OPTIMAL CONTROL PROBLEMS
WITH PARTIALLY POLYHEDRIC CONSTRAINTS

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Abstract. This paper discusses a class of state constrained optimal control problems, for which it is possible to formulate second order necessary or sufficient conditions for local optimality or quadratic growth that do not involve all curvature terms for the constraints. This kind of result is classical in the case of polyhedric control constraints. Our theory of optimization problems with partially polyhedric constraints allows to extend these results to the case when the control constraints are polyhedric, in the presence of state constraints satisfying some specific hypotheses. The analysis is based on the assumption that some strict semilinearized qualification condition is satisfied. We apply the theory to some optimal control problems of elliptic equations with state and control constraints.

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1. Introduction. This paper discusses a class of optimal control problems that have local control constraints and a finite number of state constraints. The problem was considered recently in [15], where second order necessary optimality conditions were obtained. The aim of this paper is to generalize this type of result to more general optimal control problems that have two types of constraints, the first of them being polyhedric. We also prove that this type of second order conditions allow to state second order sufficient conditions, andin fact that they allow to characterize quadratic growth. Our basic tools are the second order necessary conditions based on second-order tangent sets and polyhedricity theory.

The approach of second order necessary conditions based on second-order tangent sets was renewed in the paper [22], where the computation of the contribution of the curvature of the feasible set to second order necessary conditions, was done in the case of nonnegative continuous functions of time. This approach was extended in [16, 20, 21, 31], for abstract optimization problems, and more recently for optimal control problems in [29, 30].

Polyhedricity theory for convex sets is a classical tool for obtaining formulas for the directional derivative of the projection over a convex set [18, 28], was applied to nonlinear control problems [36, 26], and has been linked to the recent work on sensitivity analysis for abstract optimization problems [6, 8, 10].

The paper is organized as follows. Section 2 presents a theory of second order necessary or sufficient optimality conditions for abstract optimization problems that satisfy the strict semilinearized qualification condition. In the case corresponding to an optimal control problem with polyhedric control constraints and a finite number of additional inequality constraints, the theory is complete in the sense that there is no gap between the necessary and sufficient conditions. More precisely, we obtain a characterization of the quadratic growth condition.

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In section 3, assuming a weak second order sufficient condition, and the strict semilinearized qualification condition, we provide a formula for computing the directional derivative of the optimal control (as well as a second order expansion of the value function) with respect to a perturbation.

The last section discusses the application of the previous results to some optimal control problems of elliptic equations. We consider the case of nonnegative control subject to a finite number of state constraints.

Notations Let (P) be an optimization problem. By $F(P)$, $\varepsilon\text{-}S(P)$ and $\mathrm{val}(P)$, we denote the feasible set, set of ε solutions and value of problem (P), respectively.

2. Second order abstract optimality conditions. In this section we discuss the theory of second order optimality conditions for optimization problems of the following type:

$$(AP) \qquad\qquad \mathrm{Min}_x f(x); \ x \in K_X; \ G(x) \in K_Y.$$

Here X and Y are Banach spaces, K_X and K_Y are closed convex subsets of X and Y, respectively, and f and G are twice continuously differentiable mappings from X into $I\!R$ and Y. We remind that, if $\mathcal{K}$ is a convex subset of a Banach space $\mathcal{X}$, and $x \in \mathcal{K}$, then the tangent and normal cones, $T_\mathcal{K}$ and $N_\mathcal{K}$, and the cone of feasible directions $\mathcal{R}_\mathcal{K}$, are defined as

$$\begin{aligned}
T_\mathcal{K}(x) &:= \{y \in \mathcal{X}; \ \exists\, x(\sigma) = x + \sigma y + o(\sigma) \in \mathcal{K}, \ \sigma \geq 0\}, \\
N_\mathcal{K}(x) &:= \{x^* \in \mathcal{X}^*; \ \langle x^*, y \rangle \leq 0, \forall y \in T_\mathcal{K}(x)\}, \\
\mathcal{R}_\mathcal{K}(x) &:= \{y \in \mathcal{X}; \ \exists\, t > 0; \ x + ty \in \mathcal{K}\},
\end{aligned}$$

with the convention that these sets are empty if $x \notin \mathcal{K}$. An interesting case is when K_X is *polyhedric* in the following sense [28, 18].

DEFINITION 2.1. *Let $x_0 \in K_X$ and $x^* \in N_{K_X}(x_0)$. We say that K_X is polyhedric at x_0 for the direction x^* if*

$$(2.1) \qquad\qquad T_{K_X}(x_0) \cap (x^*)^\perp = \overline{\mathcal{R}_{K_X}(x_0) \cap (x^*)^\perp}.$$

If K_X is polyhedric at each $x_0 \in K_X$ for all $x^* \in N_{K_X}(x_0)$, we say that K_X is polyhedric.

By setting

$$\mathcal{K} := K_X \times K_Y, \quad \mathcal{Y} := X \times Y, \quad \mathcal{G}(x) := (x, G(x)),$$

we can write the abstract optimization problem (AP) under the form

$$(AP2) \qquad\qquad \mathrm{Min}_x f(x) \ \text{s.t.} \ \mathcal{G}(x) \in \mathcal{K},$$

with $\mathcal{G}(x)$, twice continuously differentiable mapping from X into $\mathcal{Y}$, and $\mathcal{K}$ closed convex subset of $\mathcal{Y}$. We will use several times the relationship between the two formats, in order to use the results that were derived for problem $(AP2)$. For instance, the standard constraint qualification condition for $x_0 \in F(AP2)$, due to Robinson [32], is as follows:

$$(2.2) \qquad\qquad 0 \in \ \mathrm{int}\{D\mathcal{G}(x_0)X - (\mathcal{K} - \mathcal{G}(x_0))\}.$$

LEMMA 2.2. (*Robinson [32]*). *Let $x_0 \in F(AP2)$ satisfy (2.2). Then the following metric regularity property holds. There exists $\varepsilon > 0$ and $\alpha > 0$ such that, for all $x \in B(x_0, \varepsilon)$ there exists $\hat{x} \in \mathcal{G}^{-1}(\mathcal{K})$ satisfying*

$$\|\hat{x} - x\| \leq \alpha \mathrm{dist}(\mathcal{G}(x), \mathcal{K}).$$

It is easy to show (e.g. [10]) that the qualification condition for a problem of the form (AP) (after it has been put under the form $(AP2)$) is equivalent to

$$(2.3) \qquad 0 \in \mathrm{int}\{DG(x_0)(K_X - x_0) - (K_Y - G(x_0))\}.$$

The *critical cone* at $x_0 \in F(AP)$ is defined as the set of directions of non increase of the cost function that are tangent to the feasible set. More precisely,

$$C(x_0) := \{h \in T_{K_X}(x_0); Df(x_0)h \leq 0; \ DG(x_0)h \in T_{K_Y}[G(x_0)]\}.$$

The Lagrangian function and the set of Lagrange multipliers are defined as

$$\mathcal{L}(x, \lambda) := f(x) + \langle \lambda, G(x) \rangle,$$
$$\Lambda(x) := \{(q, \lambda) \in N_{K_X}(x) \times N_{K_Y}[G(x)]; \ D_x\mathcal{L}(x, \lambda) + q = 0\}.$$

LEMMA 2.3. (*Zowe-Kurcyusz [37]*). *Let x_0 be a local solution of (AP) satisfying the qualification hypothesis (2.3). Then with x_0 is associated a non empty and bounded set of Lagrange multipliers.*

It is convenient to use the following well-known characterization of the critical cone.

LEMMA 2.4. *Let $\Lambda(x) \neq \emptyset$, say contains (q, λ). Then $Df(x)h = 0$ whenever $h \in C(x)$, and*

$$(2.4) \qquad C(x) = \{h \in T_{K_X}(x) \cap q^\perp; \ DG(x)h \in T_{K_Y}[G(x)] \cap \lambda^\perp\}.$$

Proof. Let $h \in X$ be tangent to the feasible set of (AP), in the sense that $h \in T_{K_X}(x)$ and $DG(x)h \in T_{K_Y}[G(x)]$. By definition of $\Lambda(x)$, we have

$$0 = \langle D_x\mathcal{L}(x, \lambda) + q, h \rangle = Df(x)h + \langle \lambda, DG(x)h \rangle + \langle q, h \rangle.$$

Since the last two terms are nonpositive, we have $Df(x)h \geq 0$, and $Df(x)h \leq 0$ iff the last two terms are zero. The result follows. $\square$

Let $x \in F(AP)$. Using the above lemma, we may view the critical cone as a linearization of the following set

$$(2.5) \qquad A(q, \lambda) := \{h \in (K_X - x) \cap q^\perp; \ DG(x)h \in T_{K_Y}[G(x)] \cap \lambda^\perp\}.$$

Note that in this expression we chose to "linearize" the constraint $G(x) \in K_Y$, but not the relation $x \in K_X$. The set $A(q, \lambda)$ is the inverse image, through the linear continuous mapping $h \to (h, DG(x)h)$, of the closed convex set

$$\left((K_X - x) \cap q^\perp\right) \times \left(T_{K_Y}[G(x)] \cap \lambda^\perp\right).$$

We will use the associated qualification condition, that we will call the *strict semilinearized qualification condition* (we justify this terminology below). From the above

discussion, it follows that the expression of the strict semilinearized qualification condition is

$$(CQA) \qquad 0 \in \text{ int} \left\{ DG(x) \left((K_X - x) \cap q^\perp \right) - T_{K_Y}[G(x)] \cap \lambda^\perp \right\}.$$

We may compare this condition to the more classical *strict qualification condition*, introduced in [34] (see also [4]), whose expression, for problem (AP), is

$$(2.6) \qquad 0 \in \text{ int} \left\{ DG(x) \left((K_X - x) \cap q^\perp \right) - (K_Y - G(x)) \cap \lambda^\perp \right\}.$$

LEMMA 2.5. (i) *Condition* (2.6) *implies* (CQA), *and both conditions are equivalent if* K_Y *is a polyhedron in the Banach space* Y.
(ii) *Assume that the standard constraint qualification* (2.3) *holds. Then condition* (CQA) *implies existence and uniqueness of the Lagrange multiplier.*

Proof. (i) Since $(K_Y - G(x)) \subset T_{K_Y}(G(x))$, we obviously have that (2.6) implies (CQA). Assume now that K_Y is a polyhedron in Y, and that (CQA) holds. Since $\mathbb{R}_+(K_Y - G(x)) = \mathcal{R}_{K_Y}(x)$ is equal to $T_{K_Y}(G(x))$, conditions (CQA) and (2.6) are obviously equivalent.
(ii) It is known that the strict qualification condition (2.6) implies existence and uniqueness of the Lagrange multiplier, see [34]. Since (CQA) is nothing but the strict qualification condition after linearization of the second constraint (that leaves invariant the set of Lagrange multipliers), we obtain that the set of Lagrange multipliers, that by (2.2) is non empty, is in fact a singleton. □

It is possible to express a second order necessary optimality condition for problem (AP), using the result of [16], in term of the *second order tangent set* to $K_Y \subset Y$, at $y \in K_Y$ in direction $z \in T_{K_Y}(y)$, that is defined as

$$T_{K_Y}^2(y, z) := \{ w \in Y; \ y + tz + \frac{t^2}{2} w + o(t^2) \in K_Y, t \geq 0 \}.$$

Let x_0 be a local minimum of (AP) satisfying (2.3). Set

$$\mathcal{T}(h) := T_{K_Y}^2[G(x_0), DG(x_0)h].$$

In all the sequel, we shall use the following definiton of support function.
DEFINITION 2.6. *Let* $\mathcal{K}$ *a subset of a Banach space* $\mathcal{X}$, *and let* x^* *be in* $\mathcal{X}^*$. *The support function of* $\mathcal{K}$ *at* x^* *is* $\sigma(x^*, \mathcal{K}) := sup\{\langle x^*, x \rangle \ : \ x \in \mathcal{K}\}$.

The following theorem is obtained by combining the result of [16] with some polyhedricity properties. Note that, if $\mathcal{T}(h) = \emptyset$, then $\sigma(\cdot, \mathcal{T}(h))$ is identically equal to $-\infty$; therefore, in that case, the conclusion of case (i) is trivially satisfied.
THEOREM 2.7. *Assume that* K_X *is polyhedric. Let* x_0 *be a local minimum of* (AP) *satisfying* (2.3) *and the strict semilinearized qualification condition* (CQA). *Then*
(i) $C(x_0) \cap \mathcal{R}_{K_X}(x_0)$ *is a dense subset of* $C(x_0)$, *and each* $h \in C(x_0) \cap \mathcal{R}_{K_X}(x_0)$ *satisfies*

$$(2.7) \qquad D_{x^2}^2 \mathcal{L}(x_0, \lambda)(h, h) - \sigma(\lambda, \mathcal{T}(h)) \geq 0,$$

where λ *is the Lagrange multiplier.*
(ii) *If in addition* $h \to \sigma(\lambda, \mathcal{T}(h))$ *is lower semi continuous over* $C(x_0)$, *then* (2.7) *holds for all critical direction* h.
(iii) *If* K_Y *is a polyhedron, then for all critical direction* h *we have*

$$(2.8) \qquad D_{x^2}^2 \mathcal{L}(x_0, \lambda)(h, h) \geq 0.$$

Proof. (i) Step a. We claim that $C(x_0) \cap \mathcal{R}_{K_X}(x_0)$ is a dense subset of $C(x_0)$. Let $h \in C(x_0)$, and fix $\varepsilon > 0$. Since K_X is polyhedric, there exists $\hat{h}_\varepsilon \in \mathcal{R}_{K_X}(x_0) \cap q^\perp$ such that $\|\hat{h}_\varepsilon - h\| \leq \varepsilon$. Let $t_\varepsilon > 0$ be such that $x_0 + t_\varepsilon \hat{h}_\varepsilon \in K_X$. We use the metric regularity property that, by lemma 2.2, follows from (CQA):

> There exists $\gamma > 0$ and $\alpha > 0$ such that, if $\hat{w} \in X$, and $\|\hat{w}\| \leq \gamma$,
>
> then there exists $w \in (K_X - x_0) \cap q^\perp$ such that
>
> $DG(x_0)w \in T_{K_Y}[G(x_0)] \cap \lambda^\perp$, and
>
> $\|w - \hat{w}\| \leq \alpha[\text{dist}(\hat{w}, (K_X - x_0) \cap q^\perp) + \text{dist}(DG(x_0)\hat{w}, T_{K_Y}[G(x_0)] \cap \lambda^\perp)]$.

Reducing t_ε if necessary, we have that $\hat{w}_\varepsilon := t_\varepsilon \hat{h}_\varepsilon$ is such that $\|\hat{w}_\varepsilon\| \leq \gamma$ and $\hat{w}_\varepsilon \in (K_X - x_0) \cap q^\perp$. Since

$$\text{dist}(DG(x_0)\hat{w}_\varepsilon, T_{K_Y}[G(x_0)] \cap \lambda^\perp) = O(\varepsilon t_\varepsilon),$$

it follows that there exists $w_\varepsilon \in (K_X - x_0) \cap q^\perp$ such that

$$DG(x)w_\varepsilon \in T_{K_Y}[G(x_0)] \cap \lambda^\perp,$$
$$\|w_\varepsilon - \hat{w}_\varepsilon\| = \alpha \, \text{dist}(DG(x_0)\hat{w}_\varepsilon, T_{K_Y}[G(x_0)] \cap \lambda^\perp) = O(t_\varepsilon \varepsilon).$$

Set $h_\varepsilon := t_\varepsilon^{-1} w_\varepsilon$. Then $h_\varepsilon \in C(x_0) \cap \mathcal{R}_{K_X}(x_0)$, and $\|h_\varepsilon - h\| = O(\varepsilon)$. This proves our claim.

Step b. Since x_0 is a qualified local solution of (AP), and the Lagrange multiplier is unique, by [16, Thm 4.2], the following second order necessary condition holds: for any critical direction h, we have

(2.9) $$D_{x^2}^2 \mathcal{L}(x_0, \lambda)(h, h) - \sigma(q, T_{K_X}^2(x_0, h)) - \sigma(\lambda, \mathcal{T}(h)) \geq 0.$$

Note that we have used here the fact that the support function of a set in product form is the sum of the corresponding support function.

On the other hand, since $q \in N_{K_X}(x_0)$, we have $\sigma(q, T_{K_X}^2(x_0, h)) \leq 0$ [16, Section 4]. If $h \in \mathcal{R}_{K_X}(x_0)$, then $0 \in T_{K_X}^2(x_0, h)$, so that $\sigma(q, T_{K_X}^2(x_0, h)) = 0$. Point (i) follows.

(ii) Let $h \in C(x_0)$. By (i) there exists $h_k \to h$, $h_k \in C(x_0) \cap \mathcal{R}_{K_X}(x_0)$, and

$$D_{x^2}^2 \mathcal{L}(x_0, \lambda)(h_k, h_k) \geq \sigma(\lambda, \mathcal{T}(h_k)).$$

Since the right hand side is l.s.c. by hypothesis, and $D_{x^2}^2 \mathcal{L}(x_0, \lambda)(\cdot, \cdot)$ is a continuous function, we may pass to the limit in this inequality. Point (ii) follows.

(iii) If K_Y is a polyhedron, then it is well known that $0 \in \mathcal{T}(h)$ (e.g. [10]), whence $\sigma(\lambda, \mathcal{T}(h)) = 0$, for all critical direction h. The result follows then from (ii). □

In order to formulate second order sufficient conditions, we need the following concept.

DEFINITION 2.8. (*See e.g. [19]*). *We say that a quadratic form Q on a Hilbert space X is a Legendre form if Q is weakly l.s.c. and, whenever a sequence $\{x_k\} \subset X$ satisfies $x_k \xrightarrow{w} x$ and $Q(x_k) \to Q(x)$, then $x_k \to x$.*

The function $x \to \|x\|^2$ is the simplest example of a Legendre form. More generally, if $N > 0$ and Q is a weakly continuous quadratic form, it is easy to check that $x \to N\|x\|^2 + Q(x)$ is a Legendre form, see e.g. [11].

DEFINITION 2.9. *We say that x_0 is a local solution of (AP) satisfying the* quadratic growth condition *if*

$$(2.10) \quad \exists \alpha > 0; \quad f(x) \geq f(x_0) + \alpha \|x - x_0\|^2 + o(\|x - x_0\|^2), \quad \forall x \in F(AP),$$

where $F(AP)$ is the feasible set of the problem (AP).

THEOREM 2.10. *Assume that K_X is a polyhedric subset of the Hilbert space X. Let x_0 be a qualified local minimum of (AP) satisfying the strict semilinearized qualification condition (CQA), and let (q_0, λ_0) be the unique associated Lagrange multiplier. If $Q_0(h) := D^2_{x^2}\mathcal{L}(x_0, \lambda_0)(h, h)$ is a Legendre form, and K_Y is a polyhedron, then the following condition is necessary and sufficient for quadratic growth,*

$$(2.11) \qquad\qquad D^2_{x^2}\mathcal{L}(x_0, \lambda_0)(h, h) > 0, \quad \text{for all } h \in C(x_0)\backslash\{0\}.$$

Proof. Let x_0 satisfy the quadratic growth condition. Then there exists $\alpha > 0$ such that x_0 is a local solution of the problem

$$(AP_\alpha) \qquad\qquad \text{Min}_x f(x) - \frac{\alpha}{2}\|x - x_0\|^2; \; x \in K_X; \; G(x) \in K_Y.$$

Since K_Y is a polyhedron, and therefore $\sigma(\lambda, \mathcal{T}(h)) = 0$, (2.11) follows from theorem 2.7.

Conversely, assume that (2.11) holds, while the quadratic growth condition is not satisfied. Then there exists $x_k \to x_0$ such that

$$(2.12) \qquad\qquad f(x_k) \leq f(x_0) + o(\|x_k - x_0\|^2).$$

Set $t_k := \|x_k - x_0\|$. Extracting a subsequence if necessary, we may assume that $x_k = x_0 + t_k h_k$, $\|h_k\| = 1$, and $h_k \xrightarrow{w} \bar{h}$. Also $\bar{h} \in T_{K_X}(x_0)$ since $h_k \in \mathcal{R}_{K_X}(x_0)$, (2.12) implies that $Df(x_0)\bar{h} \leq 0$, and from $G(x_k) \in K_Y$ we deduce that $DG(x_0)\bar{h} \in T_{K_Y}(G(x_0))$. It follows that $\bar{h}$ is a critical direction.

By the first order optimality condition we have
$$\langle q_0, x_k - x_0 \rangle \leq 0 \text{ and } \langle \lambda_0, G(x_k) - G(x_0) \rangle \leq 0.$$
Combining with $D_x\mathcal{L}(x_0, \lambda_0) + q_0 = 0$, we deduce that

$$f(x_k) - f(x_0) \geq \mathcal{L}(x_k, \lambda_0) - \mathcal{L}(x_0, \lambda_0) + \langle q_0, x_k - x_0 \rangle,$$
$$= \frac{t_k^2}{2}Q_0(h_k) + o(t_k^2).$$

Combining with (2.12), it follows that $Q_0(h_k) \leq o(1)$. Since $Q_0(\cdot)$ is l.s.c., we have $Q_0(\bar{h}) \leq 0$. Since $\bar{h}$ is critical, this with (2.11) imply that $\bar{h} = 0$. It follows that $Q_0(h_k) \to Q_0(\bar{h})$. Due to $\|h_k\| = 1$ and $\bar{h} = 0$, this contradicts the fact that $Q_0(h_k)$ is a Legendre form. □

3. Abstract sensitivity analysis. This section is devoted to the study of the family of perturbed optimization problems

$$(AP_u) \qquad\qquad \text{Min}_x f(x, u) \text{ s.t. } x \in K_X; \; G(x, u) \in K_Y.$$

Here u belongs to a Banach space U, K_X is a closed convex subset of the Hilbert space X, K_Y is a polyhedron included in the finite dimensional space Y, so that (CQA) is equivalent to the strict qualification condition (2.6), f and G are twice continuously

differentiable mappings from $X \times U$ into $I\!R$ and Y. The Lagrangian of this problem is

$$\mathcal{L}(x, \lambda, u) := f(x, u) + \langle \lambda, G(x, u) \rangle.$$

We perform a sensitivity analysis along a path of perturbation variables of the form

$$u(t) := u_0 + tu_1 + \frac{t^2}{2} u_2 + o(t^2) \quad \text{with } u_i(t) \in U, i = 0, 1, 2.$$

Let x_0 be a local solution of (AP_{u_0}). The following problems may be interpreted as the linearization and the second order expansion of problem (AP_u) at (x_0, u_0) along the path $u(t)$, respectively:

$$(LP) \qquad \begin{aligned} &\text{Min}_{h \in X} Df(x_0, u_0)(h, u_1) \text{ s.t. } h \in T_{K_X}(x_0); \\ &DG(x_0, u_0)(h, u_1) \in T_{K_Y}[G(x_0, u_0)], \end{aligned}$$

and, (q_0, λ_0) being the Lagrange multiplier associated with x_0:

$$(SP) \qquad \text{Min}_{h \in S(LP)} D_u \mathcal{L}(x_0, \lambda_0, u_0) u_2 + D^2_{(x,u)^2} \mathcal{L}(x_0, \lambda_0, u_0)((h, u_1), (h, u_1));$$

LEMMA 3.1. *Let x_0 satisfy (CQA). Then* (i) *$S(LP)$ is non empty, and*

$$(3.13) \qquad \text{val } (LP) = D_u \mathcal{L}(x_0, \lambda_0, u_0) u_1,$$

where (q_0, λ_0) is the unique Lagrange multiplier associated with x_0, and
(ii) *The set $S(LP) \cap \mathcal{R}_{K_X}(x_0)$ is a dense subset of $S(LP)$.*
   Proof. The dual, in the sense of convex analysis, to the linearized problem (LP), is known to be (e.g. [8])

$$(LD) \qquad \text{Max}_{(q,\lambda)} D_u \mathcal{L}(x_0, \lambda, u_0) u_1; \quad (q, \lambda) \in \Lambda(x_0).$$

By lemma 2.5, we know that there exists a unique Lagrange multiplier (q_0, λ_0), and that the primal and dual values are equal. This proves (3.13). It follows that $h \in X$ is solution of (LP) iff $h \in F(LP)$ and the complementarity conditions

$$\langle q_0, h \rangle = \langle \lambda_0, DG(x_0, u_0)(h, u_1) \rangle = 0$$

are satisfied. In other words, $h \in S(LP)$ iff

$$h \in T_{K_X}(x_0) \cap (q_0)^\perp; \quad DG(x_0, u_0)(h, u_1) \in T_{K_Y}[G(x_0, u_0)] \cap (\lambda_0)^\perp.$$

By (CQA) the set of such h is not empty, hence $S(LP)$ is not empty.
(ii) This is a consequence of theorem 2.7(i) applied to problem (LP), once we have checked that problem (LP) itself satisfies the strict semilinearized qualification condition. The expression of the latter (for problem (LP)) is

$$0 \in \text{int} \left\{ DG(x)[(T_{K_X(x_0)} - h) \cap q_0^\perp] - (T_{T_{K_Y}[G(x_0)]} DG(x_0)h) \cap \lambda_0^\perp \right\}.$$

Since $(K_X - x_0) \subset T_{K_X(x_0)} - h$ and $T_{K_Y}[G(x_0)] \subset T_{T_{K_Y}[G(x_0)]} DG(x_0)h$, this is an obvious consequence of (CQA). □
   THEOREM 3.2. *Assume that*
(i) *For small enough $t > 0$, there exists $x(t)$, $o(t^2)$-solution of $(AP_{u(t)})$, such that*

$x(t) \to x_0$.

(ii) *The point x_0 is the unique solution of (AP_{u_0}), and satisfies (CQA) and the second order sufficient optimality condition (2.11),*

(iii) *The Hessian $Q_0(h) := D^2_{x^2}\mathcal{L}(x_0, \lambda_0, u_0)(h, h)$ is a Legendre form over the Hilbert space X,*

Then

(a) *The following expansion for the value function of $(AP_{u(t)})$ holds:*

$$(3.14) \qquad \mathrm{val}\,(AP_{u(t)}) = \mathrm{val}\,(AP_{u_0}) + t\mathrm{val}\,(LP) + \frac{t^2}{2}\mathrm{val}\,(SP) + o(t^2),$$

(b) *One has $x(t) = x_0 + O(t)$. Any weak limit-point of $t^{-1}(x(t) - x_0)$ is a strong limit-point, and is solution of (SP). In particular, if (SP) has a unique solution $\bar{h}$, then $x(t) = x_0 + t\bar{h} + o(t)$.*

Proof. Let $Q_{u_1}(h) := D^2_{x^2}\mathcal{L}(x_0, \lambda_0)((h, u_1), (h, u_1))$. (Note that this notation is coherent with the definition of $Q_0(\cdot)$ given before.) Consider the subproblem

$$(SP_\sigma) \qquad \mathrm{Min}_{h \in S(LP)} D_u \mathcal{L}(x_0, \lambda, u_0)u_2 + Q_{u_1}(h) - \sigma(q_0, T^2_{K_X}(x_0, h)).$$

Since K_Y is a polyhedron, we have $\sigma(\lambda_0, T^2_{K_Y}[G(x_0, u_0), DG(x_0, u_0)(h, u_1)]) = 0$. It follows from [8, Prop. 2.1] that

$$\mathrm{val}(AP_{u(t)}) \leq \mathrm{val}(AP_{u_0}) + t\mathrm{val}(LP) + \frac{t^2}{2}\mathrm{val}(SP_\sigma) + o(t^2),$$

while [8, Prop. 4.3] imply that the right-hand-side of (3.14) is a lower estimate of $\mathrm{val}(AP_{u(t)})$. We now prove (3.14) by checking that $\mathrm{val}(SP) \geq \mathrm{val}(SP_\sigma)$. By lemma 3.1, the set $S(LP) \cap \mathcal{R}_{K_X}(x_0)$ is a dense subset of $S(LP)$. Also on $S(LP) \cap \mathcal{R}_{K_X}(x_0)$ the cost functions of (SP) and (SP_σ) coincide. Since $\sigma(q_0, T^2_{K_X}(x_0, h)) \leq 0$, it follows that

$$\begin{aligned}
\mathrm{val}(SP) &= \inf_{h \in S(LP) \cap \mathcal{R}_{K_X}(x_0)} \{D_u\mathcal{L}(x_0, \lambda_0, u_0)u_2 + Q_{u_1}(h)\}, \\
&= \inf_{h \in S(LP) \cap \mathcal{R}_{K_X}(x_0)} \{D_u\mathcal{L}(x_0, \lambda_0, u_0)u_2 + Q_{u_1}(h) - \sigma(q_0, T^2_{K_X}(x_0, h))\}, \\
&\geq \mathrm{val}(SP_\sigma),
\end{aligned}$$

as was to be proved.

(b) By [8, Prop 5.3], we have $x(t) = x_0 + O(t)$. Let us prove that any weak limit-point of $t^{-1}(x(t) - x_0)$ is a strong limit-point. Let $t_k \to 0^+$, $x_k := x(t_k)$, and $h_k := t_k^{-1}(x_k - x_0)$ be such that $h_k \xrightarrow{w} \bar{h}$. By [8, Prop 5.3], we know that

$$Q_{u_1}(h_k) \to Q_{u_1}(\bar{h}).$$

Since $Q_0(\cdot)$ is a Legendre form, we have $h_k \to \bar{h}$, as was to be proved. Finally if (SP) has a unique solution $\bar{h}$, it follows that $t^{-1}(x(t) - x_0)$ converges to $\bar{h}$. The conclusion follows. $\square$

4. Application to state constrained optimal control problems.

4.1. General results. In this section we apply the results of the previous sections to some optimal control problems for semilinear elliptic equations. In the sequel of this paper, we denote by Ω a bounded open subset of $I\!\!R^n$ ($n \leq 3$) with Lipschitz boundary Γ. Given a function $u \in L^2(\Omega)$, (we take in this section the standard notations for optimal control problems) we consider the following boundary value problem:

$$(4.15) \qquad -\Delta y + \phi(x, y) = u \ \text{ in } \Omega, \qquad y(x) = 0 \ \text{ on } \Gamma,$$

where $\phi : \overline{\Omega} \times I\!\!R \longrightarrow I\!\!R$ is a continuous function which is of class C^2, and such that $\phi'_y(x, \cdot) \geq 0$, for all $x \in \Omega$.

From now on, the weak solution of (4.15) associated with u will be denoted y_u. Under the above assumption, we can prove the existence and uniqueness of a solution of (4.15).

THEOREM 4.1. *For every $u \in L^2(\Omega)$, equation (4.15) admits a unique weak solution y_u in $H_0^1(\Omega) \cap C(\overline{\Omega})$, this solution is Hölder continuous and we have*

$$\|y_u\|_{C(\overline{\Omega})} \leq C_1(1 + \|u\|_{L^2(\Omega)}),$$

where $C_1 = C_1(\Omega)$ is independent of u. Moreover, if we denote by $\mathcal{A} : L^2(\Omega) \longrightarrow C(\overline{\Omega})$ the mapping which associates with every control u the weak solution y_u of (4.15), then $\mathcal{A}$ is twice continuously Fréchet differentiable, and for every $u, h \in L^2(\Omega)$, if we denote $y_u = \mathcal{A}(u)$ and $z_h = \mathcal{A}'(u)h$, then z_h is the weak solution of

$$(4.16) \qquad -\Delta z_h + \phi'_y(x, y_u)z_h = h \quad \text{in } \Omega, \qquad z_h = 0 \quad \text{on } \Gamma.$$

Proof. The above theorem is a collection of known results for semilinear elliptic equations (see [7, 6, 15] and the general references [1, 2, 5]). □

Consider the following control constraints:

$$L_+^2(\Omega) := \{u \in L^2(\Omega) \mid u(x) \geq 0 \ \text{ a.e. on } \ x \in \Omega\}.$$

Let us also consider a family of functions G_j of class $C^2 : L^2(\Omega) \to I\!\!R$, for $1 \leq j \leq m$. We consider the following optimal control problem:

$$(\mathcal{P}) \qquad \text{Min}\{F(u) \mid u \in L_+^2(\Omega), \ G_j(u) \leq 0 \text{ for } 1 \leq j \leq m\},$$

where

$$(4.17) \qquad F(u) := \frac{1}{2} \int_\Omega (y_u(x) - y_d(x))^2 \, dx + \frac{N}{2} \int_\Omega u(x)^2 \, dx,$$

with y_d is a given function in $L^2(\Omega)$, and $N > 0$. The adjoint state p_u^0 associated with u is defined as the unique solution in $H^2(\Omega)$ of the system

$$-\Delta p_u^0 + \phi'_y(x, y_u)p_u^0 = y_u - y_d \ \text{ in } \Omega, \quad p_u^0 = 0 \ \text{ on } \Gamma.$$

It is known that $u \to F(u)$ is a C^2 mapping with derivative

$$DF(u) = Nu + p_u^0.$$

We will detail later the cases when $G_j(u)$ are some punctual or integral functions of the state.

Let $\bar{u}$ be an optimal solution of problem $(\mathcal{P})$. Set

$$J_+ = \{j \in \{1, \cdots, m\} \mid G_j(\bar{u}) < 0\},$$
$$J_0 = \{j \in \{1, \cdots, m\} \mid G_j(\bar{u}) = 0\},$$
$$J_- = \{j \in \{1, \cdots, m\} \mid G_j(\bar{u}) = 0, \ \bar{\lambda}_j > 0\}.$$

Then $J_0 \cup J_- \cup J_+ = \{1, \cdots, m\}$. Problem $(\mathcal{P})$ can be written as follows

$$\text{Min}\{F(u) \mid u \in L_+^2(\Omega), G(u) \in I\!\!R_-^m\}.$$

In addition (see e.g. [10, 3]) Robinson's constraint qualification assumption is equivalent to

(4.18) $\exists v \in L_+^2(\Omega), \ \ G_j(\bar{u}) + DG_j(\bar{u})(v - \bar{u}) < 0.$

Therefore we obtain the following (classical) expression of the first order optimality system.

THEOREM 4.2. *Assume that $\bar{u}$ is a local solution of (P) satisfying (4.18). Denote by $\bar{y}$ and $\bar{p}$ the state and adjoint state associated with $\bar{u}$. Then there exist Lagrange multipliers $(\bar{q}, \bar{\lambda}) \in L^2(\Omega) \times I\!\!R^m$ such that:*

(4.19) $\bar{\lambda}_j \geq 0, \ 1 \leq j \leq m, \ and \ \bar{\lambda}_j = 0 \ if \ G_j(\bar{u}) < 0,$

(4.20) $N\bar{u} + \bar{p} + \sum_{j=1}^m \bar{\lambda}_j DG_j(\bar{u}) + \bar{q} = 0; \quad \langle \bar{q}, u - \bar{u} \rangle \leq 0, \ \ \forall u \in L_+^2(\Omega).$

Since $L_+^2(\Omega)$ is polyhedric in $L^2(\Omega)$ (see e.g. [18, 28]), $(\mathcal{P})$ is of the form (AP), with
$$X = L^2(\Omega), \ Y = I\!\!R^m, \ K_X = L_+^2(\Omega), \ K_Y = I\!\!R_-^m.$$
We now discuss the strict semilinearized qualification condition (CQA). We need a notation for the contact set of $\bar{u}$ and its complement (defined up to a null measure set):

$$\Omega_-(\bar{u}) := \{x \in \Omega; \ \bar{q}(x) < 0\}, \qquad \Omega_0(\bar{u}) := \{x \in \Omega; \ \bar{u}(x) = 0\},$$

$$\Omega_+(\bar{u}) := \{x \in \Omega; \ \bar{u}(x) > 0\}.$$

Since $\Omega_-(\bar{u}) \subset \Omega_0(\bar{u})$, we have:

$$T_{L_+^2(\Omega)}(\bar{u}) \cap \bar{q}^\perp = \left\{h \in L^2(\Omega); \ h \geq 0 \ \text{on} \ \Omega_0(\bar{u}); \ h = 0 \ \text{on} \ \Omega_-(\bar{u})\right\}.$$

Therefore, the strict qualification condition is identical to the qualification condition of the problem

$$\text{Min}\{F(u) \mid u \in L^2(\Omega), \ u = 0 \ \text{on} \Omega_-(\bar{u}), \ G_j(u) = 0 \ \text{for} \ j \in J_-\}.$$

LEMMA 4.3. *Let $\bar{u} \in F(\mathcal{P})$, with associated Lagrange multiplier $(\bar{q}, \bar{\lambda})$. Then the three conditions below are equivalent:*
(i) *The strict semilinearized qualification condition (CQA) is satisfied.*
(ii) *The following conditions hold:*

(4.21) $\begin{cases} \text{(i)} \ \{DG_i(\bar{u})h; \ i \in J_-; \ h \in (T_{L_+^2(\Omega)}(\bar{u}) \cap \bar{q}^\perp)\} \ is \ onto, \\ \text{(ii)} \ \exists h \in (\mathcal{R}_{L_+^2(\Omega)}(\bar{u}) \cap \bar{q}^\perp); \ DG_i(\bar{u})h = 0, i \in J_-; \ DG_i(\bar{u})h < 0, i \in J_0 \backslash J_-. \end{cases}$

(iii) *There exists no $(\tilde{q}, \tilde{\lambda}) \in L^2(\Omega) \times \mathbb{R}^m$, with $\tilde{\lambda} \neq 0$, satisfying the following relations:*

(4.22)
$$\begin{cases} \text{(i)} \ \tilde{\lambda}_i = 0, i \in J_+; \ \tilde{\lambda}_i \geq 0, i \in J_0 \backslash J_-, \\ \text{(ii)} \ \tilde{q}(x) = 0 \quad on \ \Omega_+(\bar{u}), \quad \tilde{q}(x) \leq 0 \quad on \ \Omega_0(\bar{u}) \backslash \Omega_-(\bar{u}); \\ \text{(iii)} \ \tilde{q} + \sum_{1 \leq i \leq m} \tilde{\lambda}_i DG_i(\bar{u}) = 0. \end{cases}$$

Proof. By the definition, (CQA) holds iff, for any $z \in \mathbb{R}^m$, close enough to 0, there exists $h \in \left(L^2_+(\Omega) - \bar{u}\right) \cap \bar{q}^\perp$ satisfying the following relations:

(4.23)   (i) $DG_i(\bar{u})h = z_i, \ i \in J_-$;   (ii) $DG_i(\bar{u})h \leq z_i, \ i \in J_0 \backslash J_-$.

It follows from (4.23(i)) that the set

$$\{DG_i(\bar{u})h; \ i \in J_-; \ h \in (\mathcal{R}_{L^2_+(\Omega)}(\bar{u}) \cap \bar{q}^\perp)\} \text{ is onto.}$$

This implies that (4.21(i)) is a necessary condition for (CQA). Then taking $z_i = 0$, $i \in J_-$, and $z_i < 0$, $i \in J_0 \backslash J_-$, we deduce that (4.21) is a necessary condition for (CQA), i.e. $(CQA) \Rightarrow (4.21)$. We end the proof by showing that $(4.21) \Rightarrow (4.22) \Rightarrow (CQA)$.

Assume that (CQA) does not hold. Then the convex cone

$$E := \left\{ DG(\bar{u})h - z; \ h \in \mathcal{R}_{L^2_+(\Omega)}(\bar{u}) \cap \bar{q}^\perp; \ z_i \leq 0, i \in J_0, \ z_i = 0, i \in J_- \right\}$$

is not equal to $\mathbb{R}^m$. Since the latter is a finite dimensional space, the closure of E is not equal to $\mathbb{R}^m$. By the Hahn-Banach theorem, since E is a cone, there exists $\tilde{\lambda} \in \mathbb{R}^m$, $\tilde{\lambda} \neq 0$, such that $\langle \tilde{\lambda}, y \rangle \geq 0$, for all $y \in E$. It follows that (4.22(i)) holds, while $\tilde{q}$ defined by (4.22(iii)) is such that

(4.24) $-\langle \tilde{\lambda}, DG(\bar{u})h \rangle = \int_\Omega \tilde{q}(x)h(x)dx \leq 0, \quad \forall h \in (L^2_+(\Omega) - \bar{u}) \cap \bar{q}^\perp.$

Since the polar of the intersection of two closed convex cones is the closure of the sum of their polar cones, we have that

$$\tilde{q} \in \left(\left(L^2_+(\Omega) - \bar{u}\right) \cap \bar{q}^\perp \right)^- = \overline{(L^2_+(\Omega) - \bar{u})^- + \mathbb{R}\bar{q}}.$$

Relation (4.22(ii)) follows.

Finally, suppose that (4.21) holds, but (4.22) does not hold. Let $(\tilde{q}, \tilde{\lambda})$ satisfy (4.22). Then (4.24) holds. It follows that for each $h \in \mathcal{R}_{L^2_+(\Omega)}(\bar{u}) \cap \bar{q}^\perp$

$$0 \leq - \int_\Omega \tilde{q}(x)h(x)dx = \sum_{1 \leq i \leq m} \tilde{\lambda}_i DG_i(\bar{u})h.$$

This and (4.21(ii)) imply $\tilde{\lambda}_i = 0$, for all $i \in J_0 \backslash J_-$. Then, since $L^2_+(\Omega)$ is polyhedric, with (4.21(i)) we obtain $\tilde{\lambda}_i = 0$, for all $i \in J_-$, in contradiction with the fact that $\tilde{\lambda} \neq 0$. $\square$

Denote by $L^2(\Omega_+(\bar{u}))$ the Hilbert space of functions of $L^2(\Omega)$ that are a.e. null outside $\Omega_+(\bar{u})$. ¿From the above lemma, we deduce the following corollary, similar to [15, Thm 5.2].

COROLLARY 4.4. *A sufficient condition for* (CQA) *is that the restriction of* $DG(\bar{u})$ *to* $L^2(\Omega_+(\bar{u}))$, *with image space* $\mathbb{R}^m$, *is onto.*

We now discuss second order optimality conditions. Since $Q_0(\cdot)$ is a Legendre form, we have the following result, that is an immediate consequence of Theorem 2.10. Note that the assumption, that the Hessians $D^2G(\bar{u})$ are weakly continuous, is typically satisfied if G represents state constraints, as will be the case in the examples to be seen later. The expression of the Lagrangian for problem $(\mathcal{P})$ is

$$\mathcal{L}(u, \lambda) := F(u) + \sum_{i=1}^{m} \lambda_i G_i(u).$$

THEOREM 4.5. *Let* $\bar{u}$ *be an optimal solution of* $(\mathcal{P})$, *with associated Lagrange multiplier* $(\bar{q}, \bar{\lambda})$, *satisfy condition* (CQA). *Assume that the Hessians* $D^2G_i(\bar{u})$ *(for* $i = 1, \cdots, m$) *are weakly continuous. Then* $\bar{u}$ *satisfies the quadratic growth condition iff*

$$D_{uu}^2 \mathcal{L}(\bar{u}, \bar{\lambda})(h, h) > 0, \ \ \forall h \in C(\bar{u}), \ \ h \neq 0.$$

4.2. Problems with finitely many punctual state constraints. We consider in this subsection the case when the functions G_j, $1 \leq j \leq m$, are defined by

$$G_j(u) = y_u(x_j) - b_j.$$

Here $b \in \mathbb{R}^m$ and $x_j \in \Omega$, $1 \leq j \leq m$, are given. We denote $\bar{y} := y_{\bar{u}}$. A simple consequence of lemma 4.3 follows:

LEMMA 4.6. *Assume that* $\Omega_+(\bar{u})$ *has a non empty interior. Then the strict semilinearized qualification condition* (CQA) *is satisfied.*

Proof. If the conclusion does not hold, then by lemma 4.3, there exists $(\tilde{q}, \tilde{\lambda}) \in L^2(\Omega) \times \mathbb{R}^m$, with $\tilde{\lambda} \neq 0$, satisfying (4.22). It is a classical result (see e.g. [12]) that $\tilde{q} \in L^2(\Omega) \cap W^{1,s}(\Omega)$, for all $s < n/(n-1)$, and is the unique solution in $W^{1,1}(\Omega)$ of

$$(4.25) \qquad -\Delta\tilde{q} + \phi_y'(x, \bar{y})\tilde{q} = -\sum_{1 \leq i \leq m} \lambda_i \delta(x_i) \ \text{ in } \Omega, \quad \tilde{q} = 0 \ \text{ on } \Gamma.$$

Here $\delta(x_i)$ stands for the Dirac measure at point x_i. Since $\tilde{q} = 0$ on the interior of Ω_+, and the latter is non empty, we have by the unique extension theorem [35] that $\tilde{q} = 0$ over Ω except perhaps at the points x_j. But this implies $\tilde{\lambda} = 0$, in contradiction with the hypothesis. $\square$

We now state the characterization of quadratic growth. By z_h we denote the solution of the linearized equation (4.16) with $y_u = \bar{y}$ and r.h.s. h. As a consequence of theorem 4.5, we have:

THEOREM 4.7. *Let* $\bar{u}$ *be a feasible point of* $(\mathcal{P})$, *with associated Lagrange multiplier* $(\bar{q}, \bar{\lambda})$, *and assume that the interior of* Ω_+ *is non empty. Then* $\bar{u}$ *satisfies the quadratic growth condition iff there exists* $\bar{p} \in W^{1,s}(\Omega)$, *for all* $s < n/(n-1)$, *such that*

$$(4.26) \qquad \bar{\lambda}_j \geq 0, \ 1 \leq j \leq m, \ \text{ and } \bar{\lambda}_j = 0 \ \text{ if } G_j(\bar{u}) < 0,$$

$$(4.27) \qquad -\Delta\bar{p} + \phi_y'(x, \bar{y})\bar{p} = \bar{y} - y_d + \sum_{j=1}^{m} \bar{\lambda}_j \delta(x_j) \ \text{ in } \Omega, \quad \bar{p} = 0 \ \text{ on } \Gamma;$$

$$(4.28) \qquad \int_{\Omega} (N\bar{u}(x) + \bar{p}(x))(u(x) - \bar{u}(x)) \, dx \geq 0, \ \ \forall u \in L_+^2(\Omega),$$

and such that, for all $h \in C(\bar{u})$, $h \neq 0$, and z_h solution of (4.16) (in which $\bar{y} = y_{\bar{u}}$):

(4.29)
$$\int_\Omega (Nh(x)^2 + z_h^2(x)^2 - \bar{p}(x)\phi_y''(x,\bar{y})z_h(x)^2)\,dx > 0.$$

We now discuss sensitivity of the solution of the optimal control problem with respect to the target y_d. Therefore we denote

$$F(u, y_d) := \frac{1}{2}\int_\Omega (y_u(x) - y_d(x))^2\,dx + \frac{N}{2}\int_\Omega u(x)^2\,dx.$$

Consider a target path, where $t \geq 0$,

$$y_d(t) = y_{d0} + ty_{d1} + \frac{t^2}{2}y_{d2} + o(t^2).$$

Note that

$$D_{y_d}F(u, y_{d0})y_{d1} = -\int_\Omega (\bar{y} - y_{d0})(x)y_{d1}(x)dx.$$

The subproblems to be considered here, corresponding to (LP) and (SP), are

$(\overline{LP})$
$$\mathrm{Min}_{h \in L^2(\Omega)} \int_\Omega (N\bar{u} + \bar{p})(x)h(x)dx - \int_\Omega (\bar{y} - y_{d0})(x)y_{d1}(x)dx$$
$$\text{s.t. } h \geq 0 \text{ on } \Omega_0(\bar{u}); z_h(x_i) \leq 0, \ i \in J_0,$$

and, denoting by z_h the solution of (4.16) (in which $y_u = \bar{y}$):

$(\overline{SP})$
$$\mathrm{Min}_{h \in S(\overline{LP})}D^2F(\bar{u}, y_{d0})((h, y_{d1}),(h, y_{d1})) - \int_\Omega (\bar{y} - y_{d0})(x)y_{d2}(x)dx.$$

(An expression of the Hessian of F in term of $\bar{p}$ and z_h is given in [6].)

THEOREM 4.8. *Assume that $\bar{u}$ is the unique solution of (AP), and satisfies (CQA) as well as the second order sufficient optimality condition (4.29). Then*
(a) *The following expansion for the value function of $(AP_{u(t)})$ holds:*

(4.30)
$$\mathrm{val}\,(AP_{u(t)}) = \mathrm{val}\,(AP_{\bar{u}}) + t\mathrm{val}\,(\overline{LP}) + \frac{t^2}{2}\mathrm{val}\,(\overline{SP}) + o(t^2),$$

(b) *Let $u(t)$ be a path of $o(t^2)$-solutions. Then one has $u(t) = \bar{u} + O(t)$. Any weak limit-point of $t^{-1}(u(t) - \bar{u})$ is a strong limit-point, and is solution of $(\overline{SP})$. In particular, if $(\overline{SP})$ has a unique solution $\bar{h}$, then $u(t) = \bar{u} + t\bar{h} + o(t)$.*

Proof. It is easy to check that the solutions of the perturbed problem are uniformly bounded, and that they strongly converge in $L^2(\Omega)$ to $\bar{u}$, see e.g. [6]. In addition the Hessian of the Lagrangian, that is equal to the Hessian of the cost, is a Legendre form. Therefore the conclusion is a consequence of theorem 3.2.
□

4.3. Problems with integral state constraints. We consider in this subsection the case when the functions $G_j(u)$ $1 \leq j \leq m$, are defined by

$$G_j(u) = \int_\Omega g_j(y_u(x), x)\,dx.$$

The functions $g_j(u)$ are assumed to be twice continuously differentiable functions $I\!R \times \bar{\Omega} \to I\!R$. Then $G(\cdot)$ is itself a C^2 mapping. We know that the derivative of $u \to G_j(u)$, viewed as a function $L^2(\Omega) \to I\!R$, is $p_j(u) \in H^2(\Omega)$ solution of

$$-\Delta p_j + \phi'_y(x, y_u)p_j = D_y g_j(y_u(x), x) \ \text{ in } \Omega, \quad p_j = 0 \ \text{ on } \Gamma.$$

A simple consequence of lemma 4.3 follows:

LEMMA 4.9. *The strict qualification condition* (CQA) *is satisfied iff the following system has no solution* $(\tilde{q}, \tilde{\lambda}) \in W^{1,s}(\Omega) \times I\!R^m$:

$$\begin{aligned} &-\Delta\tilde{q} + \phi'_y(x, \bar{y})\tilde{q} = -\sum_{1 \leq i \leq m} \lambda_i D_y g_j(\bar{y}(x), x) \ \text{ in } \Omega, \quad \tilde{q} = 0 \ \text{ on } \Gamma. \\ (4.31) \quad &\tilde{q}(x) = 0, \quad \text{ on } \Omega_+(\bar{u}), \quad \tilde{q}(x) \geq 0 \ \text{ on } \Omega_0(\bar{u})\backslash\Omega_-(\bar{u}), \\ &\lambda \neq 0; \quad \lambda_i = 0, i \in J_+, \quad \lambda_i \geq 0, i \in J_0\backslash J_-. \end{aligned}$$

Let us give an example of such integral constraints for which condition (CQA) can be checked. Let $b \in I\!R^m$, and $a_j \in C(\bar{\Omega})$, for $1 \leq j \leq m$, with $a_j(x)$ of constant sign over its support $\Omega_j := \text{supp}(a_j)$. Assume that these supports satisfy the following geometric relation:

$$(4.32) \qquad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j; \quad \Omega \setminus (\cup_{1 \leq j \leq m}\Omega_j) \ \text{ is connected.}$$

We consider the case when

$$g_j(u) := \int_\Omega a_j(x)y_u(x)dx - b_j.$$

We assume also the following:

(4.33) There exists Ω_*, open subset of $\Omega_+(u)$, such that $\Omega_* \cap \Omega_j = \emptyset$, $1 \leq j \leq m$.

LEMMA 4.10. *Under the above hypotheses, the strict qualification condition* (CQA) *is satisfied.*

Proof. If the conclusion does not hold, then there exists $(\tilde{q}, \tilde{\lambda})$ satisfying the condition of lemma 4.9, and in particular

$$(4.34) \qquad -\Delta\tilde{q} + \phi'_y(x, \bar{y})\tilde{q} = -\sum_{j \in J_0 \cup J_-} \tilde{\lambda}_j a_j \ \text{ in } \Omega, \quad \tilde{q} = 0 \ \text{ on } \Gamma,$$

as well as $\tilde{q} = 0$ on Ω_*. Set $A := \Omega \setminus (\cup_{1 \leq j \leq m}\Omega_j)$. Then A is a connected open set that contains Ω_*. Since Ω_* is open, and $\tilde{q} = 0$ on A, by the unique extension theorem [35] we obtain $\tilde{q} = 0$ on A, hence on $\partial\Omega_j$, for all $1 \leq j \leq m$. Let j be such that $\tilde{\lambda}_j \neq 0$. Let A_j be the interior of Ω_j, $1 \leq j \leq m$, and let

$$B_j := \{x \in \Omega; \ \text{dist}(x, A_j) \leq \varepsilon\}.$$

Take $\varepsilon > 0$ so small that $B_j\backslash A_j$ does not intersect Ω_i, for $i \neq j$. Then $\tilde{q}$ satisfies

$$-\Delta\tilde{q} + \phi'_y(x, y_u)\tilde{q} = -\tilde{\lambda}_j a_j \ \text{ in } B_j, \quad \tilde{q} = 0 \ \text{ on } \partial B_j.$$

This equation has a unique solution in $H^1_0(B_j)$. Since a_j is of constant sign, $\tilde{q}$ is nonzero over the interior of B_j. But this is impossible, since the latter contains a nonempty open set included in A. $\square$

Whenever (CQA) holds, we can state a characterization of quadratic growth. We omit the statement since it is similar to theorem 4.2.

5. Conclusion and possible extensions. Our theoretical results extend those in [6], that discuss problems with polyhedric control constraints only. We were able to give an application of these results for control and state constrained optimal control problems, when the number of state constraints is finite.

For technical reasons we discussed only the case when the space dimension n is less or equal 3. Extension of these results in the case $n > 3$ seems possible by combining the technique of this paper with the two norms approach [6, 25]. The latter would also allow to extend our results to the case of boundary control, or to problems with a parabolic state equation.

It seems also possible to extend our results to the case when K_Y is not a polyhedron, taking advantage of the results in [9]. For instance, the set of semi definite positive matrices is a closed convex set that satisfies hypothesis (ii) of theorem 2.7. On the other hand, the case of a punctual state constraint at every point of the domain Ω seems out of reach, since the strict qualification condition is probably not satisfied in that case.

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