restriction of \([\phi'(\bar{y})^*\bar{\mu}]\) to \(\mathcal{P}_0\), \([\phi'(\bar{y})^*\bar{\mu}]\) is the Radon measure on \(\overline{Q}\) defined by \(z \mapsto \langle \bar{\mu}, \phi'(\bar{y})z \rangle_{\mathcal{M}(\mathcal{D}) \times C(\mathcal{D})}\) for \(z \in C(\overline{Q})\), \(\mathcal{L}^N\) denotes the \(N\)-dimensional Lebesgue measure.

Moreover, if (A8) is satisfied, we can take \(\bar{v} = 1\) in (7), (8), (9).

The meaning of weak solutions for (7), regularity results for \(\bar{p}\), and the definition of \(\bar{p}(0)\) are given in Section 3.

The notion of stability is closely related to the notion of calmness introduced by F. H. Clarke [15]. In the above setting this notion is due to J. V. Burke [9]. It has first been used in control problems by J. F. Bonnans and J. F. Bonnans and E. Casas. We do not know sufficient conditions ensuring that a state-constrained control problem is strongly stable on the right. However, even if \((P)\) is not strongly stable on the right, \((P_\gamma)\) will be strongly stable for all \(\gamma > 0\), except on a subset of \(\mathbb{R}^+\) of zero Lebesgue measure [8], [12]. In some situations Pontryagin’s principles in qualified form may be derived from a non qualified form. Consider the example described below.

**Example 2.3.** We suppose that \(\Omega\) is a connected. The state equation is

\[
\frac{\partial y}{\partial t} - \Delta y = 0 \quad \text{in } Q, \quad \frac{\partial y}{\partial n} + y^4 = v \quad \text{on } \Sigma, \quad y(0) = y_0 \quad \text{on } \Omega,
\]

where \(y_0\) is a given function in \(C(\overline{\Omega})\). We set \(V_{ad} = \{v \in L^\infty(\Sigma) \mid v(s, t) \geq 0 \ \text{a.e. on } \Sigma\}\). The state constraints are defined by

\[
0 \leq y(x, t) \leq \gamma_d \quad \text{on } \overline{Q},
\]

for some given \(\gamma_d > 0\). (We suppose that \(0 \leq y_0(x) < \gamma_d\) on \(\overline{\Omega}\).) Since for \(v \in V_{ad}\), the solution \(y\) of (10) is nonnegative, then we can restrict the state constraints to \(y(x, t) \leq \gamma_d\). We denote by \(J\) a cost
functional defined as in (3), with \( w \equiv y_0 \), for which the assumptions (A3) to (A5) are satisfied. We suppose that the control problem

\[
(P_{\text{ex}}) \quad \inf \{ J(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), \ v \in V_{ad}, \ (y,v) \text{satisfies (10)}, \ y(x,t) \leq \gamma_d \text{ on } \overline{Q} \},
\]

admits a solution \((\bar{y}, \bar{v})\). The adjoint equation (7) for \((P_{\text{ex}})\) corresponding to \((\bar{y}, \bar{v})\) and to \(\bar{v} = 0\) is

\[-\frac{\partial \bar{p}}{\partial t} - \Delta \bar{p} = \bar{\mu}_Q \text{ in } Q, \quad \frac{\partial \bar{p}}{\partial n} + 4\bar{y}^3 \bar{p} = \bar{\mu}_\Sigma \text{ on } \Sigma, \quad \bar{p}(T) = \bar{\mu}_{\Gamma_T} \text{ on } \overline{\Omega},\]

where the measure \(\bar{\mu} = \bar{\mu}_Q + \bar{\mu}_\Sigma + \bar{\mu}_{\Gamma_T}\) obeys

\[
\bar{\mu} \geq 0, \quad \bar{\mu} \neq 0, \quad \langle \bar{\mu}, \bar{y} - \gamma_d \rangle_{M(\overline{Q}) \times C(\overline{Q})} = 0. \tag{11}
\]

(Observe that (11) is nothing else than (6) for \((P_{\text{ex}})\) and \(\bar{v} = 0\).) The Pontryagin principle in nonqualified form is expressed:

\[
\bar{p}(s,t)(v - \bar{v}(s,t)) \geq 0 \quad \text{for all } v > 0 \quad \text{and for almost all } (s,t) \in \Sigma. \tag{12}
\]

We set \(\bar{t} = \inf \{ t \in [0,T] \mid \bar{\mu}(\overline{\Omega} \times [t,T]) \neq 0 \}.\) Following [32] (Theorems 4.2, 4.3 and Remark 4.7), we can define \(\bar{\mu}_{\Gamma_T}(t^+)\) as a function in \(L^1(\Omega)\) which satisfies the Green formula

\[
\int_{\Omega \times [t,T]} \bar{p}(\frac{\partial y}{\partial t} - \Delta y) \, dx \, dt + \int_{\Gamma \times [t,T]} \bar{p}(\frac{\partial y}{\partial n} + 4\bar{y}^3 y) \, ds \, dt
\]

\[
= \langle \bar{\mu}_Q, y \rangle_{M(\Omega \times [t,T]) \times C_b(\Omega \times [t,T])} + \langle \bar{\mu}_\Sigma, y \rangle_{M_b(\Omega \times [t,T]) \times C_b(\Omega \times [t,T])}
\]

\[
+ \langle p(T), y(T) \rangle_{M(\overline{\Omega}) \times C(\overline{\Omega})} - \langle p_{\Gamma_T}(t^+), y(t) \rangle_{M(\overline{\Omega}) \times C(\overline{\Omega})},
\]

for all \(y \in C^{2,1}(\overline{Q})\) (\(C^{2,1}\) means \(C^2\) with respect to \(x\) and \(C^1\) with respect to \(t\)). With this definition for \(\bar{\mu}_{\Gamma_T}(t^+)\), the function \(\bar{p}\) is, on \(\overline{\Omega} \times [0,t[\), the unique solution of

\[-\frac{\partial \bar{p}}{\partial t} - \Delta \bar{p} = \bar{\mu}_{\Omega \times [0,t[} \text{ in } \Omega \times [0,t[], \quad \frac{\partial \bar{p}}{\partial n} + 4\bar{y}^3 \bar{p} = \bar{\mu}_{\Gamma \times [0,t[} \text{ on } \Gamma \times [0,t[, \quad \bar{p}(t) = \bar{\mu}_{\Gamma_T}(t^+) + \bar{\mu}_{\Gamma \times \{t\}} \text{ on } \overline{\Omega}.\]

We can easily prove that if \(\bar{\mu}_{\Gamma_T}(t^+) = 0\), then \(\bar{\mu}_{\Gamma \times \{t\}} = 0\). From the definition of \(\bar{t}\) we see that, for every \(\varepsilon > 0\), \(\bar{\mu}_{\Gamma \times \{t\}}((\bar{t} - \varepsilon)^+) \geq 0\) and \(\bar{\mu}_{\Gamma_T}((\bar{t} - \varepsilon)^+) \neq 0\). Let \(\bar{p}\) be the solution of

\[-\frac{\partial \bar{p}}{\partial t} - \Delta \bar{p} = 0 \text{ in } \Omega \times [0,\bar{t} - \varepsilon[, \quad \frac{\partial \bar{p}}{\partial n} + kp = 0 \text{ on } \Gamma \times [0,\bar{t} - \varepsilon[, \quad \bar{p}(t) = \bar{\mu}_{\Gamma_T}((\bar{t} - \varepsilon)^+) \text{ on } \overline{\Omega},\]

where \(k = \max \{4\bar{y}^3(x,t) \mid (x,t) \in \overline{Q}\}\). By a comparison principle, we can verify that \(\bar{p} \geq \bar{p}\) on \(\Gamma \times [0,\bar{t} - \varepsilon[\). Moreover \(\bar{p} \in C^2(\Gamma \times [0,\bar{t} - \varepsilon[), \bar{p}(\bar{t} - 2\varepsilon)\) is nonnegative and non identically zero.

From the maximum principle for classical solutions of parabolic equations, we deduce that \(\bar{p} > 0\) on \(\overline{\Omega} \times [0,\bar{t} - 2\varepsilon[\). Thus \(\bar{p}(s,t) > 0\) almost everywhere on \(\Gamma \times [0,\bar{t}\).\) With (12), this implies \(\bar{v} \equiv 0\) on \(\Gamma \times [0,\bar{t}\). Thus \(0 \leq y(x,t) \leq \max_{\Omega \times [0,\bar{t}} < \gamma_d \) on \(\overline{\Omega} \times [0,\bar{t}].\) From (11), it follows \(\bar{\mu}_{\Gamma \times [0,\bar{t}} = 0.\) Thus \(\mu \equiv 0\) and we get a contradiction. In this simple example we see that the Pontryagin principle in qualified form follows from the Pontryagin principle in nonqualified form.
3 State equation and Adjoint equation.

3.1 Existence, uniqueness and regularity of the state variable.

**Theorem 3.1** Under assumptions (A1) and (A2), if \( v \in L^\sigma(\Sigma) \) and \( w \in C(\overline{\Omega}) \), then the equation (1) admits a unique weak solution \( y_{vw} \) in \( W(0, T) \cap C(\overline{Q}) \). This solution satisfies

\[
\|y_{vw}\|_{\infty, Q} \leq C_1(\|v\|_{\sigma, \Sigma} + \|w\|_{\infty, \Omega} + 1),
\]

where \( C_1 = C_1(T, \Omega, N, q, \sigma, C_0) \). Moreover, the mapping \((v, w) \mapsto y_{vw}\) is continuous from \( L^\sigma(\Sigma) \times C(\overline{\Omega}) \) into \( C(\overline{Q}) \).

**Proof.** The existence of a unique weak solution \( y_{vw} \) in \( W(0, T) \cap C(\overline{Q}) \) for equation (1), is proved in ([35], Theorem 3.1). The last part of Theorem can be proved as in ([35], Proposition 4.3). \( \square \)

**Corollary 3.1** For every \( k > 0 \) and for every \( \varepsilon > 0 \), there exist \( C_2 = C_2(T, \Omega, N, q, \sigma, C_0, k), C_3 = C_3(T, \Omega, N, q, \sigma, C_0, k, \varepsilon) \) and \( \alpha > 0 \) such that, for every \((v, w) \in V_{ad} \times W_{ad} \) satisfying \( \|v\|_{L^\sigma(\Sigma)} + \|w\|_{\infty, \Omega} \leq k \), the weak solution \( y_{vw} \) of (1) corresponding to \((v, w)\) is Hölder continuous on \([\varepsilon, T] \times \overline{\Omega}\) and obeys

\[
\|y_{vw}\|_{C(\overline{Q})} \leq C_2, \quad \|y_{vw}\|_{C^{\alpha, \frac{2}{3}}(\overline{\Omega} \times [\varepsilon, T])} \leq C_3.
\]

Moreover, if \( w \) is Hölder continuous on \( \overline{\Omega} \), then \( y_{vw} \) is Hölder continuous on \( \overline{Q} \).

**Proof.** Since \( y_{vw} \) belongs to \( C(\overline{Q}) \), thanks to (A1)-(A2), we see that \( y_{vw} \) is also the unique weak solution of

\[
\frac{\partial y}{\partial t} + Ay = \tilde{f} \quad \text{in } Q, \quad \frac{\partial y}{\partial n_A} = \tilde{g} \quad \text{on } \Sigma, \quad y(0) = w \quad \text{in } \Omega,
\]

where

\[
\tilde{f}(\cdot) = -f(\cdot, y_{vw}(\cdot)), \quad \tilde{g}(\cdot) = -g(\cdot, y_{vw}(\cdot), v(\cdot)) \in L^\sigma(\Sigma).
\]

If we denote by \( \gamma_0 \) the trace operator from \( L^\sigma(0, T; W^{1,\nu}(\Omega)) \) into \( L^\sigma(0, T; W^{1,\frac{1}{2},\nu}(\Gamma)) \) (with \( \nu = N/(\sigma N - N + 1) \)) and by \( i \) the continuous embedding from \( L^\sigma(0, T; W^{1,\nu}(\Gamma)) \) into \( L^\sigma(0, T; L^\nu(\Gamma)) \), we can write:

\[
\langle \tilde{g}, (i \circ \gamma_0)z \rangle_{L^\sigma(\Sigma) \times L^\sigma(\Sigma)} = \langle (\gamma_0^i \circ i^0)\tilde{g}, z \rangle_{L^\sigma(0,T;W^1,\nu(\Omega)) \times L^\sigma(0,T;W^1,\nu(\Omega))}
\]

for every \( z \in L^\sigma(0, T; W^{1,\nu}(\Omega)) \). We can identify \( (\gamma_0^i \circ i^0)\tilde{g} \) with \( (f_0, f_1, ..., f_N) \in (L^\sigma(0, T; L^\nu(\Omega)))^{N+1} (\nu = \frac{N}{N-1}) \) in the following manner :

\[
\langle (\gamma_0^i \circ i^0)\tilde{g}, z \rangle_{L^\sigma(0,T;W^1,\nu(\Omega)) \times L^\sigma(0,T;W^1,\nu(\Omega))} = \int_Q (f_0z + \sum_i f_iD_i(z)) \, dx \, dt
\]

for every \( z \in L^\sigma(0, T; W^{1,\nu}(\Omega)) \). Therefore, \( y_{vw} \) is the weak solution (in the sense of [16]) of the initial boundary value problem

\[
\frac{\partial y}{\partial t} - \div(a(x, t, \nabla y)) = \tilde{f} + f_0 \quad \text{in } Q, \quad a(x, t, \nabla y) \cdot n = 0 \quad \text{on } \Sigma, \quad y(0) = w \quad \text{in } \Omega, \quad (13)
\]
where
\[ a(x, t, \nabla y) = \left( \sum_{j=1}^{N} a_{ij} D_j y - f_i \right)_{i=1, \ldots, N}. \]

Now we can easily verify that assumptions of ([16], Theorem 1.3, Chapter 3) are satisfied by the system (13) and H"older continuity results of Corollary follow from this theorem. \( \square \)

### 3.2 Metric space of controls.

To apply the Ekeland variational principle, we have to define a metric space of controls in order that the mapping \((v, w) \mapsto y_{vw}\) be continuous from this metric space to \(C(\Omega)\). Thanks to Theorem 3.1, this continuity condition will be realized if convergence in the metric space of controls implies convergence in \(L^\sigma(\Sigma) \times C(\overline{\Omega})\).

In the case when boundary controls are bounded, convergence in \((V_{ad}, d_E)\) (where \(d_E\) is the so-called Ekeland distance) implies convergence in \(L^\sigma(\Sigma)\). This condition is no longer true for unbounded controls (see [23], p. 227). To overcome this difficulty, we define a new metric space in the following way.

Let \(\tilde{v}\) be in \(V_{ad}\) (in Section 5, \(\tilde{v}\) will be an optimal boundary control that we want to characterize). For \(0 < k < \infty\), we define the set:

\[ V_{ad}(\tilde{v}, k) = \{ v \in V_{ad} \mid |v(s, t) - \tilde{v}(s, t)| \leq k \text{ for a.e. } (s, t) \in \Sigma \}. \]

We endow the set \(V_{ad}(\tilde{v}, k) \times W_{ad}\) with the following metric:

\[ d((v_1, w_1), (v_2, w_2)) = \mathcal{L}^N(\{(s, t) \mid v_1(s, t) \neq v_2(s, t)\}) + \|w_1 - w_2\|_{\infty, \Omega}. \]

**Remark 3.1.** Thanks to [18], we know that the mapping

\[ d_E : (v_1, v_2) \mapsto \mathcal{L}^N(\{(s, t) \mid v_1(s, t) \neq v_2(s, t)\}) \]

is a distance on \(V_{ad}(\tilde{v}, k)\). Moreover, if \((v_n)_n \subset V_{ad}(\tilde{v}, k)\), if \(v \in V_{ad}(\tilde{v}, k)\) and if \(\lim_n d_E(v_n, v) = 0\), then \((v_n)_n\) converges to \(v\) in \(L^\sigma(\Sigma)\). This is no longer true for any sequence \((v_n)_n\) included in \(V_{ad}\).

**Lemma 3.1** \((V_{ad}(\tilde{v}, k) \times W_{ad}, d)\) is a complete metric space, and the mapping which associates \((y_{vw}, J(y_{vw}, v, w))\) with \((v, w)\) is continuous from \((V_{ad}(\tilde{v}, k) \times W_{ad}, d)\) into \(C(\overline{\Omega}) \times \mathbb{R}\).

**Proof.** i) To prove that \((V_{ad}(\tilde{v}, k) \times W_{ad}, d)\) is a complete metric space, it remains to prove that \((V_{ad}(\tilde{v}, k), d_E)\) is complete. Let \((v_n)_n\) be a Cauchy sequence in \((V_{ad}(\tilde{v}, k), d_E)\). Following [18], we can prove that \((v_n)_n\) converges for \(d_E\) to some measurable function \(v\) such that \(v(s, t) \in K_V(s, t)\) and \(|v(s, t) - \tilde{v}(s, t)| \leq k\) for almost all \((s, t) \in \Sigma\). Therefore \(v \in L^\sigma(\Sigma)\) and \(v \in V_{ad}(\tilde{v}, k)\).

ii) Now, we consider \((v_n, w_n)_{n \geq 1} \subset V_{ad}(\tilde{v}, k) \times W_{ad}\) and \((v, w) \in V_{ad}(\tilde{v}, k) \times W_{ad}\) such that \((v_n, w_n)_n\) converges to \((v, w)\) for the metric \(d\). We denote by \(y_n\) \((n \geq 1)\) the solution of (1) corresponding
respectively to \((v, w)\) and to \((v_n, w_n)\). To prove the continuity result, it remains to prove that the sequence \((y_n, J(y_n, v_n, w_n))\) converges to \((y, J(y, v, w))\) in \(C(\overline{Q}) \times \mathbb{R}\). For this, we remark that \((w_n)_n\) converges to \(w\) in \(C(\overline{Q})\) and \((v_n)_n\) converges to \(v\) in \(L^\sigma(\Sigma)\). We complete the proof thanks to the continuity assumptions on \(F, G, L\) and thanks to the continuity results stated in Theorem 3.1. \(\square\)

### 3.3 Adjoint equation.

Let \((a, b)\) be in \(L^q(Q) \times L^\sigma(\Sigma)\) with \(a \geq C_0\) and \(b \geq C_0\). We consider the following terminal boundary value problem:

\[
-\frac{\partial p}{\partial t} + Ap + ap = \mu_Q \quad \text{in } Q, \quad \frac{\partial p}{\partial n_A} + bp = \mu_\Sigma \quad \text{on } \Sigma, \quad p(T) = \mu_{\overline{T}_T} \quad \text{on } \overline{Q},
\]

where \(\mu = \mu_Q + \mu_\Sigma + \mu_{\overline{T}_T}\) is a bounded Radon measure on \(\overline{Q} \setminus \overline{\Omega}_0\), \(\mu_Q\) is the restriction of \(\mu\) to \(Q\), \(\mu_\Sigma\) is the restriction of \(\mu\) to \(\Sigma\) and \(\mu_{\overline{T}_T}\) is the restriction of \(\mu\) to \(\overline{T}_T\).

**Definition 3.1** We shall say that \(p\) is a weak solution of (14) in \(L^1(0, T; W^{1,1}(\Omega))\) if and only if the two following conditions are verified:

(i) \(ap \in L^1(Q)\) and \(bp \in L^1(\Sigma)\),

(ii) For every \(\varphi \in C^1(\overline{Q})\) verifying \(\varphi(x, 0) = 0\) on \(\overline{Q}\), we have

\[
\int_Q \left\{ \frac{\partial \varphi}{\partial t} + \sum_{i,j} a_{ij} D_j \varphi D_i p + a_p \varphi \right\} dx \, dt + \int_\Sigma b \varphi p ds \, dt = \langle \varphi, \mu \rangle_{C_b(\overline{Q} \setminus \overline{\Omega}_0) \times \mathcal{M}_b(\overline{Q} \setminus \overline{\Omega}_0)}.
\]

\((C_b(\overline{Q} \setminus \overline{\Omega}_0))\) denotes the space of bounded continuous functions on \(\overline{Q} \setminus \overline{\Omega}_0\), while \(\mathcal{M}_b(\overline{Q} \setminus \overline{\Omega}_0)\) denotes the space of bounded Radon measures on \(\overline{Q} \setminus \overline{\Omega}_0\), that is, the topological dual of \(C_0(\overline{Q} \setminus \overline{\Omega}_0)\).

In all the sequel, we shall say that a pair \((\delta, d)\) verifies the condition \((C_{q\sigma})\) if and only if

\[
\begin{cases}
\frac{N\sigma}{\sigma - 2} < d \leq \frac{N\sigma}{\sigma - 1} & \text{if } \sigma \leq q, \\
\frac{Nq}{q - 2} < d \leq \frac{Nq}{q - 1} & \text{if } N \leq q < \sigma, \\
\frac{Nq}{q - 2} < d \leq \inf\left(\frac{Nq}{q - 1}, \frac{Nq}{q - q}\right) & \text{and } \frac{2d}{\sigma - N} < q \quad \text{if } q < N.
\end{cases}
\]

Since \(q > \frac{N}{2} + 1\), \(\sigma > N + 1\) and \(q \sigma + q > qN + 2\sigma\), we remark that the set of pairs \((\delta, d)\) satisfying \((C_{q\sigma})\) is nonempty. These conditions appear in a natural manner when we study the equation (14). Indeed, if \((\delta, d)\) satisfies \((C_{q\sigma})\), if \((a, b)\) belongs to \(L^q(Q) \times L^\sigma(\Sigma)\) and if \(p\) belongs to \(L^{\delta}(0, T; W^{1,d}(\Omega))\), then \(ap \in L^1(Q)\) and \(bp \in L^1(\Sigma)\). We now recall an existence theorem for parabolic equations with measures as data stated in [32].

**Theorem 3.2** Let \((a, b)\) be in \(L^q(Q) \times L^\sigma(\Sigma)\) satisfying \(a \geq C_0\), \(b \geq C_0\) and let \(\mu\) be in \(\mathcal{M}_b(\overline{Q} \setminus \overline{\Omega}_0)\). Equation (14) admits a unique weak solution \(p \in L^1(0, T; W^{1,1}(\Omega))\). For every \((\delta, d)\) satisfying \((C_{q\sigma})\), \(p \in L^\delta(0, T; W^{1,d}(\Omega))\) and we have:

\[
\|p\|_{L^\delta(0, T; W^{1,d}(\Omega))} \leq C_{4\delta} \|\mu\|_{\mathcal{M}_b(\overline{Q} \setminus \overline{\Omega}_0)},
\]

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where $C_4 = C_4(T, \Omega, N, \delta, d, C_0)$ is independent of $a$ and $b$.

Moreover, there exists a Radon measure on $\Omega$, denoted by $p(0)$ such that:

$$\int_Q p \left( \frac{\partial y}{\partial t} + Ay + ay \right) \, dxdt + \int_\Sigma p \left( \frac{\partial y}{\partial n_a} + by \right) \, dsdt = \langle y, \mu \rangle_{C_b(\overline{Q} \setminus \Omega_0) \times M(\Omega)} - \langle y(0), p(0) \rangle_{C(\overline{Q}) \times M(\Omega)}$$

for every $y \in Y = \left\{ y \in W(0,T) \cap C(Q) \mid \frac{\partial y}{\partial t} + Ay \in L^q(Q), \frac{\partial y}{\partial n_a} \in L^q(\Sigma) \right\}$.

**Remark 3.2.** If $p \in L^\delta(0,T; W^{1,d'}(\Omega))$ (where $(\delta, d)$ satisfies $(C_{qa})$), and if

$$\text{div}_{xt} ((\Sigma_j a_{ij} D_j p)_{1 \leq i \leq N, p}) = \frac{\partial p}{\partial t} - Ap \quad \text{belongs to } M_b(Q),$$

then we can define the normal trace of the vector field $((\Sigma_j a_{ij} D_j p)_{1 \leq i \leq N, p})$, in the space $W^{1-m, \delta}(\partial Q)$ (for some $1 < m < \frac{N+1}{N}$). If we denote by $\gamma_n((\Sigma_j a_{ij} D_j p)_{1 \leq i \leq N, p})$ this normal trace, we can prove (see Theorem 4.2 in [32]) that this normal trace belongs to $M(\partial Q)$ and the restriction of $\gamma_n((\Sigma_j a_{ij} D_j p)_{1 \leq i \leq N, p})$ to $\overline{\Omega}_T$ is equal to $\mu_{\overline{\Omega}_T}$, the restriction of $\gamma_n((\Sigma_j a_{ij} D_j p)_{1 \leq i \leq N, p})$ to $\Sigma$ is equal to $\mu_{\Sigma_T} - bp$ and if $p(0)$ is the measure on $\overline{Q}$ which satisfies the Green formula of Theorem 3.2, then $-p(0)$ is the restriction of $\gamma_n((\Sigma_j a_{ij} D_j p)_{1 \leq i \leq N, p})$ to $\overline{\Omega}_0$. In fact it can be proved that $p(0)$ belongs to $L^1(\Omega)$ (see Theorem 4.3 in [32]).

**Remark 3.3.** Let us explain the origin of the condition $(C_{qa})$. In [32], the existence of a weak solution in $L^1(0,T; W^{1,1}(\Omega))$ for equation (14) is proved by duality arguments and an approximation process. The condition $\delta > 2d/(d - N)$ appears to get $C^0$-regularity results for some adjoint equation associated with (14) (see ([32], Theorem 4.1)). The condition $\delta > 2d/(d - N)$, together with the conditions $p \in L^\delta(0,T; W^{1,d'}(\Omega)), a \in L^\delta(Q), b \in L^\delta(\Sigma), ap \in L^1(Q), bp \in L^1(\Sigma)$ are equivalent to the condition: "$(\delta, d)$ satisfies $(C_{qa})$". In the case when $a \in L^\infty(Q)$ and $b \in L^\infty(\Sigma)$, the condition $(C_{qa})$ can be replaced by the only condition $\delta > 2d/(d - N)$.

### 4 Existence of diffuse perturbations.

In Section 5, we consider control problems in which the state constraints are penalized. The penalization is chosen in such a way that the solution of $(P)$ that we want to characterize will be an $\varepsilon$-solution of the penalized problem. In order to exploit optimality conditions deduced from the Ekeland’s variational principle, we need to construct admissible perturbations of approximate optimal solutions. For this we use a kind of perturbations that we call "diffuse perturbations" and which goes back to Yao [38] and Li [26]. A diffuse perturbation of a control $\bar{v} \in V_{ad}$ is a function $v_\rho$ defined by

$$v_\rho(s,t) = \begin{cases} \bar{v}(s,t) & \text{on } \Sigma \setminus E_\rho, \\ v(s,t) & \text{on } E_\rho, \end{cases}$$

where $v \in V_{ad}$ and $E_\rho$ is some measurable subset of $\Sigma$. It is clear that $v_\rho \in V_{ad}$. Contrary to spike perturbations or multispike perturbations, where $E_\rho$ is precisely defined, here $E_\rho$ must satisfy some
relations such as (23), (24) and (25). As explained in Lemma 4.1, the existence of $E_\rho$ follows from the Lyapunov convexity Theorem.

To get optimality conditions we need some differential calculus rules, for this type of perturbation, stated in the following theorem.

**Theorem 4.1** Let $\rho$ be positive constant such that $0 < \rho < 1$. For every $v_1, v_2 \in V_{ad}$ and for every $w_1, w_2 \in C(\overline{\Omega})$, there exists a measurable subset $E_\rho \subset \Sigma$ such that:

$$L^N(E_\rho) = \rho L^N(\Sigma),$$  

(15)

$$\int_{E_\rho} (G(s, t, y_1, v_2) - G(s, t, y_1, v_1)) \, ds \, dt = \rho \int_{\Sigma} (G(s, t, y_1, v_2) - G(s, t, y_1, v_1)) \, ds \, dt,$$  

(16)

$$y_\rho = y_1 + \rho z + r_\rho, \quad \text{with} \quad \lim_{\rho \to 0} \frac{1}{\rho} \| r_\rho \|_{C(\overline{\Omega})} = 0,$$  

(17)

$$J(y_\rho, v_\rho, w_\rho) = J(y_1, v_1, w_1) + \rho \Delta J + o(\rho),$$  

(18)

where $v_\rho, w_\rho$ are the controls defined by:

$$v_\rho(s, t) = \begin{cases} v_1(s, t) & \text{on } \Sigma \setminus E_\rho \\ v_2(s, t) & \text{on } E_\rho \end{cases},$$  

(19)

$$w_\rho = w_1 + \rho w_2,$$  

(20)

$y_\rho, y_1$ are the solutions of (1) corresponding respectively to $(v_\rho, w_\rho)$ and to $(v_1, w_1)$, $z$ is the weak solution of

$$\frac{\partial z}{\partial t} + Az + f'_y(x, t, y_1)z = 0 \quad \text{in } Q,$$  

$$\frac{\partial z}{\partial n_A} + g'_y(s, t, y_1, v_1)z = g(s, t, y_1, v_1) - g(s, t, y_1, v_2) \quad \text{on } \Sigma,$$  

(21)

$$z(0) = w_2 \quad \text{in } \Omega,$$  

and

$$\Delta J = J'_y(y_1, v_1, w_1)z + J(y_1, v_2, w_1) - J(y_1, v_1, w_1) + \int_{\Omega} L'_w(x, y_1(T), w_1)w_2 \, dx.$$  

(22)

The proof relies on the following Lemma.

**Lemma 4.1** Let $v_1, v_2$ be in $V_{ad}$ and let $y$ be in $C(\overline{\Omega})$. For every $\rho \in ]0, 1[$, there exists a sequence of measurable subsets $(E_\rho^n)_n$ in $\Sigma$ such that:

$$L^N(E_\rho^n) = \rho L^N(\Sigma),$$  

(23)

$$\int_{E_\rho^n} (G(s, t, y_1, v_1) - G(s, t, y_1, v_2)) \, ds \, dt = \rho \int_{\Sigma} (G(s, t, y_1, v_1) - G(s, t, y_1, v_2)) \, ds \, dt,$$  

(24)

$$\frac{1}{\rho} \chi_{E_\rho^n} \rightharpoonup 1 \text{ weakly star in } L^\infty(\Sigma), \text{ when } n \to \infty,$$  

(25)

where $\chi_{E_\rho^n}$ is the characteristic of $E_\rho^n$.  

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Remark 4.1. A statement similar to (15), (17), (18) is given in [28], [29], [24] and in [12] (conditions (23), (25) are also stated in [12]). In [28] the proof relies on an extension of Uhl’s theorem. The proofs in [24] and in [12] are constructive. Since the existence of $E^n_\rho$, satisfying together the conditions (23), (24), (25), is not proved neither in [12] nor in [28]. We here give a short proof of Lemma 4.1 based on the Lyapunov convexity theorem.

Proof of Lemma 4.1. We consider a family $(\varphi_n)_n$ dense in $L^1(\Sigma)$. For $n \geq 0$, we set

$$f^n = (1, G(\cdot, y, v_1) - G(\cdot, y, v_2), \varphi_0, \varphi_1, ..., \varphi_n) \in (L^1(\Sigma))^{n+3}.$$

Thanks to Lyapunov’s convexity theorem, for every $n \geq 0$ and every $\rho \in ]0, 1[\$, there exists a measurable subset $E^n_\rho \subset \Sigma$ satisfying

$$\int_{E^n_\rho} f^n \, dsdt = \rho \int_{\Sigma} f^n \, dsdt.$$

Thus, for every $n \geq 0$, $E^n_\rho$ satisfies (23), (24) and

$$\int_{E^n_\rho} \varphi_m \, dsdt = \rho \int_{\Sigma} \varphi_m \, dsdt,$$

(26)

for every $m \in \{0, \ldots, n\}$. Now, for any fixed $\varphi$ in $L^1(\Sigma)$, we have

$$\left| \int_{\Sigma} (1 - \rho \chi_{E^n_\rho}) \varphi \, dsdt \right| \leq \left| \int_{\Sigma} (1 - \rho \chi_{E^n_\rho}) (\varphi - \varphi_m) \, dsdt \right| + \left| \int_{\Sigma} (1 - \rho \chi_{E^n_\rho}) \varphi_m \, dsdt \right|$$

$$\leq \left( \frac{1}{\rho} + 1 \right) \| \varphi - \varphi_m \|_{1, \Sigma} + \int_{\Sigma} (1 - \rho \chi_{E^n_\rho}) \varphi_m \, dsdt.$$

Since $(\varphi_m)_m$ is dense in $L^1(\Sigma)$, for $\epsilon > 0$ there exists $m_0 > 0$ such that $\| \varphi - \varphi_{m_0} \|_{1, \Sigma} \leq \frac{\epsilon}{\frac{1}{\rho} + 1}$. Thanks to (26), for every $n \geq m$, we have $\int_{\Sigma} (1 - \rho \chi_{E^n_\rho}) \varphi_m \, dsdt = 0$. Thus, it follows that

$$\lim_n \int_{\Sigma} (1 - \rho \chi_{E^n_\rho}) \varphi \, dsdt = 0,$$

and the proof is complete. \hfill \Box

Proof of Theorem 4.1. The existence of $E_\rho$ satisfying (15), (16) is an easy consequence of Lemma 4.1. The only delicate point is the proof of (17). This kind of result is already given in ([12], Theorem 5.2) and in ([24], Theorem 3.3). Since we deal with unbounded controls and nonmonotone operators, our assumptions are different from those in [12] and in [24]. However, the proof of (17) can be adapted from the ones given in [12] and in [24].

Let $\rho$ be in $]0, 1[$ and let $(E^n_\rho)_n$ be the sequence of measurable subsets defined in Lemma 4.1. We set

$$v^n_\rho(s, t) = \begin{cases} v_1(s, t) & \text{on } \Sigma \setminus E^n_\rho, \\ v_2(s, t) & \text{on } E^n_\rho, \end{cases} \quad w_\rho = w_1 + \rho w_2.$$
Let \( y^n_\rho \) be the solution of (1) corresponding to \((v^n_\rho, w^n_\rho)\) and let \( z \) be the weak solution of (21). It is clear that \( \xi^n_\rho = (y^n_\rho - y_1)/\rho - z \) is the weak solution in \( C(\overline{\Omega}) \cap W(0, T) \) of

\[
\frac{\partial \xi}{\partial t} + A\xi + a^n_\rho \xi = f^n_\rho \quad \text{in } Q, \quad \frac{\partial \xi}{\partial n_A} + b^n_\rho \xi = g^n_\rho + h^n_\rho \quad \text{on } \Sigma, \quad \xi(0) = 0 \quad \text{in } \Omega,
\]

where

\[
a^n_\rho(x,t) = \int_0^1 f'_y(x,t, (y_1 + \theta(y^n_\rho - y_1))(x,t)) \, d\theta,
\]

\[
b^n_\rho(s,t) = \int_0^1 g'_y(s,t, (y_1 + \theta(y^n_\rho - y_1))(s,t), v^n_\rho(s,t)) \, d\theta,
\]

\[
f^n_\rho = (f'_y(x,t,y_1) - a^n_\rho)z,
\]

\[
g^n_\rho = (g'_y(s,t,y_1,v_1) - b^n_\rho)z,
\]

\[
h^n_\rho = (1 - \frac{1}{\rho} \chi_{E^n_\rho})(g(s,t,y_1,v_2) - g(s,t,y_1,v_1)),
\]

and \( \chi_{E^n_\rho} \) is the characteristic function of \( E^n_\rho \). We denote by \( \xi^{n,1}_\rho \) the solution in \( C(\overline{\Omega}) \cap W(0, T) \) of

\[
\frac{\partial \xi}{\partial t} + A\xi + a^n_\rho \xi = f^n_\rho \quad \text{in } Q, \quad \frac{\partial \xi}{\partial n_A} + b^n_\rho \xi = g^n_\rho \quad \text{on } \Sigma, \quad \xi(\cdot,0) = 0 \quad \text{in } \Omega,
\]

by \( \xi^{n,2}_\rho \) the solution in \( C(\overline{\Omega}) \cap W(0, T) \) of

\[
\frac{\partial \xi}{\partial t} + A\xi + a^n_\rho \xi = 0 \quad \text{in } Q, \quad \frac{\partial \xi}{\partial n_A} + b^n_\rho \xi = h^n_\rho \quad \text{on } \Sigma, \quad \xi(\cdot,0) = 0 \quad \text{in } \Omega,
\]

and by \( \xi^n_\rho \) the solution in \( C(\overline{\Omega}) \cap W(0, T) \) of

\[
\frac{\partial \xi}{\partial t} + A\xi + a_\rho \xi = 0 \quad \text{in } Q, \quad \frac{\partial \xi}{\partial n_A} + b_\rho \xi = h^n_\rho \quad \text{on } \Sigma, \quad \xi(\cdot,0) = 0 \quad \text{in } \Omega,
\]

where \( a(x,t) = f'_y(x,t,y_1(x,t)), b(s,t) = g'_y(s,t,y_1(s,t),v_1(s,t)) \). We also have

\[
\frac{\partial (\xi^{n,2}_\rho - \xi^n_\rho)}{\partial t} + A(\xi^{n,2}_\rho - \xi^n_\rho) + a^n_\rho(\xi^{n,2}_\rho - \xi^n_\rho) = (a - a^n_\rho)\xi^n_\rho \quad \text{in } Q,
\]

\[
\frac{\partial (\xi^{n,2}_\rho - \xi^n_\rho)}{\partial n_A} + b^n_\rho(\xi^{n,2}_\rho - \xi^n_\rho) = (b - b^n_\rho)\xi^n_\rho \quad \text{on } \Sigma,
\]

\[
(\xi^{n,2}_\rho - \xi^n_\rho)(\cdot,0) = 0 \quad \text{in } \Omega.
\]

Thanks to ([35], Proposition 3.3), there exists \( C = C(T, \Omega, N, q, \sigma, C_0) > 0 \) (independent of \( n \) and \( \rho \)) such that

\[
\|\xi^{n,2}_\rho - \xi^n_\rho\|_{C(\overline{\Omega})} \leq C(\|a - a^n_\rho\|_{q,Q} + \|b - b^n_\rho\|_{q,\sigma,\Sigma})\|\xi^n_\rho\|_{C(\overline{\Omega})},
\]

\[
\|\xi^{n,1}_\rho\|_{C(\overline{\Omega})} \leq C(\|f^n_\rho\|_{q,Q} + \|g^n_\rho\|_{q,\sigma,\Sigma}).
\]

The operator \( T \) which associates \( \zeta \), the solution in \( C(\overline{\Omega}) \cap W(0, T) \) of

\[
\frac{\partial \zeta}{\partial t} + A\zeta + a_\rho \zeta = \varphi \quad \text{in } Q, \quad \frac{\partial \zeta}{\partial n_A} + b_\rho \zeta = \psi \quad \text{on } \Sigma, \quad \zeta(0) = 0 \quad \text{in } \Omega,
\]
with \((\varphi, \psi)\) is continuous from \(L^q(Q) \times L^\sigma(\Sigma)\) into \(C^{\alpha,\frac{q}{q-1}}(\overline{Q})\) for some \(0 < \alpha < 1\) (as for Corollary 3.1, this continuity result can be deduced from Theorem 1.3, Chapter 3 in [16]). Since the embedding from \(C^{\alpha,\frac{q}{q-1}}(\overline{Q})\) into \(C(\overline{Q})\) is compact, \(T\) may also be considered as a compact operator from \(L^q(Q) \times L^\sigma(\Sigma)\) into \(C(\overline{Q})\). Because of (25), for every \(0 < \rho < 1\) the sequence \((h^\rho_n)_n\) converges to zero for the weak topology of \(L^\sigma(\Sigma)\). Therefore, since \(T\) is compact from \(L^q(Q) \times L^\sigma(\Sigma)\) into \(C(\overline{Q})\), the sequence \((\zeta^\rho_n)_n\) converges to zero in \(C(\overline{Q})\). There then exists an integer depending on \(\rho\), denoted by \(n(\rho)\), such that

\[\|\zeta^\rho_{n(\rho)}\|_{C(\overline{Q})} \leq \rho.\] (29)

Notice that \((v^{n(\rho)}_\rho)_\rho\) converges to \(v_1\) in \(L^\sigma(\Sigma)\) and \((w_\rho)_\rho\) converges to \(w\) in \(C(\overline{\Omega})\) as \(\rho\) tends to zero. Theorem 3.1 yields that \((y^{n(\rho)}_\rho)_\rho\) uniformly converges to \(y_1\) on \(\overline{Q}\) as \(\rho\) tends to zero. Therefore, thanks to (A1) and (A2), \(f^{n(\rho)}_\rho\) and \((a - a^{n(\rho)}_\rho)\) both converge to zero in \(L^q(Q)\) when \(\rho\) tends to zero and \(g^{n(\rho)}_\rho\), \((b - b^{n(\rho)}_\rho)\) both converge to zero in \(L^\sigma(\Sigma)\) when \(\rho\) tends to zero. Thus, thanks to (27)-(29), we get

\[\lim_{\rho \to 0} \|C^{n(\rho)}_\rho\|_{C(\overline{Q})} \leq \lim_{\rho \to 0} \|C^{n(\rho)}_{\rho,1}\|_{C(\overline{Q})} + \lim_{\rho \to 0} \|C^{n(\rho),2}_{\rho,2} - C^{n(\rho),2}_{\rho,2}\|_{C(\overline{Q})} + \lim_{\rho \to 0} \|C^{n(\rho)}_{\rho,1}\|_{C(\overline{Q})} = 0.\]

Now we set \(E_\rho = E^{n(\rho)}_\rho\), \(v_\rho = v^{n(\rho)}_\rho\) and \(1_{\partial d_\rho} = \zeta^{n(\rho)}_\rho\). Conditions (15) to (17) are clearly satisfied; moreover, taking into account (16), (17) and the definition of \((v_\rho, w_\rho)\), we easily verify (18).

\[\square\]

5 Proof of Pontryagin’s principle.

5.1 Penalized problem.

We first give the proof of optimality conditions in qualified form (the case \(\bar{\nu} = 1\) in Theorem 2.1). The proof of the nonqualified form can be obtained with slight modifications that we give in Section 5.3. For notational simplicity, throughout what follows we set

\[H_\Sigma(s, t, y, v, p, 1) = H_\Sigma(s, t, y, v, p)\]

for every \((s, t, y, v, p) \in \Gamma \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\).

Following [28], [29], since \(C(\overline{D})\) is separable, there exists a norm \(|\cdot|_{C(\overline{D})}\), which is equivalent to the norm \(\|\cdot\|_{C(\overline{D})}\) such that \((C(\overline{D}), |\cdot|_{C(\overline{D})})\) is strictly convex and \(M(\overline{D})\), endowed with the dual norm of \(|\cdot|_{C(\overline{D})}\) (denoted by \(|\cdot|_{M(\overline{D})}\)), is also strictly convex (see [17], Corollary 2 p. 148, or Corollary 2 p. 167). We define the distance function to \(C\) (for the new norm \(|\cdot|_{C(\overline{D})}\)) by

\[d_C(\varphi) = \inf_{z \in C} |\varphi - z|_{C(\overline{D})}.\]

Since \(C\) is convex, then \(d_C\) is convex and Lipschitz of rank 1, and we have

\[\lim_{\varphi \to 0} \sup_{\varphi' \to \varphi} \frac{d_C(\varphi' + \rho z) - d_C(\varphi')}{\rho} = \max\{\langle \xi, z\rangle_{M(\overline{D})} \times C(\overline{D})} \mid \xi \in \partial d_C(\varphi)\} \]

(30)
for every $\varphi, z \in C(\overline{D})$, where $\partial d_C$ is the subdifferential in the sense of convex analysis (see [15]). Therefore, for a given $\varphi \in C(\overline{D})$ we have

$$\langle \xi, z - \varphi \rangle_{M(\overline{D}) \times C(\overline{D})} + d_C(\varphi) \leq d_C(z) \quad \text{for every } \xi \in \partial d_C(\varphi) \text{ and every } z \in C(\overline{D}), \quad (31)$$

$$|\xi|_{M(\overline{D})} \leq 1 \quad \text{for every } \xi \in \partial d_C(\varphi).$$

Moreover it is proved in ([29], Lemma 3.4) that, since $C$ is a closed convex subset of $C(\overline{D})$, for every $\varphi \notin C$, and every $\xi \in \partial d_C(\varphi)$, $|\xi|_{M(\overline{D})} = 1$. Since $\partial d_C(\varphi)$ is convex in $M(\overline{D})$ and $(M(\overline{D}), |\cdot|_{M(\overline{D})})$ is strictly convex, then if $\varphi \notin C$, $\partial d_C(\varphi)$ is a singleton and $d_C$ is Gâteaux-differentiable at $\varphi$.

Let $(\bar{y}, \bar{v}, \bar{w})$ be a solution of problem $(P)$. Thanks to (A8), we prove in the proposition below that $(\bar{y}, \bar{v}, \bar{w})$ is also a local solution of some related penalized problems.

**Proposition 5.1** For every $k > 0$, there exists $\lambda = \lambda(k)$ such that $(\bar{y}, \bar{v}, \bar{w})$ is a solution of the following problem:

$$(P^{r,k}) \quad \inf \{ J_r(y, v, w) \mid y \in W(0,T) \cap C(\overline{D}), \ (v, w) \in (V_{ad}(\bar{v}, k) \times W_{ad}) \cap B^d_{\lambda(k)}(\bar{v}, \bar{w})$$

$$\text{and } (y, v, w) \text{ satisfies } (1) \}$$

where$$J_r(y, v, w) = J(y, v, w) + rd_C(\phi(y)),$$$B^d_{\lambda(k)}(\bar{v}, \bar{w}) \subset V_{ad} \times W_{ad}$ is the closed ball centered on $(\bar{v}, \bar{w})$ and with radius $\lambda(k)$ (for the distance $d$), $r$ only depends on the constant $\tilde{r}$ given in (A8) and on $\overline{D}$.

**Proof.** From (A8), there exist $\bar{\varepsilon} > 0$ and $\tilde{r} > 0$ such that

$$\inf(P) = \inf \{ J(y_{vw}, v, w) + \tilde{r} \gamma \mid (v, w) \in V_{ad} \times W_{ad}, \ \phi(y_{vw}) \in C_\gamma, \ \gamma \in [0, \bar{\varepsilon}] \}.$$

Now by writing

$$\inf(P) = \inf \{ \inf \{ J(y_{vw}, v, w) + \tilde{r} \gamma \mid \phi(y_{vw}) \in C_\gamma, \ \gamma \in [0, \bar{\varepsilon}] \} \mid (v, w) \in V_{ad} \times W_{ad} \},$$

we get

$$\inf(P) = \inf \{ J(y_{vw}, v, w) + \tilde{r} \inf_{z \in C_\varepsilon} \| \phi(y_{vw}) - z \|_{C(\overline{D})} \mid (v, w) \in V_{ad} \times W_{ad}, \ \phi(y_{vw}) \in C_\varepsilon \}.$$
Thus
\[
\inf(P) \leq \inf \{ J(y,v,w) + rdC(\phi(yvw)) \mid (v,w) \in (V_{ad}(\bar{v},k) \times W_{ad}) \cap B^d_{\lambda(k)}(\bar{v},\bar{w}) \} = \inf(P^{r,k}) \leq J(\bar{y},\bar{v},\bar{w}) = \inf(P).
\]

Now we set
\[
J_{r,n}(y,v,w) = J(y,v,w) + r[\phi(yvw)]^2 + n^{-2} \frac{1}{2},
\]
and we denote by \((P_n^{r,k})\) the problem
\[
\inf \{ J_{r,n}(y,v,w) \mid y \in W(0,T) \cap C(\bar{Q}), (v,w) \in (V_{ad}(\bar{v},k) \times W_{ad}) \cap B^d_{\lambda(k)}(\bar{v},\bar{w}) \}
\]
and \((y,v,w)\) satisfies (1).

**Proposition 5.2** For every \(k > 0\),
\[
\inf(P^{r,k}) = \lim_{n \to \infty} \inf(P_n^{r,k}) \quad \text{and} \quad \lim_{n \to \infty} J_{r,n}(y,v,w) = J_r(y,v,w)
\]
for every \(y \in C(\bar{Q})\) and every \((v,w) \in L^\sigma(\Sigma) \times C(\bar{\Omega})\). Moreover \((\bar{y},\bar{v},\bar{w})\) is a \(\varepsilon^2\)-solution of \((P_n^{r,k})\) with \(\varepsilon^2 = rn^{-1}\).

**Proof.** The first part of the proof is immediate if we observe that
\[
J_r(y,v,w) \leq J_{r,n}(y,v,w) \leq J_r(y,v,w) + rn^{-1}
\]
for every \((y,v,w) \in C(\bar{Q}) \times L^\sigma(\Sigma) \times C(\bar{\Omega})\). Moreover, since \((\bar{y},\bar{v},\bar{w})\) is solution of \((P^{r,k})\), with the previous inequalities, we get
\[
J_{r,n}(\bar{y},\bar{v},\bar{w}) \leq J_r(\bar{y},\bar{v},\bar{w}) + rn^{-1} \leq J_r(y,v,w) + rn^{-1} \leq J_{r,n}(y,v,w) + rn^{-1},
\]
for every \(y \in C(\bar{Q})\) and every \((v,w) \in (V_{ad}(\bar{v},k) \times W_{ad}) \cap B^d_{\lambda(k)}(\bar{v},\bar{w})\) such that \((y,v,w)\) obeys (1).

**5.2 Proof of Theorem 2.1 (Pontryagin principle in qualified form).**

Let \(k\) be a positive constant. Thanks to Proposition 5.2, for every \(n \geq 1\), \((\bar{y},\bar{v},\bar{w})\) is an \(\varepsilon_n^2\)-solution of \((P_n^{r,k})\), with \(\varepsilon_n^2 = rn^{-1}\). For every \(k > 0\), we choose \(n(k)\) such that
\[
\varepsilon_n(k) = \left( \frac{r}{n(k)} \right)^\frac{1}{2} \leq \min \left( \frac{1}{k^{2\sigma}}, \frac{\lambda(k)}{2} \right).
\]
The metric space \((V_{ad}(\bar{v}, k) \times W_{ad}) \cap B^d_{\Lambda(k)}(\bar{v}, \bar{w}), d)\) is complete and the functional \((v, w) \mapsto J_{r,n(k)}(y_{vw}, v, w)\) is continuous on this metric space. Thanks to the Ekeland’s variational principle, for every \(k \geq 1\), there exists \((v_k, w_k) \in (V_{ad}(\bar{v}, k) \times W_{ad}) \cap B^d_{\Lambda(k)}(\bar{v}, \bar{w})\) such that

\[
d((v_k, w_k), (\bar{v}, \bar{w})) \leq \varepsilon_{n(k)},
\]

\[
J_{r,n(k)}(y_k, v_k, w_k) \leq J_{r,n(k)}(y_{vw}, v, w) + \varepsilon_{n(k)}d((v_k, w_k), (v, w))
\]

for every \((v, w) \in (V_{ad}(\bar{v}, k) \times W_{ad}) \cap B^d_{\Lambda(k)}(\bar{v}, \bar{w})\) \((y_k\text{ and } y_{vw}\text{ being the states corresponding respectively to } (v_k, w_k)\text{ and to } (v, w))\). The proof is split into five parts.

**Step 1.** Approximate optimality conditions for the boundary control \(v_k\) satisfying (32), (33).

For fixed \(v_0\) in \(V_{ad}\), we denote by \(v_{0k}\) \((k > 0)\) the function in \(V_{ad}(\bar{v}, k)\) defined by

\[
v_{0k}(s, t) = \begin{cases} v_0(s, t) & \text{if } |v_0(s, t) - \bar{v}(s, t)| \leq k, \\ \bar{v}(s, t) & \text{if not.} \end{cases}
\]

Applying Theorem 4.1, we deduce the existence of measurable sets \(E^k_{\rho}\), such that \(L^N(E^k_{\rho}) = \rho L^N(\Sigma)\),

\[
y^k_{1, \rho} = y_k + \rho z^1_k + v^k, \quad \lim_{\rho \to 0} \frac{1}{\rho} \|z^k\|_{C(Q)} = 0,
\]

\[
J(y^k_{1, \rho}, v^k_{1, \rho}, w^k_{1, \rho}) = J(y_k, v_k, w_k) + \rho \Delta J^1_k + o(\rho),
\]

where \(v^k_{1, \rho}\) and \(w^k_{1, \rho}\) are defined by

\[
v^k_{1, \rho}(s, t) = \begin{cases} v_k(s, t) & \text{on } \Sigma \setminus E^k_{\rho}, \\ v_{0k}(s, t) & \text{on } E^k_{\rho}. \end{cases}
\]

\(y^k_{1, \rho}\) is the state corresponding to \((v^k_{1, \rho}, w^k_{1, \rho})\), \(z_k^1\) is the weak solution of

\[
\frac{\partial z_k^1}{\partial t} + Az_k^1 + f_y'(x, t, y_k)z_k^1 = 0 \quad \text{in } Q,
\]

\[
\frac{\partial z_k^1}{\partial t_A} + g_y'(s, t, y_k, v_k)z_k^1 = g(s, t, y_k, v_k) - g(s, t, y_k, v_{0k}) \quad \text{on } \Sigma,
\]

\(z_k^1(\cdot, 0) = 0 \quad \text{in } \Omega,
\]

and

\[
\Delta J^1_k = \int_Q f_y'(x, t, y_k(x, t))z_k^1(x, t) \, dx \, dt + \int_\Sigma G_y'(s, t, y_k(s, t), v_k(s, t))z_k^1(s, t) \, ds \, dt + \\
+ \int_\Sigma [G(s, t, y_k(s, t), v_{0k}(s, t)) - G(s, t, y_k(s, t), v_k(s, t))] \, ds \, dt + \\
+ \int_\Omega L_y'(x, y_k(x, T), w_k(x))z_k^1(x) \, dx.
\]

On the other hand, we have

\[
d((v^k_{1, \rho}, w^k_{1, \rho}), (\bar{v}, \bar{w})) \leq d((v^k_{1, \rho}, w^k_{1, \rho}), (v_k, w_k)) + d((v_k, w_k), (\bar{v}, \bar{w}))
\]
\[
\leq \mathcal{L}^N(E^k_{\rho}) + \varepsilon_{n(k)} \leq \rho \mathcal{L}^N(\Sigma) + \varepsilon_{n(k)}.
\]

There then exists \(\rho_k\) such that, for every \(0 < \rho < \rho_k\), we have

\[
dl((v^k_{\ell,\rho}, w^k_{\ell,\rho}), (\tilde{v}, \tilde{w})) \leq \rho \mathcal{L}^N(\Sigma) + \varepsilon_{n(k)} \leq \lambda(k).
\]

Therefore, for every \(k > 0\) and every \(0 < \rho < \rho_k\), \((v^k_{\ell,\rho}, w^k_{\ell,\rho})\) belongs to \((V_{ad}(\tilde{v}, k) \times W_{ad}) \cap B_{\lambda(k)}(\tilde{v}, \tilde{w})\).

If we set \((v, w) = (v^k_{\ell,\rho}, w^k_{\ell,\rho})\) in (33), it follows that

\[
\limsup_{\rho \to 0} \frac{J_{r,n}(k)(y_k, v_k, w_k) - J_{r,n}(k)(y^k_{\ell,\rho}, v^k_{\ell,\rho}, w^k_{\ell,\rho})}{\rho} \leq \varepsilon_{n(k)} \mathcal{L}^N(\Sigma).
\]

Taking (30), (36) and the definition of \(J_{r,n}\) into account, we get

\[
- \Delta J^1_k - (\mu_k, \phi'(y_k)z^1_k)_{M(\Sigma)} \leq C(\Sigma) \leq \varepsilon_{n(k)} \mathcal{L}^N(\Sigma)
\]

where

\[
\mu_k = \begin{cases} 
\frac{r d_c(\phi'(y_k)) \nabla d_c(\phi'(y_k))}{|d_c(\phi'(y_k))^2 + n(k)^{2s}|^{2s/2} + n(k)^{-2}} & \text{if } d_c(\phi'(y_k)) > 0, \\
0 & \text{if not}.
\end{cases}
\]

For every \(k > 0\), we consider the weak solution \(p_k\) of

\[
- \frac{\partial p_k}{\partial t} + Ap_k + f'_y(x, t, y_k)p_k = F'_y(x, t, y_k) + [\phi'(y_k)^*\mu_k]_{Q} \quad \text{in } Q, \\
\frac{\partial p_k}{\partial n_A} + g'_y(s, t, y_k, v_k)p_k = G'_y(s, t, y_k, v_k) + [\phi'(y_k)^*\mu_k]_{\Sigma} \quad \text{on } \Sigma,
\]

\[
p_k(T) = L'_y(x, y_k(T), w_k) + [\phi'(y_k)^*\mu_k]_{\Omega_T} \quad \text{in } \Omega,
\]

where \([\phi'(y_k)^*\mu_k]_{Q}\) is the restriction of \([\phi'(y_k)^*\mu_k]\) to \(Q\), \([\phi'(y_k)^*\mu_k]\) is the restriction of \([\phi'(y_k)^*\mu_k]\) to \(\Sigma\) and \([\phi'(y_k)^*\mu_k]\) is the restriction of \([\phi'(y_k)^*\mu_k]\) to \(\Omega_T\). By using the Green formula of Theorem 3.2, we obtain

\[
\int_Q F'_y(x, t, y_k)z^1_k \, dx \, dt + \int_\Sigma G'_y(s, t, y_k, v_k)z^1_k \, ds \, dt + \int_\Omega L'_y(x, y_k(T), w_k)z^1_k(T) \, dx + 
\]

\[
+ (\mu_k, \phi'(y_k)z^1_k)_{M(\Sigma)} = 
\]

\[
= \int_Q p_k \left( \frac{\partial z^1_k}{\partial t} + Az^1_k + f'_y(x, t, y_k)z^1_k \right) \, dx \, dt + \int_\Sigma p_k \left( \frac{\partial z^1_k}{\partial n_A} + g'_y(s, t, y_k, v_k)z^1_k \right) \, ds \, dt 
\]

\[
= \int_\Sigma p_k \left[ g(s, t, y_k, v_k) - g(s, t, y_k, v_{0k}) \right] \, ds \, dt.
\]

With this equality, with (39) and the definition of \(\Delta J^1_k\), we get

\[
\int_\Sigma |G(s, t, y_k, v_k) - p_kg(s, t, y_k, v_k)| \, ds \, dt \leq 
\]

\[
\leq \int_\Sigma |G(s, t, y_k, v_k) - p_kg(s, t, y_k, v_{0k})| \, ds \, dt + \varepsilon_{n(k)} \mathcal{L}^N(\Sigma)
\]

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As in Step 1, if $\rho$ Theorem 4.1 then yields

Thus $(v_k^2(w^k_v, \rho^k)$. Since $W_{ad}$ is convex, we see that

Let $w_0$ be in $W_{ad}$, we consider the sequence $(v_{2,\rho}^k, w_{2,\rho}^k)$ defined by

and we denote by $y_{2,\rho}^k$ the state corresponding to $(v_{2,\rho}^k, w_{2,\rho}^k)$. Since $W_{ad}$ is convex, we see that

Moreover, we have

As in Step 1, if $\rho_k$ is small enough, for every $0 < \rho < \rho_k$, we have:

Thus $(v_{2,\rho}^k, w_{2,\rho}^k)$ belongs to $(V_{ad}(\bar{v}, k) \times W_{ad}) \cap B_{\lambda(k)}^d(\bar{v}, \bar{w})$, for every $k > 0$ and for every $0 < \rho < \rho_k$. Theorem 4.1 then yields

where $z_{2,\rho}^k$ is the weak solution of

and

As in Step 1, from (33), (44) we deduce that

$$-\Delta J_k^2 - \langle \mu_k, o'(y_k)z_{2,\rho}^2 \rangle_{M(\overline{T}) \times C(\overline{T})} = \limsup_{\rho \to 0} \frac{J_{r,n(k)}(y_k, v_k, w_k) - J_{r,n(k)}(y_{2,\rho}^k, v_{2,\rho}^k, w_{2,\rho}^k)}{\rho}$$
\[ \leq \varepsilon_{n(k)}\|w_0 - w_k\|_{\infty, \Omega} \leq \varepsilon_{n(k)}(\varepsilon_{n(k)} + \|w_0 - \bar{w}\|_{\infty, \Omega}), \] (45)

where \( \mu_k \) is defined in (40). If we consider the weak solution \( p_k \) of (41), still by using the Green formula of Theorem 3.2, we obtain

\[ \int_Q F_y(x, t, y_k)z_k^2 \, dx dt + \int_{\Sigma} G_y(s, t, y_k, v_k)z_k^2 \, ds dt + \int_{\Omega} L'_y(x, y_k(T), w_k)z_k^2(T) \, dx + \]

\[ + (|\partial^y(y_k)^* \mu_k|_{\mathcal{T}_0, \{z_k^2\}, M_b(\mathcal{T}_0, \{z_k^2\})}) = \]

\[ = \int_Q p_k \left( \partial z_k^2 + A z_k^2 + f'_y(x, t, y_k)z_k^2 \right) \, dx dt + \int_{\Sigma} p_k \left( \partial z_k^2 + g'_y(s, t, y_k)z_k^2 \right) \, ds dt + \]

\[ + (p_k(0), z_k^2(0))_{\mathcal{M}(\mathcal{T}) \times C(\mathcal{T})}. \]

Taking (45) and the definition of \( \Delta J^2_k \) into account, we get

\[ - \int_{\Omega} L'_w(x, y_k)w_k(w_0 - w_k) \, dx - (p_k(0), w_0 - w_k)_{\mathcal{M}(\mathcal{T}) \times C(\mathcal{T})} \]

\[ - (|\partial^w(y_k)^* \mu_k|_{\mathcal{T}_0, \{w_0 - w_k\}, \mathcal{M}(\mathcal{T}) \times C(\mathcal{T})}) \leq \varepsilon_{n(k)}(\varepsilon_{n(k)} + \|w_0 - \bar{w}\|_{\infty, \Omega}) \] (46)

for every \( w_0 \in W_{ad} \).

**Step 3.** Convergence of sequences \((\mu_k)_k\) and \((p_k)_k\).

We observe that

\[ |\mu_k|_{\mathcal{M}(\mathcal{T})} \leq rd_C(\phi(y_k)) \left( (d_C(\phi(y_k)))^2 + n(k)^{-2} \right)^{-1/2} \leq r. \]

The sequence \((\mu_k)_k\) is bounded in \( \mathcal{M}(\mathcal{T}) \), so there exist \( \bar{\mu} \in \mathcal{M}(\mathcal{T}) \) and a subsequence, still denoted by \((\mu_k)_k\), such that

\[ \mu_k \rightharpoonup \bar{\mu} \text{ weak}^* \text{ in } \mathcal{M}(\mathcal{T}). \] (47)

Let \((\delta, d)\) be a pair verifying \((C_{q\sigma})\). From Theorem 3.2, we have

\[ \|p_k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \leq C_4\{\|F'_y(\cdot, y_k)\|_{1, \mathcal{Q}} + \|C'_y(\cdot, y_k, v_k)\|_{1, \Sigma} + \|L'_y(\cdot, y_k(T), w_k)\|_{1, \Omega} + \]

\[ + |\mu_k|_{\mathcal{M}(\mathcal{T})}\|\phi'(y_k)\|_{L(C(\mathcal{T}) \cap C(\mathcal{T}))}\}. \]

(Here \( L(C(\mathcal{Q}); C(\mathcal{Q})) \) denotes the space of linear continuous mappings from \( C(\mathcal{Q}) \) to \( C(\mathcal{T}) \).)

Since the sequences \((\mu_k)_k\), \((y_k)_k\), \((v_k)_k\) and \((w_k)_k\) are bounded respectively in \( \mathcal{M}(\mathcal{T}), C(\mathcal{Q}), L^p(\Sigma) \) and in \( C(\mathcal{T}) \), the sequence \((p_k)_k\) is bounded in \( L^{q'}(0,T;W^{1,q'}(\Omega)) \). Then there exist \( \tilde{p} \in L^{q'}(0,T;W^{1,q'}(\Omega)) \) and a subsequence, still denoted by \((p_k)_k\), such that \((p_k)_k\) weakly converges to \( \tilde{p} \) in \( L^{q'}(0,T;W^{1,q'}(\Omega)) \).

Let us prove that \( \tilde{p} \) is the weak solution of equation (7).

Let \( \varphi \) be in \( C^1(\mathcal{Q}) \) satisfying \( \varphi(\cdot, 0) = 0 \) in \( \mathcal{T} \), for every \( k > 0 \), we have

\[ \int_Q \left\{ p_k \frac{\partial \varphi}{\partial t} + \sum_{i,j=1}^N a_{ij} D_j p_k D_i \varphi + f'_y(x, t, y_k)p_k \varphi \right\} \, dx dt + \int_{\Sigma} \left( p_k \frac{\partial \varphi}{\partial n_A} + g'_y(s, t, y_k)p_k \varphi \right) \, ds dt = \]
\[
\int_Q F_y'(x,t,y_k)\varphi \, dxdt + \int_\Sigma G_y'(s,t,y_k,\varphi) \, dsdt + \int_{\Omega} L_y'(x,y_k(T),w_k)\varphi(T) \, dx
+ \langle \mu_k, \phi'(y_k) \varphi \rangle_{\mathcal{M}(\mathcal{D}) \times C(\mathcal{D})}.
\]

(48)

Since \((\delta, d)\) satisfies \((C_{\sigma\sigma})\), the following imbeddings are continuous:

\[L^{\sigma}(0, T; W^{1,d'}(\Omega)) \hookrightarrow L^d(Q), \quad L^{\sigma}(0, T; W^{1-\frac{1}{d'},d'}(\Gamma)) \hookrightarrow L^d(\Sigma).\]

Therefore, \((p_k)_k\) weakly converges to \(\bar{p}\) in \(L^d(Q)\) and the sequence of traces \((p_k|\Sigma)_k\) weakly converges to the trace \(\bar{p}|\Sigma\) in \(L^d(\Sigma)\). Moreover, since \((v_k)_k\) converges to \(\bar{v}\) in \(L^d(\Sigma)\) (indeed, since \(d_E(v_k, \bar{v}) \leq \varepsilon_n(k) \leq \frac{1}{k^p}\) and \(|v_k - \bar{v}| \leq k\) a.e. on \(\Sigma\), we have \(\int_\Sigma |v_k - \bar{v}|^\sigma \, dsdt \leq \frac{1}{k^p}\)), \((w_k)_k\) converges to \(\bar{w}\) in \(C(\Omega)\) and \((y_k)_k\) converges to \(\bar{y}\) in \(C(\bar{Q})\), thanks to assumptions on \(f, g, F, G, L, \phi\), we have

\[\lim_k \|f_y'(\cdot, y_k) - f_y'(\cdot, \bar{y})\|_{Q, Q} = 0, \quad \lim_k \|F_y'(\cdot, y_k) - F_y'(\cdot, \bar{y})\|_{1, Q} = 0,
\]

\[\lim_k \|g_y'(\cdot, y_k, v_k) - g_y'(\cdot, \bar{y}, \bar{v})\|_{\sigma, \Sigma} = 0, \quad \lim_k \|G_y'(\cdot, y_k, v_k) - G_y'(\cdot, \bar{y}, \bar{v})\|_{1, \Sigma} = 0,
\]

\[\lim_k \|L_y'(\cdot, y_k(T), w_k) - L_y'(\cdot, \bar{y}(T), \bar{w})\|_{1, \Omega} = 0, \quad \lim_k \|\phi'(y_k) - \phi'(\bar{y})\|_{C(\bar{Q}); C(\bar{D})} = 0.
\]

Thus, by passing to the limit in (48), it follows that

\[\int_Q \{\bar{p} \frac{\partial \varphi}{\partial t} + \sum_{i,j=1}^N a_{ij} D_j \bar{p} B_i \varphi + f_y'(x,t,\bar{y})\bar{p} \varphi\} \, dxdt + \int_\Sigma \{\bar{p} \frac{\partial \varphi}{\partial n_A} + g_y'(s,t,\bar{y},\bar{v})\bar{p} \varphi\} \, dsdt =
\]

\[= \int_Q F_y'(x,t,\bar{y})\varphi \, dxdt + \int_\Sigma G_y'(s,t,\bar{y},\bar{v})\varphi \, dsdt + \int_{\Omega} L_y'(x,\bar{y}(T),\bar{w})\varphi(T) \, dx + \langle \bar{\mu}, \phi'(\bar{y})\varphi \rangle_{\mathcal{M}(\mathcal{D}) \times C(\mathcal{D})}.
\]

(49)

for every \(\varphi \in C^1(\bar{Q})\) satisfying \(\varphi(\cdot, 0) = 0\) in \(\overline{\Omega}\). Therefore \(\bar{p}\) is the unique weak solution for (7). Since the weak solution of (7) is unique in the sense of definition 3.1, we can deduce by classical arguments that \(\bar{p}\) is independent of the pair \((\delta, d)\) (chosen after (47)) and that the original sequence \((p_k)_k\) converges weakly to \(\bar{p}\) in \(L^d(0, T; W^{1,d'}(\Omega))\) for every \((\delta, d)\) satisfying \((C_{\sigma\sigma})\).

To pass to the limit in (46), we prove that

\[(p_k(0) + [\phi'_y(y_k)^* \mu_k]|_{\mathcal{D}_0})_k \rightharpoonup \bar{p}(0) + [\phi'_y(\bar{y})^* \bar{\mu}]|_{\mathcal{D}_0} \quad \text{weakly star in } \mathcal{M}(\overline{\Omega}).
\]

(50)

For this, let \(\varphi\) be in \(C(\overline{\Omega})\) and let \(y\) be the solution of

\[\frac{\partial y}{\partial t} + Ay = 0 \quad \text{in } Q, \quad \frac{\partial y}{\partial n_A} = 0 \quad \text{on } \Sigma, \quad y(0) = \varphi \quad \text{in } \Omega.
\]

Thanks to the Green formula of Theorem 3.2, we get

\[\int_Q \{\bar{p} f_y'(x,t,\bar{y}) - p_k f_y'(x,t,y_k)y\} \, dxdt + \int_\Sigma \{\bar{p} g_y'(s,t,\bar{y},\bar{v}) - p_k g'_y(s,t,y_k,v_k)y\} \, dsdt +
\]

\[\int_{\Omega} \{\bar{p} \phi'(\bar{y}) - p_k \phi'(y_k)\} \, dsdt = 0.
\]

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Now (50) follows from the previous convergence results.

**Step 4.** Integral Pontryagin’s principle.

Notice that \((v_0)_k\) tends to \(v_0\) in \(L^o(\Sigma)\) and \((v_k)\) tends to \(\bar{v}\) in \(L^o(\Sigma)\). By passing to the limit when \(k\) tends to infinity in (42) and (46), and by using the convergence results stated in Step 3, we obtain

\[
\int_{\Sigma} H_\Sigma(s, t, \bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)) \, ds dt \leq \int_{\Sigma} H_\Sigma(s, t, \bar{y}(s, t), v_0(s, t), \bar{p}(s, t)) \, ds dt
\]

(51)

for every \(v_0 \in V_{ad}\), and

\[
\int_{\Omega} L'(x, \bar{y}(T), w), (\bar{w} - w_0) \, dx + (\bar{p}(0) + [\phi'_y(\bar{y})^* \bar{\mu}])|_{\Omega_0}, \bar{w} - w_0)_{M(\Omega) \times C(\Omega)} \leq 0
\]

(52)

for every \(w_0 \in W_{ad}\). On the other hand, from the definition of \(\mu_k\) and from (31), we deduce

\[
\langle \mu_k, z - \phi(y_k) \rangle_{M(\Omega) \times C(\Omega)} \leq 0 \quad \forall z \in C.
\]

By passing to the limit in this expression, we obtain (6).

**Step 5.** Pointwise Pontryagin’s principle. Now we prove the pointwise boundary Pontryagin principle. The functions

\[
(s, t) \mapsto \bar{v}(s, t), \quad (s, t) \mapsto H_\Sigma(s, t, \bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)),
\]

are measurable on \(\Sigma\) and the function

\[
(s, t, v) \mapsto H_\Sigma(s, t, \bar{y}(s, t), v, \bar{p}(s, t)),
\]

is a Carathéodory function from \(\Sigma \times R\) into \(R\). Thanks to Lusin’s Theorem and Scorza-Dragoni’s Theorem, for every \(\epsilon > 0\), there exist a compact subset \(\Sigma_\epsilon \subset \Sigma\), continuous mappings \(\varphi_0, \varphi_1\) from \(\Sigma_\epsilon\) into \(R\) and a continuous mapping \(\varphi_2\) from \(\Sigma_\epsilon \times R\) into \(R\) such that

\[
\mathcal{L}^N(\Sigma \setminus \Sigma_\epsilon) \leq \epsilon,
\]

\[
\varphi_0(s, t) = \bar{v}(s, t) \quad \text{on} \quad \Sigma_\epsilon,
\]

\[
\varphi_1(s, t) = H_\Sigma(s, t, \bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)) \quad \text{on} \quad \Sigma_\epsilon,
\]

\[
\varphi_2(s, t, v) = H_\Sigma(s, t, \bar{y}(s, t), v, \bar{p}(s, t)) \quad \text{on} \quad \Sigma_\epsilon \times R.
\]

Since \(\bar{v}\) is continuous on \(\Sigma_\epsilon\), \(\bar{v}\) is bounded on \(\Sigma_\epsilon\) and, for \(M > \|\bar{v}\|_{\infty, \Sigma_\epsilon}\), the multimapping

\[
(s, t) \mapsto K_M(s, t) := K_V(s, t) \cap [-M, M]
\]

has nonempty compact values. From a Lusin type Theorem for measurable multimappings with compact values (see for example [2], Theorem 1.4.1), for every integer \(M > \|\bar{v}\|_{\infty, \Sigma_\epsilon}\), there exists a
measurable subset $\Sigma_{eM} \subset \Sigma_e$ such that $\mathcal{L}^N(\Sigma_e \setminus \Sigma_{eM}) \leq \frac{1}{2M}$ and the restriction of $K_M$ to $\Sigma_{eM}$ is continuous. Let us denote by $\tilde{\Sigma}_{eM}$ the set of Lebesgue points in $\Sigma_{eM}$, of the characteristic function of $\Sigma_{eM}$.

Now let $(s_0, t_0)$ be in $\tilde{\Sigma}_{eM}$ and let $v \in K_M(s_0, t_0)$. Since the multimapping $K_M$ is continuous on $\Sigma_{eM}$ (in fact we only use the lower semicontinuity of $K_M$), for every integer $n > 0$, the multimapping $K_M$ admits a measurable selection $v_n$ and there exists an increasing function $\gamma$ from $R^+$ into $R^+$ such that

$$|v_n(s, t) - v| \leq \frac{1}{n} \text{ on } B((s_0, t_0), \gamma(\frac{1}{n})) \cap \Sigma_{eM} \text{ and } \lim_{n \to \infty} \gamma(\frac{1}{n}) = 0,$$

where $B((s_0, t_0), \gamma(\frac{1}{n}))$ is the ball in $R^N$ centered on $(s_0, t_0)$ and of radius $\gamma(\frac{1}{n})$. We now set

$$S_{e,M,n} = \tilde{\Sigma}_{eM} \cap B((s_0, t_0), \gamma(\frac{1}{n})),$$

and we consider the variation

$$v_M(s, t) = \begin{cases} \bar{v}(s, t) & \text{on } \Sigma \setminus S_{e,M,1}, \\ v_n(s, t) & \text{on } S_{e,M,n} \setminus S_{e,M,n+1} \text{ for every } n > 0, \\ v & \text{if } (s, t) = (s_0, t_0). \end{cases}$$

It is clear that $v_M \in V_{ad}$ and that

$$\lim_{(s, t) \to (s_0, t_0)} v_M(s, t) = v.$$

If we take $v_0 = \chi_{S_{e,M,n}}v_M + (1 - \chi_{S_{e,M,n}})\bar{v}$ in (51) (where $\chi_{S_{e,M,n}}$ is the characteristic function of $S_{e,M,n}$), it follows

$$\frac{1}{\mathcal{L}^N(S_{e,M,n})} \int_{S_{e,M,n}} \varphi_1(s, t) \, dsdt \leq \frac{1}{\mathcal{L}^N(S_{e,M,n})} \int_{S_{e,M,n}} \varphi_2(s, t, v_M(s, t)) \, dsdt.$$

(Note that for every $n \geq 1$, $\mathcal{L}^N(S_{e,M,n}) \neq 0$ because $(s_0, t_0) \in \tilde{\Sigma}_{eM}$.) By passing to the limit in the above inequality when $n$ tends to infinity and using the continuity of $\varphi_1$ and $\varphi_2$, we obtain

$$\varphi_1(s_0, t_0) = H_\Sigma(s_0, t_0, \bar{v}(s_0, t_0), \bar{p}(s_0, t_0)) \leq \varphi_2(s_0, t_0, v) = H_\Sigma(s_0, t_0, \bar{v}(s_0, t_0), v, \bar{p}(s_0, t_0))$$

for every $(s_0, t_0) \in \tilde{\Sigma}_{eM}$ and for every $v \in K_V(s_0, t_0)$ such that $|v| \leq M$. We set

$$\tilde{\Sigma}_e = \bigcap_{M \in N^*} \bigcap_{M > \|v\|_{L^\infty, \Sigma_e}} \tilde{\Sigma}_{eM}$$

and we observe that $\mathcal{L}^N(\Sigma \setminus \tilde{\Sigma}_e) \leq 2\epsilon$. For every $(s, t) \in \tilde{\Sigma}_e$ and every $v \in K_V(s, t)$ we have

$$H_\Sigma(s, t, \bar{v}(s, t), \bar{v}(s, t), \bar{p}(s, t)) \leq H_\Sigma(s, t, \bar{v}(s, t), v, \bar{p}(s, t)).$$

Upon setting $\tilde{\Sigma} = \bigcup_{\epsilon > 0} \tilde{\Sigma}_\epsilon$, we have $\mathcal{L}^N(\tilde{\Sigma}) = \mathcal{L}^N(\Sigma)$, the pointwise Pontryagin’s principle is satisfied on $\tilde{\Sigma}$ and the proof is complete. $\square$
5.3 Proof of Pontryagin principle in nonqualified form.

In this case, as in [20], [29], [40], [24], [12], we can consider the penalized functional

\[ J_n(y, v, w) = \left\{ \left[ (J(y, v, w) - J(\bar{y}, \bar{v}, \bar{w}) + \frac{1}{n^2})^2 + (d_\mathcal{C}(\phi(y)))^2 \right] \right\}^{1/2}. \]

With such a choice, for every \( k > 0 \), \((\bar{y}, \bar{v}, \bar{w})\) is a \( \frac{1}{n^2} \)-solution of the penalized problem

\[ (P_n^k) \quad \inf\{ J_n(y, v, w) \mid y \in W(0, T) \cap C(\mathcal{Q}), (v, w) \in V_{ad}(\bar{v}, k) \times W_{ad}, (y, v, w) \text{ satisfies (1)} \}. \]

As in Section 5.2, for every \( k > 0 \), we choose \( n(k) \) such that

\[ \frac{1}{n(k)} \leq \frac{1}{k^{2\sigma}}. \]

Thanks to the Ekeland’s principle, there exists \((v_k, w_k) \in V_{ad}(\bar{v}, k) \times W_{ad}\) such that

\[ d((v_k, w_k), (\bar{v}, \bar{w})) \leq \frac{1}{n(k)}, \]

\[ J_n(k)(y_k, v_k, w_k) \leq J_n(k)(y_{ew}, v, w) + \frac{1}{n(k)} d((v_k, w_k), (v, w)) \]

for every \((v, w) \in V_{ad}(\bar{v}, k) \times W_{ad}\) \((y_k)\) is the solution of (1) corresponding to \((v_k, w_k)\). With calculations similar to those in [24], [40], [12] by using diffuse perturbations, we get

\[ \int_\Sigma H(s, t, y_k(s, t), v_k(s, t), p_k(s, t), \nu_k) ds dt \leq \int_\Sigma H(s, t, y_k(s, t), v_{0k}(s, t), p_k(s, t), \nu_k) + \frac{1}{n(k)} \mathcal{L}^N(\Sigma) \]

for every \( k > 0 \) and every \( v_0 \in V_{ad} \) \((v_{0k}\) is defined in function of \( v_0\) in (34)) and

\[ \int_\Omega \nu_k L_w'(x, y_k(T), w_k)(w_0 - w_k) dx - (p_k(0) + [\phi'_y(y_k)^* \mu_k])|_{\overline{\Omega}}, w_0 - w_k)_{\mathcal{M}(\overline{\Omega}) \times C(\overline{\Omega})} \leq \]

\[ \frac{1}{n(k)} \left( \frac{1}{n(k)} + \|w_0 - \bar{w}\|_{\infty, \Omega} \right) \]

for every \( w_0 \in W_{ad}\), where

\[ \nu_k = \left( \frac{J(y_k, v_k, w_k) - J(\bar{y}, \bar{v}, \bar{w}) + \frac{1}{n(k)^2}}{J_n(k)(y_k, v_k, w_k)} \right)^+, \]

\[ \mu_k = \begin{cases} \frac{d_\mathcal{C}(\phi(y_k))\nabla d_\mathcal{C}(\phi(y_k))}{J_n(k)(y_k, v_k, w_k)} & \text{if } \phi(y_k) \notin \mathcal{C}, \\ 0 & \text{otherwise}, \end{cases} \]

and \( p_k \) is the weak solution of

\[ \begin{array}{ll} -\frac{\partial p_k}{\partial t} + Ap_k + f'_y(x, t, y_k)p_k = \nu_k F'_{y}(x, t, y_k) + [\phi'_{y}(y_k)^* \mu_k]Q & \text{in } Q, \\ \frac{\partial p_k}{\partial n_A} + g'_y(s, t, y_k, v_k)p_k = \nu_k G'_{y}(s, t, y_k, v_k) + [\phi'_{y}(y_k)^* \mu_k]\Sigma & \text{on } \Sigma, \end{array} \]

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\[ p_k(T) = \nu_k L_y(x, y_k(T), w_k) + [\varphi'_y(y_k) \mu_k] \mathbb{I}_T \]

in \( \Omega \).

By passing to the limit when \( k \) tends to infinity, as in Section 5.2, we finally get the Pontryagin principle in nonqualified form with \( \bar{\nu} = \lim_k \nu_k \) and \( \bar{\mu} \) the weak star limit of \( \mu_k \). To prove that \( (\bar{\nu}, \bar{\mu}) \) is nonzero, we remark that

\[ \nu_k^2 + |\mu_k|^2_{\mathcal{M}(\overline{D})} = 1. \]

If \( \bar{\nu} > 0 \), the proof is complete. If \( \bar{\nu} = 0 \), we can prove that \( |\bar{\mu}|_{\mathcal{M}(\overline{D})} > 0 \) by using \( \lim_k |\mu_k|_{\mathcal{M}(\overline{D})} = 1 \) and \( \text{int}_{C(\overline{D})} C \neq \emptyset \). Indeed, if \( \text{int}_{C(\overline{D})} C \) is nonempty, there exists a ball \( B(z; \rho) \subset C \) with \( \rho > 0 \) (where \( B(z; \rho) \) is the ball in \( C(\overline{D}) \), centered at \( z \) and with radius \( \rho \)). We can choose \( z_k \in B(0; \rho) \) such that \( \langle \mu_k, z_k \rangle_{\mathcal{M}(\overline{D}) \times C(\overline{D})} = \frac{1}{2} \rho |\mu_k|_{\mathcal{M}(\overline{D})} \). Since \( z + z_k \in C \), from the definition of \( \mu_k \) and from (31), we have

\[ \langle \mu_k, z + z_k - \phi(y_k) \rangle_{\mathcal{M}(\overline{D}) \times C(\overline{D})} \leq 0. \]

By passing to the limit, we get

\[ \frac{1}{2} \rho + \langle \bar{\mu}, z - \phi(\bar{y}) \rangle_{\mathcal{M}(\overline{D}) \times C(\overline{D})} \leq 0, \]

thus \( \bar{\mu} \neq 0 \).

References


