

PONTRYAGIN MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL OF VARIATIONAL INEQUALITIES*

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Abstract. In this paper we investigate optimal control problems governed by variational inequalities. We present a method for deriving optimality conditions in the form of Pontryagin's principle. The main tools used are the Ekeland's variational principle combined with penalization and spike variation techniques.

Key words. variational inequalities, optimal control, Pontryagin principle

AMS subject classifications. 49J20, 49M29

PII. S0363012997328087

1. Introduction. The purpose of this paper is to present a method for deriving a Pontryagin-type maximum principle as a first-order necessary condition of optimal controls for problems governed by variational inequalities. We allow various kinds of constraints to be imposed on the state. To be more precise, we consider the following variational inequality:

$$(1.1a) \quad \frac{\partial y}{\partial t} + Ay + f(y) + \partial\varphi(y) \ni u \quad \text{in } Q = \Omega \times]0, T[,$$

$$(1.1b) \quad y = 0 \quad \text{on } \Sigma = \Gamma \times]0, T[,$$

$$(1.1c) \quad y(0) = y_o \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$, $T > 0$, u is a distributed control, A is a second-order elliptic operator, and $\frac{\partial y}{\partial t}$ denotes the derivative of y with respect to t ; $\partial\varphi(y)$ is the subdifferential of the function φ at y . We shall give all the definitions we need in section 3 and (1.1) will be made clear as well. The control variable u and the state variable y must satisfy constraints of the form

$$(1.2a) \quad u \in U_{ad} = \{ u \in L^p(Q) \mid u(x, t) \in \mathcal{K}_U(x, t) \text{ almost everywhere (a.e.) in } Q \} \subset L^p(Q),$$

where \mathcal{K}_U is a measurable set-valued mapping from Q with closed values in $\mathcal{P}(\mathbb{R})$ ($\mathcal{P}(\mathbb{R})$ being the set of all subsets of \mathbb{R}), and where

$$(1.2b) \quad \Phi(y) \in \mathcal{C}$$

with $1 < p < \infty$, Φ is a \mathcal{C}^1 mapping from $C(\overline{Q})$ into $C(\overline{Q})$, and $\mathcal{C} \subset C(\overline{Q})$ is a closed convex subset with finite codimension.

*Received by the editors October 1, 1997; accepted for publication (in revised form) July 6, 1998; published electronically May 28, 1999.

<http://www.siam.org/journals/sicon/37-4/32808.html>

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The control problem is

$$(\mathcal{P}) \quad \inf\{J(y, u) \mid y \in C(\overline{Q}), u \in U_{ad}, (y, u) \text{ satisfies (1.1), (1.2)}\},$$

where the cost functional is defined by

$$(1.3) \quad J(y, u) = \int_Q F(x, t, y(x, t), u(x, t)) dx dt + \int_\Omega L(x, y(x, T)) dx.$$

Many authors (for example, Barbu [2], Mignot–Puel [17], Yong [23], Bonnans–Tiba [6], Bonnans–Casas [5], and Bergounioux [3]) have already considered control problems for variational inequalities from the theoretical or numerical point of view. Here we are interested in optimality conditions in the form of Pontryagin’s principle. The existence of an optimal solution is assumed a priori. The novelty of this paper is twofold: We obtain the optimality conditions in Pontryagin’s form and we think that our hypotheses seem to be minimal. In essence we ask for the state equation to be well posed and assume differentiability of data with respect to the state. We allow various kinds of constraints to be added on the control u and on the state. However, we restrict the study to the case in which φ is the indicator function of the closed convex set $K_o = \{z \in C(\overline{Q}) \mid z \geq 0\}$ so that the variational inequality (1.1) becomes the so-called obstacle problem.

To get Pontryagin’s principle, we use a method based on penalization of state constraints and on Ekeland’s principle combined with diffuse perturbations [16, 20]. These techniques already have been used by many authors in the case of optimal control of parabolic or elliptic equations [5, 16, 21]. Some of these techniques also have been used for control problems governed by variational inequalities [5, 23, 4]. In those papers, the variational inequality is approximated via the Moreau–Yosida approximation of the maximal monotone operator $\partial\varphi$.

Here we use another idea based on the formulation of (1.1) with a slackness variable and the regularity of its solution. In fact, the solution of (1.1) is also a weak solution of

$$(1.4) \quad \frac{\partial y}{\partial t} + Ay + f(y) = u + \xi \quad \text{in } Q, \quad y = 0 \quad \text{on } \Sigma, \quad y(0) = y_o \quad \text{in } \Omega,$$

where ξ is the Lagrange multiplier associated with the variational inequality and is introduced as an additional control variable. Therefore we obtain a problem $(\tilde{\mathcal{P}})$ equivalent to (\mathcal{P}) , with constraints on both the control variable and the state variable as well as coupled state-control constraints. We first give a Pontryagin’s principle for $(\tilde{\mathcal{P}})$. For this, we adapt the proof given in [21, 24, 7] to problem $(\tilde{\mathcal{P}})$. Next we derive optimality conditions for (\mathcal{P}) from those for $(\tilde{\mathcal{P}})$.

2. Assumptions. Let Ω be an open, smooth (with a \mathcal{C}^2 boundary Γ for example), and bounded domain of \mathbb{R}^n ($2 \leq n$). In this paper we suppose that

$$p > n.$$

Remark 2.1. We must emphasize that this choice of p is not optimal. Indeed, we should distinguish the integers p (for the L^p -space of the distributed control u) and q (for the L^q -space of the initial value y_o). The optimal choice should be $u \in L^p(Q)$ with $p > \frac{n}{2} + 1$ and $y_o \in W_o^{1,q}(\Omega)$ with $q > n$; at each occurrence we note how the assumptions that follow could be weakened from this point of view. To make the presentation clearer we simply assume that $p = q > n$.

In addition we make the following assumptions.

(A1) A is a linear elliptic differential operator defined by

$$(2.1) \quad \begin{aligned} Ay &= - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j}y) + a_0(x)y \quad \text{with} \\ a_{ij} &\in C^2(\bar{\Omega}) \text{ for } i, j = 1 \cdots n, \\ a_0 &\in L^\infty(\Omega), \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq m_o \sum_{i=1}^n \xi_i^2 \quad \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n, m_o > 0. \end{aligned}$$

(A2) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a monotone increasing, globally Lipschitz C^1 -function.

Remark 2.2. The monotonicity assumption on f can be relaxed and replaced by

$$\exists c_o \in \mathbb{R} \quad f'_y \geq c_o.$$

An appropriate translation shows that we retrieve the case where f is monotonically increasing, so we assume this for the sake of simplicity.

On the other hand one could consider a mapping f from $\mathbb{R} \times \mathbb{R}$ to \mathbb{R} depending on both y and u . The method would work in the same way. (In what follows, we denote the real function $f : \mathbb{R} \rightarrow \mathbb{R}$ and the Nemytski operator associated to $f : y(\cdot) \mapsto f(y(\cdot))$ in $L^p(Q)$ by the same symbol f .)

(A3) $\varphi : W_o^{1,p}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper (i.e., nonidentically equal to $+\infty$), convex, lower semicontinuous function such that $0 \in \text{dom } \varphi$.

(A4) $y_o \in \text{dom } \varphi$.

(A5) For every $(y, u) \in \mathbb{R}^2$, $F(\cdot, y, u)$ is measurable on Q . For almost every $(x, t) \in Q$, for every $u \in \mathbb{R}$, $F(x, t, \cdot, u)$ is C^1 on \mathbb{R} . For almost every $(x, t) \in Q$, $F(x, t, \cdot)$ and $F'_y(x, t, \cdot)$ are continuous on \mathbb{R}^2 . The following estimate holds:

$$|F(x, t, y, u)| + |F'_y(x, t, y, u)| \leq (M_1(x, t) + m_1|u|^p)\eta(|y|),$$

where $M_1 \in L^1(Q)$, $m_1 \geq 0$, and η is a nondecreasing function from \mathbb{R}^+ to \mathbb{R}^+ .

(A6) For every $y \in \mathbb{R}$, $L(\cdot, y)$ is measurable on Ω . For almost every $x \in \Omega$, $L(x, \cdot)$ is C^1 on \mathbb{R} . The following estimate holds:

$$|L(x, y)| + |L'_y(x, y)| \leq M_2(x)\eta(|y|),$$

where $M_2 \in L^1(\Omega)$, η is as in (A5).

(A7) Φ is a C^1 mapping from $C(\bar{Q})$ into $C(\bar{Q})$, and \mathcal{C} is a closed convex subset of $C(\bar{Q})$ with finite codimension.

We recall that for $p \in \mathbb{N}$

$$\begin{aligned} W^{1,p}(\Omega) &= \{y \in L^p(\Omega) \mid \nabla y \in L^p(\Omega)^n\} \quad \text{and} \\ W^{2,1,p}(Q) &= \left\{ y \in L^p(Q) \mid Dy, D^2y, \frac{\partial y}{\partial t} \in L^p(Q) \right\}. \end{aligned}$$

3. Existence and regularity of solutions to the variational inequality.

Let V and H be Hilbert spaces such that $V \subset H \subset V'$ with continuous and dense injections. We denote by $(\cdot, \cdot)_V$ the V -scalar product, $\langle \cdot, \cdot \rangle$ the duality product between V and V' , and $\|\cdot\|_V$ the V -norm. We consider a linear, continuous V -elliptic operator \mathcal{A} from V to V' and ϕ a convex, proper, and lower semicontinuous function from V to $\mathbb{R} \cup \{+\infty\}$. Then we may define the variational inequality

$$(3.1) \quad \begin{cases} \frac{\partial y}{\partial t}(t) + \mathcal{A}y(t) + \partial\phi(y)(t) \ni u(t) \text{ a.e. } t \text{ in } [0, T], \\ y(0) = y_o \end{cases}$$

in the following (variational) sense:

$$(3.2) \quad \left\langle \frac{\partial y}{\partial t}(t) + \mathcal{A}y(t), y(t) - z \right\rangle + \phi(y(t)) - \phi(z) \leq \langle f(t), y(t) - z \rangle \quad \text{a.e. } t \in (0, T) \quad \forall z \in V.$$

Here $\partial\phi(y(t))$ denotes the subdifferential of ϕ at $z = y(t) \in V$ [8]:

$$(3.3) \quad \partial\phi(z) = \{ z^* \in V' \mid \phi(z) - \phi(\zeta) \leq \langle z - \zeta, z^* \rangle \quad \forall \zeta \in V \}.$$

Now we set $V = H^1_o(\Omega)$ and $H = L^2(\Omega)$; we let g be a primitive function of f (such that $g(0) = 0$ for example) and define

$$(3.4) \quad \phi = \varphi + g,$$

where φ is given by (A3). Then $\partial\phi = g' + \partial\varphi = f + \partial\varphi$ (g is the regular part of ϕ). Therefore (1.1) makes sense in the (3.1) form with $\mathcal{A} = A$ and we may give a first existence and regularity result as in the following theorem.

THEOREM 3.1. *Set $p \geq 2$; let $u \in L^p(Q)$ and $y_o \in W^{1,p}_o(\Omega)$. Assume that*

$$(3.5) \quad \exists \gamma \in L^p(\Omega) \cap \partial\varphi(y_o);$$

then (1.1) has a unique solution $y \in W^{2,1,p}(Q)$.

Proof. We first use a result of Tiba [22, Theorem 4.5, p. 26] that ensures that if β is a maximal monotone graph $\subset \mathbb{R} \times \mathbb{R}$, $u \in L^p(Q)$ and $y_o \in W^{1,p}_o(\Omega)$, then the parabolic variational inequality

$$\begin{cases} \frac{\partial y}{\partial t} + Ay + \beta(y) & \ni u & \text{a.e. in } Q, \\ y(0, x) & = y_o(x) & \text{a.e. on } \Omega, \\ y(t, x) & = 0 & \text{a.e. on } \Sigma \end{cases}$$

has a unique solution in $W^{2,1,p}(Q)$ if the compatibility relation

$$(3.6) \quad \begin{aligned} &0 \in \text{dom } \beta, y_o(x) \in \text{dom } \beta \text{ a.e. in } \Omega, \\ &\exists \gamma \in L^p(\Omega) \text{ such that } \gamma(x) \in \beta(y_o(x)) \text{ a.e. in } \Omega \end{aligned}$$

is fulfilled. One can apply this result to $\beta = f + \partial\varphi$, which is a maximal monotone graph since f is monotone increasing and φ is convex, lower semicontinuous, and proper. It remains to check (3.6), that is,

$$\exists \gamma \in L^p(\Omega) \text{ such that } \gamma(x) \in f(y_o(x)) + \partial\varphi(y_o(x)) \quad \text{a.e. in } \Omega.$$

This is equivalent to

$$\exists \gamma \text{ such that } \gamma + f(y_o) \in L^p(\Omega), \text{ and } \gamma(x) \in \partial\varphi(y_o(x)) \text{ a.e. in } \Omega.$$

Since f is globally Lipschitz then $f(y_o) \in L^p(\Omega)$ and we get the result. \square

We set

$$\xi = u - \frac{\partial y}{\partial t} - Ay - f(y) \in L^p(Q)$$

(since f is globally Lipschitz and $y \in W^{2,1,p}(Q)$). In addition, $\xi(t) \in \partial\varphi(y(t))$ almost everywhere in $]0, T[$; using the characterization of the subdifferential of a function in Banach spaces this gives

$$(3.7) \quad \varphi(y(t)) + \varphi^*(\xi(t)) - \langle y(t), \xi(t) \rangle = 0 \quad \text{a.e. in }]0, T[.$$

In this last relation $\langle \cdot, \cdot \rangle$ denotes the duality product between $V = W^{1,p}_o(\Omega)$ and V' , and φ^* is the conjugate function of φ . For more details refer to Barbu–Precupanu [1] or Ekeland–Temam [13]. It follows that the variational inequality (1.1) is equivalent to

$$(3.8) \quad \begin{aligned} \frac{\partial y}{\partial t} + Ay + f(y) &= u + \xi && \text{in } Q, \\ y &= 0 && \text{on } \Sigma, \\ y(x, 0) &= y_o(x) && \text{in } \Omega \end{aligned}$$

and (3.7). Because $y_o \in W^{1,p}_o(\Omega)$ and $(u, \xi) \in L^p(Q) \times L^p(Q)$, the solution y of equation (3.8) belongs to $C(\overline{Q}) \cap W^{2,1,p}(Q)$. More precisely, we have the following theorem.

THEOREM 3.2. (i) *If $p > n/2 + 1$ and $(u, \xi, y_o) \in L^p(Q) \times L^p(Q) \times C(\overline{\Omega})$, then (3.8) has a unique weak solution $y_{u\xi}$ in $W(0, T) \cap C(\overline{Q})$ which satisfies*

$$\|y_{u\xi}\|_{\infty, Q} \leq C_1(\|u\|_{p, Q} + \|\xi\|_{p, Q} + \|y_o\|_{\infty, \Omega} + 1),$$

where $C_1 = C_1(T, \Omega, m_0, n, p)$. Moreover, for every $\varepsilon > 0$, $y_{u\xi}$ is Hölder continuous on $[\varepsilon, T] \times \overline{\Omega}$ and belongs to $W^{2,1,p}(\Omega \times]\varepsilon, T[)$.

(ii) *If $p > n$ and $(u, \xi, y_o) \in L^p(Q) \times L^p(Q) \times W^{1,p}_o(\Omega)$, (3.8) has a unique weak solution $y_{u\xi}$ in $W^{2,1,p}(Q) \cap C(\overline{Q})$.*

Proof. The existence of a unique weak solution $y_{u\xi}$ in $W(0, T) \cap C(\overline{Q})$ for (3.8) can be proved as in the case of the Robin boundary condition (see Raymond–Zidani [20, 21]). The Hölder continuity result holds thanks to [9]. Point (ii) can be found in Bergounioux–Tröltzsch [4]. \square

4. Optimal control of the obstacle problem.

4.1. The obstacle problem. Now we focus on the very case of control of the obstacle problem, where

$$(4.1) \quad K_o = \{z \in W^{1,p}_o(\Omega) \mid z \geq 0 \text{ a.e. in } \Omega\}$$

and φ is the indicator function of K_o :

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in K_o, \\ +\infty & \text{else.} \end{cases}$$

It is clear that $0 \in \text{dom } \varphi = K_o$. Moreover, the compatibility condition (3.5) is fulfilled with $\gamma = 0$ so that Theorem 3.1 is valid. On the other hand, the (classical) calculus of φ^* shows that relation (3.7) is equivalent to

$$(4.2) \quad \begin{aligned} y(t) &\geq 0 \text{ in } \Omega \quad \forall t \in]0, T[, \quad \xi(t) \geq 0 \text{ in } \Omega, \\ \text{and } \int_{\Omega} y(t, x) \xi(t, x) \, dx &= 0 \quad \text{a.e. } t \in]0, T[, \end{aligned}$$

that is, at last

$$y \geq 0 \text{ in } Q, \xi \geq 0 \text{ a.e. in } Q, \text{ and } \int_Q y(t, x) \xi(t, x) dx dt = 0.$$

We may summarize in the following theorem.

THEOREM 4.1. *Assume $p > n$, $(u, y_o) \in L^p(Q) \times W_o^{1,p}(\Omega)$; then the variational inequality*

$$(4.3) \quad \frac{\partial y}{\partial t} + Ay + f(y) + \partial\varphi(y) \ni u \text{ in } Q, \quad y = 0 \text{ on } \Sigma, \quad y(0) = y_o \text{ in } \Omega,$$

where φ is the indicator function of K_o , has a unique solution $y \in \mathcal{C}(\bar{Q}) \cap W^{2,1,p}(Q)$. Moreover, it is equivalent to

$$(4.4) \quad \begin{cases} \frac{\partial y}{\partial t} + Ay + f(y) = u + \xi \text{ in } Q, & y = 0 \text{ on } \Sigma, & y(x, 0) = y_o(x) \text{ in } \Omega, \\ \xi \geq 0, \quad y \geq 0, & \int_Q y(t, x) \xi(t, x) dx dt = 0. \end{cases}$$

In the following we denote

$$(4.5) \quad V_{ad} = \{ \xi \in L^p(Q) \mid \xi \geq 0 \text{ a.e. in } Q \}.$$

4.2. Pontryagin principle. Now we consider the following problem $(\tilde{\mathcal{P}})$: Minimize $J(y, u)$ subject to

$$(4.6a) \quad \frac{\partial y}{\partial t} + Ay + f(y) = u + \xi \text{ in } Q, \quad y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_o \text{ in } \Omega,$$

$$(4.6b) \quad \tilde{\Phi}(y) \in \tilde{\mathcal{C}} \quad (\text{“pure” state constraint}),$$

$$(4.6c) \quad (u, \xi) \in U_{ad} \times V_{ad} \quad (\text{“pure” control constraints}),$$

$$(4.6d) \quad \int_Q y(t, x) \xi(t, x) dx dt = 0 \quad (\text{mixed state-control integral constraints}),$$

where

$$(4.7) \quad \tilde{\Phi}(y) = (\Phi(y), y) \text{ and } \tilde{\mathcal{C}} = \mathcal{C} \times \{ y \in \mathcal{C}(\bar{Q}) \mid y \geq 0 \}.$$

The results of section 3 yield that problems (\mathcal{P}) and $(\tilde{\mathcal{P}})$ are equivalent. In particular if (\bar{y}, \bar{u}) is a solution of (\mathcal{P}) , then there exists $\bar{\xi} \in L^p(Q)$ such that $(\bar{y}, \bar{u}, \bar{\xi})$ is an optimal solution of $(\tilde{\mathcal{P}})$ with $\bar{\xi} = \partial\bar{y}/\partial t + A\bar{y} + f(\bar{y}) - \bar{u}$. Let us mention that we are interested not in existence results (although we will give an example in the last section of this paper) but in optimality conditions for (\bar{y}, \bar{u}) . Consequently, we study optimality conditions for $(\bar{y}, \bar{u}, \bar{\xi})$ to get those for (\bar{y}, \bar{u}) .

Let us define the Hamiltonian functions by

$$(4.8) \quad H_1(x, t, y, u, q, \nu) = \nu F(x, t, y, u) + q u$$

for every $(x, t, y, u, q, \nu) \in Q \times \mathbb{R}^4$, and

$$(4.9) \quad H_2(y, \xi, q, \lambda) = q \xi + \lambda y \xi$$

for every $(y, \xi, q, \lambda) \in \mathbb{R}^4$.

THEOREM 4.2 (Pontryagin principle for $(\tilde{\mathcal{P}})$). *If (A1)–(A7) are fulfilled and if $(\bar{y}, \bar{u}, \bar{\xi})$ is a solution of $(\tilde{\mathcal{P}})$, then there exist $\bar{q} \in L^1(0, T; W_o^{1,1}(\Omega))$, $\bar{\nu} \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}$, and $(\bar{\mu}, \bar{\theta}) \in \mathcal{M}(\bar{Q}) \times \mathcal{M}(\bar{Q})$ ($\mathcal{M}(\bar{Q})$ is the space of Radon measures on \bar{Q}), such that*

$$(4.10a) \quad (\bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\theta}) \neq 0, \quad \bar{\nu} \geq 0,$$

$$(4.10b) \quad \forall z \in \{z \in C(\bar{Q}) \mid z \geq 0\} \quad \langle \bar{\mu}, z - \bar{y} \rangle_{\bar{Q}} \leq 0, \quad \text{and } \forall z \in C \quad \langle \bar{\theta}, z - \Phi(\bar{y}) \rangle_{\bar{Q}} \leq 0,$$

$$(4.10c) \quad \begin{cases} -\frac{\partial \bar{q}}{\partial t} + A^* \bar{q} + f'_y(\bar{y}) \bar{q} = \bar{\nu} F'_y(x, t, \bar{y}, \bar{u}) + \bar{\mu}|_Q + [\Phi'(\bar{y})^* \bar{\theta}]|_Q + \bar{\lambda} \bar{\xi} & \text{in } Q, \\ \bar{q} = 0 \quad \text{on } \Sigma, \quad \bar{q}(T) = \bar{\nu} L'_y(x, \bar{y}(T)) + \bar{\mu}|_{\bar{\Omega}_T} + [\Phi'(\bar{y})^* \bar{\theta}]|_{\bar{\Omega}_T} & \text{in } \Omega, \end{cases}$$

$$(4.10d) \quad \bar{q} \in L^{\delta'}(0, T; W_o^{1,d'}(\Omega)) \quad \text{for every } (\delta, d) \text{ satisfying } \frac{n}{2d} + \frac{1}{\delta} < \frac{1}{2},$$

$$(4.10e) \quad \begin{aligned} H_1(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{q}(x, t), \bar{\nu}) &= \min_{u \in \mathcal{K}_U(x, t)} \\ H_1(x, t, \bar{y}(x, t), u(x, t), \bar{q}(x, t), \bar{\nu}) &\quad \text{a.e. in } Q, \end{aligned}$$

$$(4.10f) \quad H_2(\bar{y}(x, t), \bar{\xi}(x, t), \bar{q}(x, t), \bar{\lambda}) = \min_{\xi \in \mathbb{R}^+} H_2(\bar{y}(x, t), \xi, \bar{q}(x, t), \bar{\lambda}) \quad \text{a.e. in } Q,$$

where $\bar{\mu}|_Q$ (resp., $[\Phi'(\bar{y})^* \bar{\theta}]|_Q$) is the restriction of $\bar{\mu}$ (resp., $[\Phi'(\bar{y})^* \bar{\theta}]$) to Q , $\bar{\mu}|_{\bar{\Omega}_T}$ (resp., $[\Phi'(\bar{y})^* \bar{\theta}]|_{\bar{\Omega}_T}$) is the restriction of $\bar{\mu}$ (resp., $[\Phi'(\bar{y})^* \bar{\theta}]$) to $\bar{\Omega} \times \{T\}$, $\langle \cdot, \cdot \rangle_{\bar{Q}}$ denotes the duality product between $\mathcal{M}(\bar{Q})$ and $C(\bar{Q})$, A^* is the adjoint operator of A , and $\frac{1}{d} + \frac{1}{d'} = 1$.

Remark 4.1. We briefly describe these relations: $(\bar{\mu}, \bar{\theta})$ are the multipliers associated with the state constraints; $\bar{\mu}$ corresponds to $y \geq 0$; and an immediate consequence of relation (4.10b) is the complementarity result $\bar{\mu} \leq 0$, $\langle \bar{\mu}, \bar{y} \rangle_{\bar{Q}} = 0$. $\bar{\theta}$ is associated to the (general) constraint $\Phi(y) \in \mathcal{C}$. $\bar{\lambda}$ is the multiplier associated to the integral constraint $\int_Q y(t, x) \xi(t, x) dx dt = 0$, and \bar{q} is the classical adjoint state which takes into account the cost functional via $\bar{\nu}$.

Condition (4.10a) is a nontriviality condition. We must emphasize that we get (a priori) *nonqualified* optimality conditions. If $\bar{\nu} \neq 0$, the problem is qualified.

Remark 4.2. One may note that if $\bar{\xi} = 0$, then it could happen that $\bar{\nu} = \bar{\mu} = \bar{\theta} = 0$ and $\bar{\lambda} \neq 0$, so that $\bar{q} = 0$; therefore, the optimality system could appear to be useless. However, this is the case where the solution (\bar{y}, \bar{u}) is the solution of a control problem governed by a classical semilinear parabolic equation, since we have

$\partial\bar{y}/\partial t + A\bar{y} + f(\bar{y}) = \bar{u}$ and the associated optimality systems are well known for this kind of problem. We refer for instance to [20].

THEOREM 4.3 (Pontryagin principle for (\mathcal{P})). *If (A1)–(A7) are fulfilled and if (\bar{y}, \bar{u}) is a solution of (\mathcal{P}) , then there exists $\bar{q} \in L^1(0, T; W_o^{1,1}(\Omega))$, $\bar{v} \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}$, $(\bar{\mu}, \bar{\theta}) \in \mathcal{M}(\bar{Q}) \times \mathcal{M}(\bar{Q})$ such that (4.10a), (4.10b), (4.10d), and (4.10e) hold. Moreover, we have*

$$(4.11a) \quad \left\{ \begin{array}{l} -\frac{\partial \bar{q}}{\partial t} + A^* \bar{q} + f'_y(\bar{y}) \bar{q} = \bar{v} F'_y(x, t, \bar{y}, \bar{u}) + \bar{\mu}|_Q \\ \quad + [\Phi'(\bar{y})^* \bar{\theta}]|_Q + \bar{\lambda} \left(\frac{\partial \bar{y}}{\partial t} + A\bar{y} + f(\bar{y}) - \bar{u} \right) \text{ in } Q, \\ \bar{q} = 0 \text{ on } \Sigma, \quad \bar{q}(T) = \bar{v} L'_y(x, \bar{y}(T)) + \bar{\mu}|_{\bar{\Omega}_T} + [\Phi'(\bar{y})^* \bar{\theta}]|_{\bar{\Omega}_T} \text{ in } \Omega, \end{array} \right.$$

$$(4.11b) \quad \bar{q}(x, t) \left(\frac{\partial \bar{y}}{\partial t} + A\bar{y} + f(\bar{y}) - \bar{u} \right) (x, t) = 0 \quad \text{a.e. } (x, t) \in Q.$$

Remark 4.3. Relation (4.11b) is a pointwise complementarity condition. Therefore, \bar{q} may be viewed as a Lagrange multiplier associated with the pointwise constraint

$$\left(\frac{\partial y}{\partial t} + Ay + f(y) - u \right) (x, t) \geq 0.$$

Let us recall a regularity result for a weak solution of parabolic equation with measures as data, as follows.

PROPOSITION 4.1. *Let μ be in $\mathcal{M}_b(\bar{Q} \setminus (\bar{\Omega} \times \{0\} \cup \bar{\Sigma}))$ and let a be in $L^p(Q)$ satisfying*

$$a \geq C_0, \quad \|a\|_{L^p(Q)} \leq M,$$

where $M > 0$. Consider the equation

$$(4.12) \quad -\frac{\partial q}{\partial t} + A^* q + aq = \mu_Q \quad \text{in } Q, \quad q = 0 \quad \text{on } \Sigma, \quad q(T) = \mu_{\bar{\Omega}_T} \quad \text{on } \bar{\Omega},$$

where $\mu = \mu_Q + \mu_{\bar{\Omega}_T}$ is a bounded Radon measure on $\bar{Q} \setminus (\bar{\Omega} \times \{0\} \cup \bar{\Sigma})$, μ_Q is the restriction of μ to Q , and $\mu_{\bar{\Omega}_T}$ is the restriction of μ to $\bar{\Omega} \times \{T\}$. Equation (4.12) admits a unique weak solution $q \in L^1(0, T; W_o^{1,1}(\Omega))$. For every (δ, d) satisfying $d > 2$, $\delta > 2$, $\frac{n}{2d} + \frac{1}{\delta} < \frac{1}{2}$, $q \in L^{\delta'}(0, T; W_o^{1,d'}(\Omega))$, and we have

$$\|q\|_{L^{\delta'}(0, T; W_o^{1,d'}(\Omega))} \leq C_2 \|\mu\|_{\mathcal{M}_b(\bar{Q} \setminus (\bar{\Omega} \times \{0\} \cup \bar{\Sigma}))},$$

where $C_2 = C_2(T, \Omega, n, C_0, M, p, \delta, d)$ is independent of a . Moreover, there exists a function $q(0) \in L^1(\Omega)$ such that

$$\int_Q q \left\{ \frac{\partial y}{\partial t} + Ay + ay \right\} dxdt = \langle y, \mu \rangle_b - \langle y(0), q(0) \rangle_{C(\bar{\Omega}) \times \mathcal{M}(\bar{\Omega})}$$

for every $y \in Y = \{y \in W(0, T) \cap C(\bar{Q}) \mid \frac{\partial y}{\partial t} + Ay \in L^p(Q), y = 0 \text{ in } \Sigma\}$, where $\langle \cdot, \cdot \rangle_b$ denotes the duality product between $C_b(\bar{Q} \setminus (\bar{\Omega} \times \{0\} \cup \bar{\Sigma}))$ and $\mathcal{M}_b(\bar{Q} \setminus (\bar{\Omega} \times \{0\} \cup \bar{\Sigma}))$.

$(C_b(\overline{Q} \setminus (\overline{\Omega} \times \{0\} \cup \overline{\Sigma})))$ denotes the space of bounded continuous functions on $\overline{Q} \setminus (\overline{\Omega} \times \{0\} \cup \overline{\Sigma})$, while $\mathcal{M}_b(\overline{Q} \setminus (\overline{\Omega} \times \{0\} \cup \overline{\Sigma}))$ denotes the space of bounded Radon measures on $\overline{Q} \setminus (\overline{\Omega} \times \{0\} \cup \overline{\Sigma})$, that is, the topological dual of $C_o(\overline{Q} \setminus (\overline{\Omega} \times \{0\} \cup \overline{\Sigma}))$.

Proof. The proof is the same as the one given in [19] for the Neumann boundary conditions (see also [7]). An easy adaptation of this proof yields the previous result. However, for the convenience of the reader we recall that q is the weak solution of (4.12) if and only if q belongs to $L^1(0, T; W_o^{1,1}(\Omega))$, $aq \in L^1(Q)$, and for every $\varphi \in C^1(\overline{Q})$ satisfying $\varphi(x, 0) = 0$ on $\overline{\Omega}$ and $\varphi(\cdot) = 0$ on Σ we have

$$\int_Q \left\{ q \frac{\partial \varphi}{\partial t} + \sum_{i,j} a_{ij} D_j \varphi D_i q + a \varphi q \right\} dx dt = \langle \varphi, \mu \rangle_b.$$

As in [7], we can prove that the weak solution q belongs to $L^{\delta'}(0, T; W_o^{1,\delta'}(\Omega))$ for every (δ, d) satisfying the condition

$$(4.13) \quad d > 2, \quad \delta > 2, \quad \frac{n}{2d} + \frac{1}{\delta} < \frac{1}{2}.$$

We remark that the set of pairs (δ, d) satisfying the above condition is nonempty. We remark also that if (δ, d) satisfies (4.13), if a belongs to $L^p(Q)$, and if q belongs to $L^{\delta'}(0, T; W_o^{1,\delta'}(\Omega))$, then $aq \in L^1(Q)$. Now, since $q \in L^{\delta'}(0, T; W_o^{1,\delta'}(\Omega))$ (where (δ, d) satisfies (4.24)), and since

$$\operatorname{div}_{xt}((\sum_j a_{ij} D_j q)_{1 \leq i \leq n}, q) = \frac{\partial q}{\partial t} - Aq \text{ belongs to } \mathcal{M}_b(Q),$$

then we can define the normal trace of the vector field $((\sum_j a_{ij} D_j q)_{1 \leq i \leq n}, q)$ in the space $W^{\frac{-1}{m}, m}(\partial Q)$ (for some $1 < m < \frac{n+1}{n}$). If we denote by $\gamma_o((\sum_j a_{ij} D_j q)_{1 \leq i \leq n}, q)$ this normal trace, we can prove (see Theorem 4.2 in [19]) that this normal trace belongs to $\mathcal{M}(\partial Q)$ and the restriction of $\gamma_o((\sum_j a_{ij} D_j q)_{1 \leq i \leq n}, q)$ to $\overline{\Omega} \times \{T\}$ is equal to $\mu_{\overline{\Omega}_T}$, and if $q(0)$ is the measure on $\overline{\Omega}$ which satisfies the Green formula of our Theorem 3.2, then $-q(0)$ is the restriction of $\gamma_o((\sum_j a_{ij} D_j q)_{1 \leq i \leq n}, q)$ to $\overline{\Omega} \times \{0\}$. In fact, it can be proved that $q(0)$ belongs to $L^1(\Omega)$ (see Theorem 4.3 in [19]). \square

4.3. Proof of Theorems 4.2–4.3. First we assume that Theorem 4.2 is valid. As mentioned before, if (\bar{y}, \bar{u}) is an optimal solution for (\mathcal{P}) , then $(\bar{y}, \bar{u}, \bar{\xi})$ is a solution for $(\tilde{\mathcal{P}})$, where $\bar{\xi} = \frac{\partial \bar{y}}{\partial t} + A\bar{y} + f(\bar{y}) - \bar{u} \in L^p(Q)$. Thanks to Theorem 4.2, there exist $(\bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{q})$ such that (4.10) holds. Replacing $\bar{\xi}$ by its value in (4.10c) obviously leads to (4.11a). Furthermore, relation (4.10f) implies

$$(\bar{q}(x, t) + \bar{\lambda} \bar{y}(x, t)) (\bar{\xi}(x, t) - \xi) \leq 0 \quad \text{a.e. } (x, t) \in Q \quad \forall \xi \in \mathbb{R}^+,$$

which gives

$$(\bar{q}(x, t) + \bar{\lambda} \bar{y}(x, t)) \bar{\xi}(x, t) = 0 \quad \text{a.e. } (x, t) \in Q.$$

Since $\bar{y}(x, t) \bar{\xi}(x, t) = 0$ a.e. in Q we obtain (4.11b). This concludes the proof of Theorem 4.3. \square

It remains to show that Theorem 4.2 is valid. Note that Pontryagin’s principle for a control problem with unbounded controls, with pointwise state constraints, and with state-control constraints in integral form already have been studied in [7]. For the convenience of the reader, we give the main ideas of the proof.

Step 1: Metric space of controls. In this paper we shall consider control problems for which the state constraints (4.6b) and the state-control integral constraints (4.6d) are penalized. These problems are constructed in such a way to make $(\bar{y}, \bar{u}, \bar{\xi})$ an approximate solution. The idea is to apply the Ekeland variational principle next. For this we have to define a metric space of controls, endowed with the so-called Ekeland distance d , to make the mapping $(u, \xi) \mapsto y_{u\xi}$ continuous from this metric space into $\mathcal{C}(\bar{Q})$. Thanks to Theorem 3.2, this continuity condition will be realized if convergence in the metric space of controls implies convergence in $L^p(Q) \times L^p(Q)$. Here, since we deal with (generally) unbounded controls, the convergence in $(U_{ad} \times V_{ad}, d)$ does not imply the convergence in $L^p(Q) \times L^p(Q)$ (see [14, p. 227]). To overcome this difficulty, as in [24, 20], we define a new metric as follows. For $0 < k < \infty$, we set

$$U_{ad}(\bar{u}, k) = \{u \in U_{ad} \mid |u(x, t) - \bar{u}(x, t)| \leq k \text{ a.e. } (x, t) \in Q\},$$

$$V_{ad}(\bar{\xi}, k) = \{\xi \in V_{ad} \mid |\xi(x, t) - \bar{\xi}(x, t)| \leq k \text{ a.e. } (x, t) \in Q\}.$$

We endow the control space with Ekeland’s metric:

$$d((u_1, \xi_1), (u_2, \xi_2)) = \mathcal{L}^{n+1}(\{(x, t) \mid u_1(x, t) \neq u_2(x, t)\}) + \mathcal{L}^{n+1}(\{(x, t) \mid \xi_1(x, t) \neq \xi_2(x, t)\}),$$

where \mathcal{L}^{n+1} denotes the Lebesgue measure in \mathbb{R}^{n+1} . Then, as in [24, 20], we can prove the following lemma.

LEMMA 4.1. $(U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k), d)$ is a complete metric space for the distance d , and the mapping which associates $(y_{u\xi}, J(y_{u\xi}, u))$ with (u, ξ) is continuous from $(U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k), d)$ into $\mathcal{C}(\bar{Q}) \times \mathbb{R}$.

In [7], the authors have used another method to build the metric space of controls. This construction was adapted to the type of constraints they have considered.

Step 2: Penalized problems. Since $\mathcal{C}(\bar{Q})$ is separable, there exists a norm $|\cdot|_{\mathcal{C}(\bar{Q})}$, which is equivalent to the norm $\|\cdot\|_{\mathcal{C}(\bar{Q})}$ such that $(\mathcal{C}(\bar{Q}), |\cdot|_{\mathcal{C}(\bar{Q})})$ is strictly convex and $\mathcal{M}(\bar{Q})$, endowed with the dual norm of $|\cdot|_{\mathcal{C}(\bar{Q})}$ (denoted by $|\cdot|_{\mathcal{M}(\bar{Q})}$), also is strictly convex (see [11, Corollary 2, p. 148, or Corollary 2, p. 167]). Let \mathbb{K} be a convex subset of $\mathcal{C}(\bar{Q})$. We define the distance function to \mathbb{K} (for the new norm $|\cdot|_{\mathcal{C}(\bar{Q})}$) by

$$\delta_{\mathbb{K}}(\zeta) = \inf_{z \in \mathbb{K}} |\zeta - z|_{\mathcal{C}(\bar{Q})}.$$

Since \mathbb{K} is convex, then $\delta_{\mathbb{K}}$ is convex and Lipschitz of rank 1, and we have

$$(4.14) \quad \limsup_{\substack{\rho \searrow 0 \\ \zeta' \rightarrow \zeta}} \frac{\delta_{\mathbb{K}}(\zeta' + \rho z) - \delta_{\mathbb{K}}(\zeta')}{\rho} = \max\{\langle \xi, z \rangle_{\bar{Q}} \mid \xi \in \partial\delta_{\mathbb{K}}(\zeta)\}$$

for every $\zeta, z \in \mathcal{C}(\bar{Q})$, where $\partial\delta_{\mathbb{K}}(\zeta)$ is the subdifferential of $\delta_{\mathbb{K}}$ at (ζ) . Moreover, as \mathbb{K} is a closed convex subset of $\mathcal{C}(\bar{Q})$ it is proved in [16, Lemma 3.4] that for every $\zeta \notin \mathbb{K}$, and every $\xi \in \partial\delta_{\mathbb{K}}(\zeta)$, $|\xi|_{\mathcal{M}(\bar{Q})} = 1$. Since $\partial\delta_{\mathbb{K}}(\zeta)$ is convex in $\mathcal{M}(\bar{Q})$ and $(\mathcal{M}(\bar{Q}), |\cdot|_{\mathcal{M}(\bar{Q})})$ is strictly convex, then if $\zeta \notin \mathbb{K}$, $\partial\delta_{\mathbb{K}}(\zeta)$ is a singleton and $\delta_{\mathbb{K}}$ is Gâteaux-differentiable at ζ . Let us notice that when $\mathbb{K} := \{z \in \mathcal{C}(\bar{Q}) \mid z \geq 0\}$, the distance function to \mathbb{K} is given by $\delta_{\mathbb{K}}(\zeta) = |\zeta^-|_{\mathcal{C}(\bar{Q})}$, where $\zeta^- = \min(0, \zeta)$.

Endowing $\mathcal{C}(\bar{Q}) \times \mathcal{C}(\bar{Q})$ with the product norm we have similarly $\delta_{\tilde{\mathcal{C}}}(\tilde{\Phi}(y))^2 = |y^-|_{\mathcal{C}(\bar{Q})}^2 + \delta_{\mathcal{C}}(\Phi(y))^2$ ($\tilde{\mathcal{C}}$ is defined by (4.7)). Let us consider the penalized functional

$$J_\varepsilon(y, u, \xi) = \left\{ \left[(J(y, u) - J(\bar{y}, \bar{u}) + \varepsilon^2)^+ \right]^2 + \delta_{\tilde{\mathcal{C}}}(\tilde{\Phi}(y))^2 + \left(\int_Q y(x, t) \xi(x, t) dx dt \right)^2 \right\}^{\frac{1}{2}}.$$

With such a choice, for every $\varepsilon > 0$ and $k > 0$, $(\bar{y}, \bar{u}, \bar{\xi})$ is a ε^2 -solution of the penalized problem

$$(\mathcal{P}_{k,\varepsilon}) \quad \inf\{J_\varepsilon(y, u, \xi) \mid y \in \mathcal{C}(\bar{Q}), (u, \xi) \in U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k), (y, u, \xi) \text{ satisfies (4.6a)}\},$$

i.e.,

$$\inf(\mathcal{P}_{k,\varepsilon}) \leq J_\varepsilon(\bar{y}, \bar{u}, \bar{\xi}) \leq \inf(\mathcal{P}_{k,\varepsilon}) + \varepsilon^2$$

(since $\inf(\mathcal{P}_{k,\varepsilon}) \geq 0$ and $J_\varepsilon(\bar{y}, \bar{u}, \bar{\xi}) = \varepsilon^2$).

For every $k > 0$, we choose $\varepsilon(k) = \varepsilon_k \leq \frac{1}{k^{2p}}$ and we denote by (\mathcal{P}_k) the penalized problem $(\mathcal{P}_{k,\varepsilon_k})$. Thanks to Ekeland's principle [13, p. 30], for every $k \geq 1$ there exists $(u_k, \xi_k) \in U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k)$ such that

$$(4.15a) \quad d((u_k, \xi_k), (\bar{u}, \bar{\xi})) \leq \varepsilon_k \leq \frac{1}{k^{2p}},$$

$$(4.15b) \quad J_{\varepsilon_k}(y_k, u_k, \xi_k) \leq J_{\varepsilon_k}(y_{u\xi}, u, \xi) + \varepsilon_k d((u_k, \xi_k), (u, \xi))$$

for every $(u, \xi) \in U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k)$ (y_k and $y_{u\xi}$ being the states corresponding respectively to (u_k, ξ_k) and (u, ξ)). In view of the definition of ε_k , we have $\lim_k \|u_k - \bar{u}\|_{p,Q} = \lim_k \|\xi_k - \bar{\xi}\|_{p,Q} = 0$. Indeed, $\mathcal{L}^{n+1}(\{(x, t) \mid u_k(x, t) \neq \bar{u}(x, t)\}) + \mathcal{L}^{n+1}(\{(x, t) \mid \xi_k(x, t) \neq \bar{\xi}(x, t)\}) \leq \frac{1}{k^{2p}}$, and $|u_k(x, t) - \bar{u}(x, t)| \leq k$, $|\xi_k(x, t) - \bar{\xi}(x, t)| \leq k$ a.e. on Q . Thus $\|u_k - \bar{u}\|_{p,Q} \leq \frac{1}{k}$, $\|\xi_k - \bar{\xi}\|_{p,Q} \leq \frac{1}{k}$.

To exploit the approximate optimality conditions (4.15), we introduce a particular perturbation of (u_k, ξ_k) .

Step 3: Diffuse perturbations. For fixed (u_o, ξ_o) in $U_{ad} \times V_{ad}$, we denote by (u_{ok}, ξ_{ok}) ($k > 0$) the pair of functions in $U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k)$ defined by

$$(4.16a) \quad u_{ok}(x, t) = \begin{cases} u_o(x, t) & \text{if } |u_o(x, t) - \bar{u}(x, t)| \leq k, \\ \bar{u}(x, t) & \text{if not,} \end{cases}$$

$$(4.16b) \quad \xi_{ok}(x, t) = \begin{cases} \xi_o(x, t) & \text{if } |\xi_o(x, t) - \bar{\xi}(x, t)| \leq k, \\ \bar{\xi}(x, t) & \text{if not.} \end{cases}$$

Observe that for every $k \geq 1$, (u_{ok}, ξ_{ok}) belongs to $U_{ad}(\bar{u}, k) \times V_{ad}(\bar{\xi}, k)$, and that $(u_{ok}, \xi_{ok})_k$ converges to (u_o, ξ_o) in $L^p(Q) \times L^p(Q)$. Applying Theorem 4.1 of [7] (see also [24, 21] for more details), we deduce the existence of measurable sets E_ρ^k with $\mathcal{L}^{n+1}(E_\rho^k) = \rho \mathcal{L}^{n+1}(Q)$, such that if we denote by (u_k^ρ, ξ_k^ρ) the pair of controls defined by

$$(4.17) \quad u_k^\rho(x, t) = \begin{cases} u_k(x, t) & \text{on } Q \setminus E_\rho^k, \\ u_{ok}(x, t) & \text{on } E_\rho^k, \end{cases} \quad \xi_k^\rho(x, t) = \begin{cases} \xi_k(x, t) & \text{on } Q \setminus E_\rho^k, \\ \xi_{ok}(x, t) & \text{on } E_\rho^k, \end{cases}$$

and if y_k^ρ is the state corresponding to (u_k^ρ, ξ_k^ρ) , then we have

$$(4.18a) \quad y_k^\rho = y_k + \rho z_k + r_k^\rho, \quad \lim_{\rho \rightarrow 0} \frac{1}{\rho} |r_k^\rho|_{\mathcal{C}(\bar{Q})} = 0,$$

$$(4.18b) \quad J(y_k^\rho, u_k^\rho) = J(y_k, u_k) + \rho \Delta_k J + o(\rho),$$

$$(4.18c) \quad \int_Q y_k^\rho \xi_k^\rho dx dt = \int_Q y_k \xi_k dx dt + \rho \int_Q [z_k \xi_k + y_k(\xi_{ok} - \xi_k)] dx dt + o(\rho),$$

where z_k is the weak solution of

$$\frac{\partial z_k}{\partial t} + Az_k + f'_y(y_k)z_k = u_k - u_{ok} + \xi_k - \xi_{ok} \quad \text{in } Q, \quad z_k = 0 \quad \text{on } \Sigma, \quad z_k(0) = 0 \quad \text{in } \Omega,$$

and

$$\Delta_k J = \int_Q [F'_y(x, t, y_k, u)z_k + F(x, t, y_k, u_{ok}) - F(x, t, y_k, u_k)] dx dt + \int_\Omega L'_y(x, y_k(T))z_k(T) dx.$$

Setting $(u, \xi) = (u_k^\rho, \xi_k^\rho)$ in (4.15b), it follows that

$$(4.19) \quad \limsup_{\rho \rightarrow 0} \frac{J_{\varepsilon_k}(y_k, u_k, \xi_k) - J_{\varepsilon_k}(y_k^\rho, u_k^\rho, \xi_k^\rho)}{\rho} \leq \varepsilon_k \mathcal{L}^{n+1}(Q).$$

Taking (4.18) and the definition of J_{ε_k} into account, we get

$$(4.20) \quad -\nu_k \Delta_k J - \langle \mu_k, z_k \rangle_{\overline{Q}} - \langle \theta_k, \Phi'(y_k)z_k \rangle_{\overline{Q}} - \lambda_k [\langle \xi_k, z_k \rangle_{\overline{Q}} + \langle y_k, \xi_{ok} - \xi_k \rangle_{\overline{Q}}] \leq \varepsilon_k \mathcal{L}^{n+1}(Q),$$

where

$$\nu_k = \frac{(J(y_k, u_k) - J(\bar{y}, \bar{u}) + \varepsilon_k^2)^+}{J_{\varepsilon_k}(y_k, u_k, \xi_k)}, \quad \lambda_k = \frac{\left(\int_Q y_k(x, t)\xi_k(x, t) dx dt \right)}{J_{\varepsilon_k}(y_k, u_k, \xi_k)},$$

$$\mu_k = \begin{cases} \frac{|y_k^-|_{C(\overline{Q})} \nabla |y_k^-|_{C(\overline{Q})}}{J_{\varepsilon_k}(y_k, u_k, \xi_k)} & \text{if } |y_k^-|_{C(\overline{Q})} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_k = \begin{cases} \frac{\delta_C(\Phi(y_k)) \nabla \delta_C(\Phi(y_k))}{J_{\varepsilon_k}(y_k, u_k, \xi_k)} & \text{if } \delta_C(\Phi(y_k)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For every $k > 0$, we consider the weak solution q_k of

$$(4.21) \quad \begin{cases} -\frac{\partial q_k}{\partial t} + A^*q_k + f'_y(y_k)q_k = \nu_k F'_y(x, t, y_k, u_k) + \mu_k|_Q + [\Phi'(y_k)^*\theta_k]|_Q + \lambda_k \xi_k & \text{in } Q, \\ q_k = 0 \quad \text{on } \Sigma, \quad q_k(T) = \nu_k L'_y(x, y_k(T)) + [\Phi'(y_k)^*\theta_k]|_{\overline{\Omega}_T} + \mu_k|_{\overline{\Omega}_T} & \text{in } \Omega, \end{cases}$$

where $\mu_k|_Q$ (resp., $[\Phi'(y_k)^*\theta_k]|_Q$) is the restriction of μ_k (resp., $[\Phi'(y_k)^*\theta_k]$) to Q , and $\mu_k|_{\overline{\Omega}_T}$ (resp., $[\Phi'(y_k)^*\theta_k]|_{\overline{\Omega}_T}$) is the restriction of μ_k (resp., $[\Phi'(y_k)^*\theta_k]$) to $\Omega \times \{T\}$. By using the Green formula of Proposition 4.1 with z_k , we obtain

$$\int_Q \nu_k F'_y(x, t, y_k, u_k)z_k dx dt + \lambda_k \int_Q z_k(x, t)\xi_k(x, t) dx dt + \int_\Omega \nu_k L'_y(x, y_k(T))z_k(T) dx + \langle \mu_k, z_k \rangle_{\overline{Q}} + \langle \theta_k, \Phi'(y_k)z_k \rangle_{\overline{Q}} = \int_Q q_k (u_{ok} - u_k + \xi_{ok} - \xi_k) dx dt.$$

With this equality, (4.20), and the definition of $\Delta_k J$, we get

$$\begin{aligned}
 & \int_Q [\nu_k F(x, t, y_k, u_k) + q_k u_k + q_k \xi_k + \lambda_k y_k \xi_k] ds dt \\
 & \leq \int_Q [\nu_k F(s, t, y_k, u_{ok}) + q_k u_{ok} + q_k \xi_{ok} + \lambda_k y_k \xi_{ok}] ds dt + \frac{1}{k^{2p}} \mathcal{L}^{n+1}(Q)
 \end{aligned}
 \tag{4.22}$$

for every $k > 0$ and every $(u_o, \xi_o) \in U_{ad} \times V_{ad}$ (where (u_{ok}, ξ_{ok}) is defined with respect to (u_o, ξ_o)).

Step 4. Convergence of sequence $(\nu_k, \lambda_k, \mu_k, \theta_k, q_k)_k$; Pontryagin principle. Observing that $\nu_k^2 + \lambda_k^2 + |\mu_k|_{\mathcal{M}(\bar{Q})}^2 + |\theta_k|_{\mathcal{M}(\bar{Q})}^2 = 1$, there exist $(\bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\theta}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{M}(\bar{Q}) \times \mathcal{M}(\bar{Q})$ and a subsequence, still denoted by $(\nu_k, \lambda_k, \mu_k, \theta_k)_k$, such that

$$\nu_k \rightharpoonup \bar{\nu}, \quad \lambda_k \rightharpoonup \bar{\lambda}, \quad \mu_k \rightharpoonup \bar{\mu} \text{ and } \theta_k \rightharpoonup \bar{\theta} \text{ weak* in } \mathcal{M}(\bar{Q}).$$

With the same arguments as in [21, Section 6.2, Step 4], we prove that $(q_k)_k$, or at least a subsequence of $(q_k)_k$, weakly converges to \bar{q} in $L^{\delta'}(0, T; W_0^{1, d'}(\Omega))$ for every (δ, d) such that $\frac{n}{2d} + \frac{1}{\delta} < \frac{1}{2}$. Recall that $(u_k, \xi_k)_k$ converges to $(\bar{u}, \bar{\xi})$ in $L^p(Q) \times L^p(Q)$. Hence y_k also converges to \bar{y} . Passing to the limit when k tends to infinity in (4.22) gives

$$\int_Q [H_1(x, t, \bar{y}, \bar{u}, \bar{q}, \bar{\nu}) + H_2(\bar{y}, \bar{\xi}, \bar{q}, \bar{\lambda})] dx dt \leq \int_Q [H_1(x, t, \bar{y}, u, \bar{q}, \bar{\nu}) + H_2(\bar{y}, \xi, \bar{q}, \bar{\lambda})] dx dt$$

for every $(u, \xi) \in U_{ad} \times V_{ad}$. This inequality is equivalent to

$$\begin{aligned}
 (4.23a) \quad & \int_Q H_1(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{q}(x, t), \bar{\nu}) dx dt \\
 & = \min_{u \in U_{ad}} \int_Q H_1(x, t, \bar{y}(x, t), u(x, t), \bar{q}(x, t), \bar{\nu}) dx dt
 \end{aligned}$$

$$(4.23b) \quad \int_Q H_2(\bar{y}(x, t), \bar{\xi}(x, t), \bar{q}(x, t), \bar{\lambda}) dx dt = \min_{\xi \in V_{ad}} \int_Q H_2(\bar{y}(x, t), \xi(x, t), \bar{q}(x, t), \bar{\lambda}) dx dt.$$

Now, by using Lebesgue’s points argument (see [21, 24]), we obtain (4.10e) and (4.10f). On the other hand, it is clear that $\bar{\nu} \geq 0$. Moreover, from the definitions of μ_k and θ_k , we deduce

$$(4.24) \quad \langle \mu_k, z - y_k \rangle_{\bar{Q}} \leq 0 \quad \forall z \in \{z \in \mathcal{C}(\bar{Q}) \mid z \geq 0\} \quad \text{and} \quad \langle \theta_k, z - \Phi(y_k) \rangle_{\bar{Q}} \leq 0 \quad \forall z \in \mathcal{C}.$$

When k tends to infinity, we obtain (4.10b) and a part of (4.10a). It remains to prove that $(\bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\theta})$ is nonzero; for this, we recall that $\nu_k^2 + \lambda_k^2 + |\mu_k|_{\mathcal{M}(\bar{Q})}^2 + |\theta_k|_{\mathcal{M}(\bar{Q})}^2 = 1$.

If $(\bar{\nu}, \bar{\lambda}) \neq 0$, then the proof is complete. If not, we can prove that $|\bar{\mu}|_{\mathcal{M}(\bar{Q})} + |\bar{\theta}|_{\mathcal{M}(\bar{Q})} > 0$.

First we recall that \mathcal{C} has a finite codimension in $\mathcal{C}(\overline{Q})$ and that $\{z \in \mathcal{C}(\overline{Q}) \mid z \geq 0\}$ is a subset of $\mathcal{C}(\overline{Q})$ with a nonempty interior. Then $\tilde{\mathcal{C}}$ is a subset of $\mathcal{C}(\overline{Q}) \times \mathcal{C}(\overline{Q})$ with a finite codimension. Moreover, from (4.24) we deduce that for every $(z_1, z_2) \in \tilde{\mathcal{C}}$

$$\begin{aligned} \langle \mu_k, z_2 - \bar{y} \rangle_{\overline{Q}} + \langle \theta_k, z_1 - \Phi(\bar{y}) \rangle_{\overline{Q}} &\leq \langle \mu_k, y_k - \bar{y} \rangle_{\overline{Q}} \\ &+ \langle \theta_k, \Phi(y_k) - \Phi(\bar{y}) \rangle_{\overline{Q}} \leq |y_k - \bar{y}|_{\mathcal{C}(\overline{Q})} + |\Phi(y_k) - \Phi(\bar{y})|_{\mathcal{C}(\overline{Q})}. \end{aligned}$$

The last right-hand side quantity tends to 0 as $k \rightarrow +\infty$. With this estimate and using $\lim_k |\mu_k|_{\mathcal{M}(\overline{Q})} + \lim_k |\theta_k|_{\mathcal{M}(\overline{Q})} = 1$, thanks to Lemma 3.6 of [16], we conclude that $(\bar{\mu}, \bar{\theta}) \neq 0$ when $(\bar{\nu}, \bar{\lambda}) = 0$. \square

5. Examples. Let us consider the following optimal control problem where the cost functional is defined by

$$(5.1) \quad J(y, u) = \int_0^T [g(t, y(t)) + h(u(t))] dt + \psi(y(T)),$$

where

(A5*) the function $h : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous and there exist $c_1 > 0, c_2 \in \mathbb{R}$ such that

$$(5.2) \quad \forall u \in L^2(\Omega) \quad h(u) \geq c_1 |u|_{L^2(\Omega)}^2 - c_2,$$

(A6*) the function $g : [0, T] \times L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is measurable in $t, g(\cdot, 0) \in L^1(0, T)$, and for every $r > 0$ there exists $\gamma_r > 0$ independent of t such that

$$(5.3) \quad \begin{aligned} \forall t \in [0, T] \quad |y|_{L^2(\Omega)} + |z|_{L^2(\Omega)} &\leq r, \\ |g(t, y) - g(t, z)| + |\psi(y) - \psi(z)| &\leq \gamma_r |y - z|_{L^2(\Omega)}. \end{aligned}$$

Conditions on g and ψ could be weakened. For more details one can refer to Barbu [2, p. 317].

Now we consider

$$(P) \quad \begin{cases} \text{Minimize } J(y(y_o, u), u), \\ u \in U_{ad}, \\ y(y_o, u) \text{ is the solution of (4.3)}, \end{cases}$$

where U_{ad} is a nonempty, convex subset of $L^p(Q)$, closed for the $L^2(Q)$ -topology, and p is an integer such that $n < p$. Although we are especially interested in optimality conditions for solutions of problem (P), we give an existence result in the following theorem.

THEOREM 5.1. *For any $y_o \in K_o$ (defined by (4.1)), problem (P) has at least one solution u . Moreover, the corresponding state belongs to $\mathcal{C}(\overline{Q}) \cap W^{2,1,p}(Q)$.*

Proof. One can find this result in Barbu [2, Proposition 1.1., p. 319] when $U_{ad} = L^2(Q)$. This is easily adapted to the case where U_{ad} is a closed convex subset of $L^2(Q)$. A priori estimations do not change so that we get the “suitable” convergence in the “suitable” spaces. The only modification concerns the cluster points of the control sequences. Because U_{ad} is convex and closed for the $L^2(Q)$ -topology these points belong to U_{ad} . Because $U_{ad} \subset L^p(Q)$, we can use regularity results of Theorem 4.1. \square

Remark 5.1. The assumption that U_{ad} has to be a convex subset of $L^p(Q)$ (for some $p > n$) closed for the $L^2(Q)$ -topology may be difficult to ensure: for example

$U_{ad} = L^p(Q)$ is not suitable. However, we give more precise example sets U_{ad} in what follows. Let us refine the example. We set

$$(5.4) \quad J(y, u) = \frac{1}{2} \int_{\Omega} (y(x, T) - z_d(x))^2 dx + \frac{N}{2} \int_Q u(x, t)^2 dx dt$$

(with $N > 0$) so that with the previous notations we get

$$F(x, t, y, u) = \frac{N}{2} u^2, \quad h(u(t)) = \frac{N}{2} \|u(t)\|_{L^2(\Omega)}^2,$$

$$L(x, y) = \frac{1}{2} (y - z_d(x))^2, \quad g(t, y(t)) \equiv 0, \quad \psi(y(T)) = \frac{1}{2} \|y(T) - z_d\|_{L^2(\Omega)}^2.$$

It is easy to see that both (A5*) and (A6*) are fulfilled for such a choice of h, g, ψ . Therefore the optimal control problem

$$(P_2) \quad \begin{cases} \min J(y, u), \\ \frac{\partial y}{\partial t} + Ay + f(y) \geq u \text{ in } Q, \quad y = 0 \text{ on } \Sigma, \quad y(0) = y_o \text{ in } \Omega, \\ u \in U_{ad}, \\ y(x, t) \geq 0 \quad \forall (x, t) \in \bar{Q}, \end{cases}$$

where $y_o \in W_o^{1,p}(\Omega), y_o \geq 0, z_d \in L^2(\Omega)$, and U_{ad} is a nonempty, convex subset of $L^p(Q)$ closed for the $L^2(Q)$ -topology, has an optimal solution.

We always assume, of course, that (A1) and (A2) are valid (one may choose $A = -\Delta$ for instance, where Δ is the Laplacian operator); we have already seen that (A3) and (A4) are fulfilled with the special choice of φ and y_o . It is also easy to see that (A5) and (A6) are ensured with F and L defined as above. Thus we may give optimality conditions for (P_2) , as follows.

THEOREM 5.2. *Assume (A1) and (A2) are valid. Then problem (P_2) has an optimal solution $(\bar{y}, \bar{u}) \in [W^{2,1,p}(Q) \cap C(\bar{Q})] \times L^p(Q)$. Moreover, there exist $(\bar{v}, \bar{\lambda}, \bar{\mu}, \bar{q}) \in \mathbb{R} \times \mathbb{R} \times \mathcal{M}(\bar{Q}) \times L^1(0, T; W_o^{1,1}(\Omega))$ such that the following optimality system holds:*

$$(5.5a) \quad (\bar{v}, \bar{\lambda}, \bar{\mu}) \neq 0, \quad \bar{v} \geq 0,$$

$$(5.5b) \quad \forall z \in \{z \in C(\bar{Q}) \mid z \geq 0\}, \quad \langle \bar{\mu}, z - \bar{y} \rangle_{\bar{Q}} \leq 0,$$

$$(5.5c) \quad \begin{cases} \frac{\partial \bar{y}}{\partial t} + A\bar{y} + f(\bar{y}) = \bar{u} + \bar{\xi} & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \quad \bar{y}(0) = y_o & \text{in } \Omega, \end{cases}$$

$$(5.5d) \quad \bar{y} \geq 0, \quad \bar{\xi} \in V_{ad}, \quad \bar{u} \in U_{ad}, \quad \int_{\Omega} \bar{y}(t) \bar{\xi}(t) dx = 0 \quad \text{a.e. on } [0, T],$$

$$(5.5e) \quad \begin{cases} -\frac{\partial \bar{q}}{\partial t} + A^* \bar{q} + f'(\bar{y}) \bar{q} = \bar{\mu}|_Q + \bar{\lambda} \bar{\xi} & \text{in } Q, \\ \bar{q} = 0 & \text{on } \Sigma, \quad \bar{q}(T) = \bar{v}[\bar{y}(T) - z_d] + \bar{\mu}|_{\bar{\Omega}_T} & \text{in } \Omega, \end{cases}$$

$$(5.5f) \quad [(\bar{\nu}N\bar{u} + \bar{q})(u - \bar{u})](x, t) \leq 0 \quad \forall u \in U_{ad}, \quad \text{and a.e. } (x, t) \in Q,$$

$$(5.5g) \quad \bar{q}(x, t) \bar{\xi}(x, t) = 0 \quad \text{a.e. } (x, t) \in Q,$$

where $\bar{\xi} = \frac{\partial \bar{y}}{\partial t} + A\bar{y} + f(\bar{y}) - \bar{u}$.

Proof. This is a direct consequence of Theorem 4.2 where $\Phi = Id$ and \mathcal{C} is the whole space. Considering the Hamiltonian functions and relations (4.10e) and (4.10f) give (5.5e) and (5.5f) immediately. \square

We end this section with two examples for U_{ad} .

5.1. Case where U_{ad} is bounded in $L^\infty(Q)$. Let us set

$$U_{ad} = \{ u \in L^\infty(Q) \mid a(x, t) \leq u(x, t) \leq b(x, t) \text{ in } Q \},$$

where $a, b \in L^\infty(Q)$. U_{ad} is of course a convex subset of $L^p(Q)$ for any $p > n$. Moreover, we get the following lemma.

LEMMA 5.1. *U_{ad} is closed for the $L^2(Q)$ -topology.*

Proof. Let $u_n \in U_{ad}$ converging to u in $L^2(Q)$. Then $u_n(x, t)$ converges to $u(x, t)$ a.e. in Q so that we get $a(x, t) \leq u(x, t) \leq b(x, t)$ a.e. in Q . Thus $u \in L^\infty(Q)$. It is clear that $u \in U_{ad}$. \square

Therefore, in view of Remark 5.1, we get the result stated in the next theorem for $y_o = 0$ and

$$J(y, u) = \frac{1}{2} \int_{\Omega} (y(x, T) - z_d(x))^2 dx + \frac{N}{2} \int_Q u^2(x, t) dx dt.$$

THEOREM 5.3. *Assume (A1) and (A2) are valid. Then problem (\mathcal{P}_2) has an optimal solution $(\bar{y}, \bar{u}) \in [W^{2,1,p}(Q) \cap \mathcal{C}(\bar{Q})] \times L^p(Q)$ for any $p > n$. Moreover there exists $(\bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{q}) \in \mathbb{R} \times \mathbb{R} \times \mathcal{M}(\bar{Q}) \times L^1(0, T; W_o^{1,1}(\Omega))$ such that (5.5a)–(5.5d) and (5.5g) hold with*

$$(5.6) \quad \begin{cases} -\frac{\partial \bar{q}}{\partial t} + A^* \bar{q} + f'(\bar{y}) \bar{q} = \bar{\mu}|_Q + \bar{\lambda} \bar{\xi} & \text{in } Q, \\ \bar{q} = 0 & \text{on } \Sigma, \quad \bar{q}(T) = \bar{\nu}[\bar{y}(T) - z_d] + \bar{\mu}|_{\bar{\Omega}_T} & \text{in } \Omega, \end{cases}$$

$$(5.7) \quad [(\bar{\nu}N\bar{u} + \bar{q})(u - \bar{u})](x, t) \leq 0 \quad \forall u \in U_{ad} \quad \text{and a.e. } (x, t) \in Q.$$

5.2. Case where $U_{ad} = \{u \in L^p(Q) \mid u(x, t) \geq 0 \text{ a.e. in } Q\}$. When $U_{ad} = \{u \in L^p(Q) \mid u(x, t) \geq 0 \text{ a.e. in } Q\}$ and $y_o \geq 0$ in Ω , thanks to the maximum principle for parabolic equations, the constraint $y \geq 0$ is automatically fulfilled in (4.6b) so that the corresponding multiplier $\bar{\mu}$ is equal to 0 (or at least does not appear). Therefore the corresponding Pontryagin optimality system consists of (5.5a) and (5.5c)–(5.5g), where (5.5e) is replaced by

$$(5.8) \quad \begin{cases} -\frac{\partial \bar{q}}{\partial t} + A^* \bar{q} + f'(\bar{y}) \bar{q} = \bar{\lambda} \bar{\xi} & \text{in } Q, \\ \bar{q} = 0 & \text{on } \Sigma, \quad \bar{q}(T) = \bar{\nu}[\bar{y}(T) - z_d] & \text{in } \Omega. \end{cases}$$

This implies in particular that $\bar{q} \in W^{2,1,p}(Q) \cap \mathcal{C}(\bar{Q})$.

For this simple example we can see that the optimality conditions (5.2) are not trivial because we cannot have $\bar{\nu} = \bar{\lambda} = 0$.

6. Conclusion. The optimality conditions we have obtained are given in a non-qualified form. So far it is difficult to compare precisely these results with those already existing, since they usually are in a qualified form [6, 5, 17] or they concern elliptic variational inequalities. Nevertheless we must emphasize that in this paper we obtain interesting informations about optimal solutions (at least in simple cases). Indeed, we have seen in Example 5 that (5.5e) provides precise information on the structure of the multipliers $\bar{\mu} + \bar{\xi} \bar{\lambda}$ for the distributed multiplier, for instance, and the adjoint state \bar{q} : the regular part of this adjoint state belongs to $\mathcal{C}(\bar{Q})$ while the nonsmooth part belongs to $L^1(0, T; W_o^{1,1}(\Omega))$. This information seems new, compared with that in Barbu [2, Section 5.1.4, p. 331], for example.

The method developed in [5, 23] for elliptic variational inequalities is still true for the parabolic case, but we think that this method does not allow the condition (4.11b) to be obtained. However, in [23, 5] the authors give a qualification assumption under which they can derive Pontryagin's principle in qualified form.

Since we now can preview the generic form of the Lagrange multipliers, we can check optimal control problems where the variational inequality is more general than the obstacle type or occurs on the boundary, with boundary control.

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