Hamiltonian Pontryagin's Principles for Control Problems Governed by Semilinear Parabolic Equations.

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Abstract

In this paper we study optimal control problems governed by semilinear parabolic equations. We obtain necessary optimality conditions in the form of an exact Pontryagin's minimum principle for distributed and boundary controls (which can be unbounded) and bounded initial controls. These optimality conditions are obtained thanks to new regularity results for linear and nonlinear parabolic equations.

Keywords. Optimal control, nonlinear boundary controls, semilinear parabolic equations, Pontryagin's minimum principle, unbounded controls.

1 Introduction.

In this paper we consider optimal control problems governed by semilinear parabolic equations with nonlinear boundary conditions. We obtain optimality conditions in the form of three decoupled Pontryagin minimum principles, one for distributed controls, the second one for boundary controls and the last one for initial controls.
The proof of these optimality conditions requires some new regularity results for linear and nonlinear parabolic equations, that we obtain in Section 3. Let us stress on that in nonlinear equations considered in Section 3, the nonlinear term in boundary conditions is neither monotone nor Lipschitz continuous. Moreover the distributed and the boundary controls are not necessarily bounded. To deal with such equations we first study linear equations in which some coefficients are not bounded, but only bounded from below (see Propositions 3.3 and 3.4). Since we plan to consider control problems with pointwise state constraints in future papers, we look for C^0-regularity results for the state variables when control variables belong to appropriate Lebesgue spaces. A first step to prove C^0-regularity results is often to prove L^∞-estimates.

There are considerable contributions to the study of second order parabolic equations (see for example [22], [11], [19]). However we think that the regularity results stated in Section 3 for linear or nonlinear equation are new (Proposition 3.3, Proposition 3.4, Theorem 3.1). Indeed for linear equations, L^∞-estimates can be obtained by the maximum principle ([22], Chapter 1), by a truncation method ([22], Chapter 3), ([11], Chapter 5). These results deal with Dirichlet boundary conditions or for Robin boundary conditions in the case of bounded coefficients and bounded data. Other results are given in [24] still in the case of bounded coefficients and bounded data. In the case of unbounded data, with Robin boundary conditions and an unbounded coefficient of order zero (which can be negative) in this boundary condition, the truncation method cannot be used

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to derive L^∞-estimates. Here we use a semigroup approach coupled with a comparison principle (Proposition 3.1 and Proposition 3.2). For nonlinear equations some nonmonotone perturbations are considered in [2], [3] but not in the form considered here (it is often supposed that the nonlinear boundary term $g(\cdot, y)$ satisfies $g(\cdot, y)y \geq -c$ for $|y|$ big enough). We suppose here (in the case when $y \longmapsto g(\cdot, y)$ is differentiable) that the nonlinear term in boundary conditions satisfies $g'_y(\cdot, y) \geq -c$, which corresponds to $g(\cdot, y)y \geq -c|y|^2$ for $|y|$ big enough. We next use these regularity results to establish in Section 4 some convergence properties for the state and adjoint equations when control variables are replaced by sequence of spike perturbations. We next prove pointwise Pontryagin principles.

Except for convex control problems [31] (with a convex cost functional and a linear state equation), the study of problems with unbounded controls seems to be recent. Their interest is clarified in a paper by H. O. Fattorini [15]. In the nonconvex case, for monotone equations we refer to [5]. Other optimality conditions for problems governed by abstract evolution equations are obtained in [12], [13], [14], [16]. For such problems, the Pontryagin principle is stated as for problems governed by ordinary differential equations except that the Hamiltonian is a functional defined on Banach spaces of infinite dimension. These results cannot be used for problems with nonlinear boundary controls as those considered here. Very recently H. O. Fattorini and T. Murphy have obtained a Pontryagin principle for problems with Dirichlet boundary condition [17] or with Neumann or Robin boundary conditions [18], in the presence of a terminal state constraint. They also prove C^0-regularity results for equations similar to those considered here, by using estimates on the Neumann function of the heat equation.

Let us finally mention that applications of results presented here, to optimal control problems with pointwise state constraints, are considered in [27] and [28].

2 Definition of the problem. Main results.

In all the sequel Ω denotes an open bounded subset in $\mathbb{R}^N$ ($N \geq 2$) of class $C^{2,\beta}$ for some $\beta > 0$ (that is, the boundary Γ of Ω is an $(N-1)$-dimensional manifold of class $C^{2,\beta}$ such that Ω lies locally on one side of Γ). A function is of class $C^{2,\beta}$ if it is of class C^2 and if its partial derivatives of second order are Hölder continuous of order β. An analogous definition takes place for $C^{1,\beta}$. We denote by q, r positive numbers satisfying:

$$q > N/2 + 1 \quad \text{and} \quad r > N + 1.$$

We consider a second order differential operator defined by:

$$Ay(x) = -\sum_{i,j=1}^{N} D_i(a_{ij}(x) D_j y(x))$$

with coefficients a_{ij} belonging to $C^{1,\beta}(\overline{\Omega})$ and satisfying the conditions:

$$a_{ij}(x) = a_{ji}(x) \qquad \text{for every } i, j \in \{1, ..., N\},$$

$$m_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \leq M|\xi|^2 \tag{1}$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$, with $0 < m_0 \leq M$, (D_i denotes the partial derivative with respect to x_i).

2.1 Notation.

Let T be a fixed positive constant. As usual we denote by Q the cylinder $\Omega\times]0,T[$ and by Σ the lateral surface $\Gamma\times]0,T[$. For every $1 \le \tau,\tau_1 \le \infty$, the norms in the spaces $L^\tau(\Omega)$, $L^\tau(\Gamma)$, $L^\tau(Q)$, $L^\tau(\Sigma)$, $L^\tau(0,T;L^{\tau_1}(\Omega))$, $L^\tau(0,T;L^{\tau_1}(\Gamma))$ will be denoted by $\|\cdot\|_{\tau,\Omega}$, $\|\cdot\|_{\tau,\Gamma}$, $\|\cdot\|_{\tau,Q}$, $\|\cdot\|_{\tau,\Sigma}$, $\|\cdot\|_{\tau,\tau_1,\Omega}$, $\|\cdot\|_{\tau,\tau_1,\Gamma}$. The Hilbert space $W(0,T;H^1(\Omega),(H^1(\Omega))') = \{y \in L^2(0,T;H^1(\Omega)) \mid \frac{dy}{dt} \in L^2(0,T;(H^1(\Omega))')\}$ will be denoted by $W(0,T)$.
Even if our notation is not usual, for every $0 \le t_1 \le t_2 \le T$, we denote by $C(\overline{Q}_{t_1,t_2})$ the space

$$C(\overline{Q}_{t_1,t_2}) = \{y \in L^2(Q) \mid y \text{ is continuous on } \overline{\Omega}\times[t_1,t_2]\},$$

and we define the seminorm: $\|y\|_{C(\overline{Q}_{t_1,t_2})} = \sup\{|y(x,t)| \mid (x,t) \in \overline{\Omega}\times[t_1,t_2]\}$. It is clear that $C(\overline{Q}_{t_1,t_2})$ is a Banach space with the norm $\|\cdot\|_{L^2(Q)} + \|\cdot\|_{C(\overline{Q}_{t_1,t_2})}$.

In all the sequel, we denote by C_i, for $i \in N^*$, constants that intervene in the estimates in the various propositions, while the letters K or K_i, $i \in N$, throughout the proofs denote various constants depending on known quantities.

2.2 State equation.

We consider a control system described by the parabolic equation:

$$\begin{aligned}
&\frac{\partial y}{\partial t} + Ay + f(x,t,y,u) = 0 && \text{in } Q,\\
&\frac{\partial y}{\partial n_A} + g(s,t,y,v) = 0 && \text{on } \Sigma,\\
&y(\cdot,0) = w && \text{in } \Omega,
\end{aligned} \tag{2}$$

where $\frac{\partial y}{\partial n_A}(s,t) = \sum_{i,j} a_{ij}(s)D_j y(s,t)n_i(s)$ is the conormal derivative of y associated with A, $n = (n_1,...,n_N)$ is the outward unit normal to Γ. The control variables u, v and w respectively belong to $L^q(Q)$, $L^r(\Sigma)$ and $L^\infty(\Omega)$.

We make the following assumptions on f and g.

(A1) - For every $(y,u) \in \mathbb{R}^2$, $f(\cdot,y,u)$ is measurable on Q. For almost every $(x,t) \in Q$, $f(x,t,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$. For almost every $(x,t) \in Q$ and every $u \in \mathbb{R}$, $f(x,t,\cdot,u)$ is of class C^1 on $\mathbb{R}$. The following estimates are verified

$$|f(x,t,0,u)| \le M_1(x,t) + m_1|u|,$$

$$C_0 \le f'_y(x,t,y,u) \le (M_1(x,t) + m_1|u|)\eta(|y|),$$

where M_1 belongs to $L^q(Q)$, η is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $m_1 \in \mathbb{R}^+$ and $C_0 \in \mathbb{R}$.

(A2) - For every $(y,v) \in \mathbb{R}^2$, $g(\cdot,y,v)$ is measurable on Σ. For almost every $(s,t) \in \Sigma$, $g(s,t,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbb{R}$, $g(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. The following estimates are verified

$$|g(s,t,0,v)| \le M_2(s,t) + m_1|v|,$$

$$C_0 \le g'_y(s,t,y,v) \le (M_2(s,t) + m_1|v|)\eta(|y|),$$

where M_2 belongs to $L^r(\Sigma)$, C_0, m_1 and η are as in $(A1)$.

Remark 2.1. Thanks to (A1)-(A2), we prove in Section 3.2 the existence and uniqueness in $W(0,T)\cap L^\infty(Q)$ of a weak solution for (2). In [18] existence of local solutions, for some nonlinear heat equations, is proved when the nonlinearity is locally Lipschitz continuous with respect to y. Here if we replace the above estimates on f'_y and g'_y by

$$C_0\eta(|y|) \le f'_y(x,t,y,u) \le (M_1(x,t)+m_1|u|)\eta(|y|), \quad C_0\eta(|y|) \le g'_y(s,t,y,v) \le (M_2(s,t)+m_1|v|)\eta(|y|),$$

and if $y_0 \in C(\overline{\Omega})$, then we can also prove that (2) admits a solution in $W(0,\tilde{T})\cap C(\overline{\Omega}\times[0,\tilde{T}])$ for some $0<\tilde{T}\le T$.

2.3 The control problem.

We consider the functional J defined on $(W(0,T)\cap L^\infty(Q))\times L^q(Q)\times L^r(\Sigma)\times L^\infty(\Omega)$ by

$$J(y,u,v,w) = \int_Q F(x,t,y,u)\,dxdt + \int_\Sigma G(s,t,y,v)\,dsdt + \int_\Omega \ell(x,y(x,T),w(x))dx, \tag{3}$$

where $F,\ G,\ \ell$ satisfy the following assumptions.

(A3) - For every $(y,w)\in\mathbb{R}^2$, $\ell(\cdot,y,w)$ is measurable on Ω. For almost every $x\in\Omega$, $\ell(x,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$. For almost every $x\in\Omega$ and every $w\in\mathbb{R}$, $\ell(x,\cdot,w)$ is of class C^1 on $\mathbb{R}$. The following estimates are verified

$$|\ell(x,0,w)| \le M_3(x)(1+|w|), \qquad |\ell'_y(x,y,w)| \le (M_4+m_1|w|)\eta(|y|),$$

where $M_3\in L^1(\Omega)$, $M_4\in\mathbb{R}^+$ and m_1,η are as in (A1).

(A4) - For every $(y,u)\in\mathbb{R}^2$, $F(\cdot,y,u)$ is measurable on Q. For almost every $(x,t)\in Q$, $F(x,t,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$. For almost every $(x,t)\in Q$ and every $u\in\mathbb{R}$, $F(x,t,\cdot,u)$ is of class C^1 on $\mathbb{R}$. The following estimates are verified

$$|F(x,t,0,u)| \le M_5(x,t)+m_1|u|^q, \qquad |F'_y(x,t,y,u)| \le (M_1(x,t)+m_1|u|)\eta(|y|),$$

where $M_5\in L^1(Q)$, M_1,m_1 and η are as in $(A1)$.

(A5) - For every $(y,v)\in\mathbb{R}^2$, $G(\cdot,y,v)$ is measurable on Σ. For almost every $(s,t)\in\Sigma$, $G(s,t,\cdot,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$. For almost every $(s,t)\in\Sigma$ and every $v\in\mathbb{R}$, $G(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. The following estimates are verified

$$|G(s,t,0,v)| \le M_6(s,t)+m_1|v|^r, \qquad |G'_y(s,t,y,v)| \le (M_2(s,t)+m_1|v|)\eta(|y|),$$

where $M_6\in L^1(\Sigma)$, M_2,m_1 and η are as in $(A2)$.

The sets of constraints on u, v and w are defined by

$$U_{ad} = \{u\in L^q(Q)\mid u(x,t)\in K_U \text{ for a.e. } (x,t)\in Q\},$$

$$V_{ad} = \{v\in L^r(\Sigma)\mid v(s,t)\in K_V \text{ for a.e. } (s,t)\in \Sigma\},$$

$$W_{ad} = \{w\in L^\infty(\Omega)\mid w(x)\in K_W \text{ for a.e. } x\in \Omega\},$$

where K_U, K_V and K_W are nonempty closed subsets in $\mathbb{R}$.

We study the control problem:

$$(P) \qquad \inf\{J(y,u,v,w) \mid (y,u,v,w) \in (W(0,T)\cap L^\infty(Q))\times U_{ad}\times V_{ad}\times W_{ad}, (y,u,v,w) \text{ satisfies (2)}\}.$$

Remark 2.2. Thanks to (A3)-(A5) and to Theorem 3.1, the infimum of (P) will be finite if we suppose for example that

$$F(x,t,y,u) \geq m_2|u|^q - m_3|y|^{\tau_1}, \quad G(s,t,y,v) \geq m_4|v|^r - m_5|y|^{\tau_2}, \quad K_W \text{ is bounded in } \mathbb{R},$$

where m_i (for $2 \leq i \leq 5$) are positive constants and $1 \leq \tau_1 < \min(q,r)$, $1 \leq \tau_2 < \min(q,r)$.

2.4 Pontryagin minimum principles.

We define a distributed Hamiltonian function, a boundary Hamiltonian function and an initial Hamiltonian function by:

$$H_Q(x,t,y,u,p) = F(x,t,y,u) - pf(x,t,y,u)$$

for every $(x,t,y,u,p) \in \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$,

$$H_\Sigma(s,t,y,v,p) = G(s,t,y,v) - pg(s,t,y,v)$$

for every $(s,t,y,v,p) \in \Gamma\times]0,T[\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and

$$H_\Omega(x,y,w,p) = \ell(x,y,w) + pw$$

for every $(x,y,w,p) \in \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

In Section 4, we prove optimality conditions for (P) in the form of three decoupled Pontryagin principles stated in the following Theorem.

Theorem 2.1 *If $(\bar{y},\bar{u},\bar{v},\bar{w})$ is a solution of (P), there then exists $\bar{p} \in W(0,T)\cap C_b(\bar{Q}\setminus(\bar{\Omega}\times\{T\}))$ satisfying the equation*

$$\begin{aligned}
&-\frac{\partial p}{\partial t} + Ap + f'_y(x,t,\bar{y},\bar{u})p = F'_y(x,t,\bar{y},\bar{u}) && \text{in } Q,\\
&\frac{\partial p}{\partial n_A} + g'_y(s,t,\bar{y},\bar{v})p = G'_y(s,t,\bar{y},\bar{v}) && \text{on } \Sigma,\\
&p(T) = \ell'_y(x,\bar{y},\bar{w}) && \text{in } \Omega,
\end{aligned}$$

and such that:

$$H_Q(x,t,\bar{y}(x,t),\bar{u}(x,t),\bar{p}(x,t)) = \min_{u\in K_U} H_Q(x,t,\bar{y}(x,t),u,\bar{p}(x,t)) \quad \text{for a.e. } (x,t)\in Q,$$

$$H_\Sigma(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{p}(s,t)) = \min_{v\in K_V} H_\Sigma(s,t,\bar{y}(s,t),v,\bar{p}(s,t)) \quad \text{for a.e. } (s,t)\in \Sigma,$$

$$H_\Omega(x,\bar{y}(x,T),\bar{w}(x),\bar{p}(x,0)) = \min_{w\in K_W} H_\Omega(x,\bar{y}(x,T),w,\bar{p}(x,0)) \quad \text{for a.e. } x\in \Omega.$$

3 State equation and linearized state equation.

3.1 Technical results for linear equations.

Let $\tilde{A}$ be the operator defined by

$$D(\tilde{A}) = \{y \in C^2(\overline{\Omega}) \mid \frac{\partial y}{\partial n_A} + k_2 y = 0 \text{ on } \Gamma\}, \qquad \tilde{A}y = Ay + k_1 y \;\text{ for every } y \in D(\tilde{A}),$$

where $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$ are such that

$$\sum_{i,j} \int_\Omega a_{ij}(x) D_j y(x) D_i y(x)\, dx + \int_\Omega k_1 y(x)^2\, dx + \int_\Gamma k_2 y(s)^2\, ds \geq \frac{m_0}{2} ||y||^2_{H^1(\Omega)} \tag{4}$$

for every $y \in D(\tilde{A})$, (m_0 is the constant in (1)).

For $1 \leq l < \infty$, we denote by A_l the closure of $\tilde{A}$ in $L^l(\Omega)$. The operator $(-A_l)$ is the infinitesimal generator of a strongly continuous semigroup $(S_l(t))_{t\geq 0}$. For $1 < l < \infty$, it is well known that this semigroup is analytic ([20], [4]) and the domain of A_l is $D(A_l) = \{y \in W^{2,l}(\Omega) \mid \frac{\partial y}{\partial n_A} + k_2 y = 0 \text{ on } \Gamma\}$. The case $l = 1$ is studied in [4], $(S_1(t))_{t\geq 0}$ is still analytic and the domain of A_1 is the set of functions $y \in L^1(\Omega)$ such that there exists $z \in L^1(\Omega)$ satisfying $\int_\Omega z(x)v(x)\, dx = \int_\Omega y(x)\tilde{A}v(x)\, dx$ for every $v \in D(\tilde{A})$ (remark that here the formal adjoint of $\tilde{A}$ is $\tilde{A}$ itself). Since the spectrum of A_l does not depend on $1 \leq l < \infty$ (see [4], p. 240 and Corollary 9.3), thanks to (4), 0 belongs to the resolvent of $(-A_l)$ and we can define A_l^γ (the γ-power of A_l). For $\gamma > 0$, $A_l^{-\gamma}$ is a bounded operator on $L^l(\Omega)$, it is one-to-one from $L^l(\Omega)$ onto its range; A_l^γ is defined by $A_l^\gamma = (A_l^{-\gamma})^{-1}$, as a closed and unbounded operator in $L^l(\Omega)$ whose domain X_γ^l is the range of $A_l^{-\gamma}$. Still with (4) (and because $\sigma(A_l) = \sigma(A_2)$ for $1 \leq l < \infty$), we have (see [21], Theorem 1.4.3):

$$\|A_l^\gamma S_l(t)\varphi\|_{l,\Omega} \leq C_1 t^{-\gamma} \|\varphi\|_{l,\Omega}, \tag{5}$$

where C_1 only depends on l, γ, N and Ω (let us mention that the estimate (5) holds for any analytic semigroup for which $\operatorname{Re}\sigma(A_l) \geq \delta > 0$). For $l = \infty$, A_l is the closure of $\tilde{A}$ in $C(\overline{\Omega})$, $(S_l(t))_{t\geq 0}$ is the semigroup in $C(\overline{\Omega})$ generated by $(-A_l)$. For every $t \geq 0$ and for $l = \infty$, $S_l(t)$ can be still considered as a continuous linear operator from $L^\infty(\Omega)$ into $L^\infty(\Omega)$ (but $(S_l(t))_{t\geq 0}$ is not a continuous semigroup in $L^\infty(\Omega)$).

Lemma 3.1 *For every $1 \leq l \leq \lambda \leq \infty$ with $l < \infty$, there exists a positive constant $C_2 = C_2(N, l, \lambda, \Omega, k_1, k_2)$ such that:*

$$\|S_l(t)\varphi\|_{\lambda,\Omega} \leq C_2(t^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})}) \|\varphi\|_{l,\Omega}, \tag{6}$$

for every $\varphi \in L^l(\Omega)$ and every $t > 0$. For every $1 \leq l \leq \lambda \leq \infty$ with $l < \infty$, for every $\alpha > 0$, there exists a positive constant $C_3 = C_3(N, l, \lambda, \alpha, \Omega, k_1, k_2)$ such that:

$$\|A_l^\alpha S_l(t)\varphi\|_{\lambda,\Omega} \leq C_3(t^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})-\alpha}) \|\varphi\|_{l,\Omega} \tag{7}$$

for every $\varphi \in L^l(\Omega)$ and every $t > 0$.

Remark 3.1. Here we prove (6) when $l < \infty$. From the proof of Proposition 3.1 (part (d)) it follows that (6) is still true for $l = \lambda = \infty$ (this result follows from estimates deduced from (6), (7) and from a comparison principle). But we cannot prove (7) for $l = \lambda = \infty$.

Proof of Lemma 3.1. The estimate (6) is proved in ([26] Lemma 1 and [4] Proposition 12.5) in the case when $1 \leq l < \lambda \leq \infty$. It can be extended to the case $1 \leq l \leq \lambda \leq \infty$ with $l < \infty$, by using the Hölder's inequality. Thanks to (6), we have:

$$\|A_l^\alpha S_l(t)\varphi\|_{\lambda,\Omega} = \|A_l^\alpha S_l(t/2)S_l(t/2)\varphi\|_{\lambda,\Omega} = \|S_l(t/2)A_l^\alpha S_l(t/2)\varphi\|_{\lambda,\Omega} =$$

$$\leq C_2((t/2)^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})})\|A_l^\alpha S_l(t/2)\varphi\|_{l,\Omega}.$$

According to (5), it results that

$$\|A_l^\alpha S_l(t)\varphi\|_{\lambda,\Omega} \leq C_1 C_2((t/2)^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})-\alpha})\|\varphi\|_{l,\Omega}$$

$$\leq K_1(t^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})-\alpha})\|\varphi\|_{l,\Omega} \tag{8}$$

for every $\alpha > 0$. □

Proposition 3.1 *Let ϕ be in $L^\mu(0,T;L^m(\Omega))$, ψ be in $L^\sigma(0,T;L^\nu(\Gamma))$ and let y_0 be in $L^\infty(\Omega)$. If*

$$\mu > 1, \quad m > 1, \quad \frac{m}{\mu'} > \frac{N}{2}, \quad \frac{\nu(2-\sigma')}{\sigma'} > N-1, \quad \text{and} \quad \sigma > 2,$$

if k_1 and k_2 are constants satisfying (4), then the weak solution in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ of

$$\frac{\partial y}{\partial t} + Ay + k_1 y = \phi \quad \text{in } Q, \qquad \frac{\partial y}{\partial n_A} + k_2 y = \psi \quad \text{on } \Sigma, \qquad y(0) = y_0 \quad \text{in } \Omega, \tag{9}$$

belongs to $L^\infty(Q) \cap C(\overline{Q}_{\varepsilon,T})$ (for every $\varepsilon > 0$) and verifies the estimate:

$$||y||_{\infty,\Sigma} + ||y||_{\infty,Q} \leq C_4(||\phi||_{\mu,m,\Omega} + ||\psi||_{\sigma,\nu,\Gamma} + ||y_0||_{\infty,\Omega}), \tag{10}$$

where $C_4 = C_4(N,\mu,m,\nu,\sigma,\Omega,T,k_1,k_2)$. In particular the estimate (10) is satisfied for $\mu = m = q > \frac{N}{2} + 1$ and $\sigma = \nu = r > N+1$. In this case the weak solution belongs to $W(0,T)$. Moreover, if $y_0 \in C(\overline{\Omega})$, then $y \in C(\overline{Q})$.

Remark 3.2. The existence of a unique weak solution $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ of (9) is a classical result (see [22], Chapter 3, Theorem 5.1).

Proof of Proposition 3.1. We first prove the estimate in $L^\infty(Q)$. Since the equation is linear, we split the proof of this estimate in three parts.

a) If $y_0 = 0$ and $(\phi,\psi) \neq (0,0)$, to get (10) we proceed by duality. Let φ be a regular function and z be the solution of

$$\frac{\partial z}{\partial t} + Az + k_1 z = 0 \text{ in } Q, \ \frac{\partial z}{\partial n_A} + k_2 z = 0 \text{ on } \Sigma, \ z(0) = \varphi \text{ in } \Omega.$$

Since z is regular and $y(x,0) = 0$, we can write:

$$\int_\Omega \varphi(x)y(x,t)\,dx = \int_0^t \left\{\frac{d}{d\tau}\int_\Omega z(x,t-\tau)y(x,\tau)\,dx\right\} d\tau$$

$$= \int_0^t \left\{ \int_\Omega -\frac{\partial z}{\partial t}(x, t-\tau) y(x,\tau) + z(x, t-\tau) \frac{\partial y}{\partial t}(x,\tau)\, dx \right\} d\tau$$

$$= \int_0^t \left\{ \int_\Omega [Az(x, t-\tau) y(x,\tau) - z(x, t-\tau) Ay(x,\tau) + \phi(x,\tau) z(x, t-\tau)]\, dx \right\} d\tau$$

$$= \int_0^t \int_\Omega \phi(x,\tau) z(x, t-\tau)\, dx d\tau + \int_0^t \int_\Gamma \psi(x,\tau) z(x, t-\tau)\, dx d\tau,$$

(since we only need estimates on y, by using density arguments, we can suppose that y is regular so that these calculations are justified).

b) Let us now suppose that $y_0 = 0$ and $\psi = 0$. With (6) in Lemma 3.1, we have:

$$\|z(t-\tau)\|_{m',\Omega} \le C_2 (t-\tau)^{-\frac{N}{2}(1-\frac{1}{m'})} \|\varphi\|_{1,\Omega}.$$

¿From this estimate we deduce:

$$|\int_0^t \int_\Omega z(x, t-\tau)\phi(x,\tau)\, dx d\tau| \le \int_0^t \|\phi(\tau)\|_{m,\Omega}\; C_2 (t-\tau)^{-\frac{N}{2m}}\, d\tau \|\varphi\|_{1,\Omega}$$

$$\le K \|\varphi\|_{1,\Omega} \|\phi\|_{\mu,m,\Omega}\; (t^{1-\frac{\mu' N}{2m}})^{\frac{1}{\mu'}}.$$

Since $\frac{m}{\mu'} > \frac{N}{2}$, from the equality established in (a) and from the previous estimate we get:

$$\|y(t)\|_{\infty,\Omega} = \sup\{ | \int_\Omega \varphi(x) y(x,t)\, dx| \;\; | \;\; \|\varphi\|_{1,\Omega} = 1 \}$$

$$\le K_1 \|\phi\|_{\mu,m,\Omega} \quad \text{for every } t \in [0,T],$$

where K_1 only depends on N, μ, m, Ω, T, k_1 and k_2. If we set $\mu = m$, the condition $\frac{m}{\mu'} > \frac{N}{2}$ gives $\mu = m > \frac{N}{2} + 1$.

c) Now we suppose that $y_0 = 0$ and $\phi = 0$. If $0 < \frac{1}{\rho} < 1 < 2\alpha < 2$ and if $\rho < N$, then the following imbeddings are continuous

$$X_\alpha^\rho \hookrightarrow W^{1,\rho}(\Omega), \qquad W^{1-\frac{1}{\rho},\rho}(\Gamma) \hookrightarrow L^{\frac{(N-1)\rho}{N-\rho}}(\Gamma),$$

where X_α^ρ is the domain of A_ρ^α. The first imbedding can be deduced from [21], Theorem 1.6.1 (see also [25], Theorem 8.4.3). The second imbedding is classical [1], p. 218. The trace mapping is continuous from $W^{1,\rho}(\Omega)$ onto $W^{1-\frac{1}{\rho},\rho}(\Gamma)$. With these imbeddings and with (7) in Lemma 3.1, we get:

$$\|z(t-\tau)\|_{\frac{(N-1)\rho}{N-\rho},\Gamma} \le K (t-\tau)^{-\alpha - \frac{N}{2\rho'}} \|\varphi\|_{1,\Omega}.$$

Thus we have:

$$|\int_0^t \int_\Gamma \psi(x,\tau) z(x, t-\tau)\, dx d\tau| \le K \|\varphi\|_{1,\Omega} \int_0^t (t-\tau)^{-\alpha-\frac{N}{2\rho'}} \|\psi(\tau)\|_{L^{\frac{(N-1)\rho}{N\rho-N}}(\Gamma)}\, d\tau$$

$$\le K \|\varphi\|_{1,\Omega} \|\psi\|_{L^\sigma(0,T;L^{\frac{(N-1)\rho}{N\rho-N}}(\Gamma))} (t^{1-\sigma'(\alpha+\frac{N}{2\rho'})})^{\frac{1}{\sigma'}}.$$

If we proceed as in (b), we obtain

$$\|y(t)\|_{\infty,\Omega} \le K_2 \|\psi\|_{L^\sigma(0,T;L^{\frac{(N-1)\rho}{N\rho-N}}(\Gamma))},$$

when the following conditions are satisfied:

$$1-\sigma'(\alpha+\frac{N}{2\rho'})>0, \tag{11}$$

$$\rho<N, \qquad \frac{1}{\rho}<1<2\alpha<2. \tag{12}$$

If $\psi\in L^\sigma(0,T;L^\nu(\Omega))$ with $\frac{\nu(2-\sigma')}{\sigma'}>N-1$ and $\sigma>2$, we set

$$\rho=\frac{N\nu}{N\nu-N+1}.$$

In this case we easily see that

$$\rho<N \quad \text{and} \quad \frac{1}{\rho}<1<\frac{2}{\sigma'}-\frac{N}{\rho'}<2,$$

and we can find α such that (11), (12) are satisfied. In conclusion, we have proved that

$$\|y\|_{\infty,Q}\le K\|\psi\|_{\sigma,\nu,\Gamma}$$

if $\frac{\nu(2-\sigma')}{\sigma'}>N-1$ and $\sigma>2$. If we set $\sigma=\nu$, the condition $\frac{\nu(2-\sigma')}{\sigma'}>N-1$ gives $\sigma=\nu>N+1$.

d) Now we suppose that $\phi=0$, $\psi=0$. Since $y_0\in L^\infty(\Omega)$, then $-\tilde{m}\le y_0(x)\le\tilde{m}$ for a.e. $x\in\Omega$, with $\tilde{m}=\|y_0\|_{\infty,\Omega}$. Obviously, $y_1=y+\tilde{m}$ is the weak solution of the equation:

$$\frac{\partial y_1}{\partial t}+Ay_1+k_1y_1=k_1\tilde{m} \quad \text{in } Q, \quad \frac{\partial y_1}{\partial n_A}+k_2y_1=k_2\tilde{m} \quad \text{on } \Sigma, \quad y_1(0)=y_0+\tilde{m} \quad \text{in } \Omega.$$

Since $y_0+\tilde{m}\ge 0$, from the comparison principle stated in Proposition 3.2 we deduce that $y_1\ge\tilde{y}_1$ where $\tilde{y}_1$ is the weak solution of :

$$\frac{\partial \tilde{y}_1}{\partial t}+A\tilde{y}_1+k_1\tilde{y}_1=k_1\tilde{m} \quad \text{in } Q, \quad \frac{\partial \tilde{y}_1}{\partial n_A}+k_2\tilde{y}_1=k_2\tilde{m} \quad \text{on } \Sigma, \quad \tilde{y}_1(0)=0 \quad \text{in } \Omega.$$

¿From (b) and (c) (with $\mu=m=\nu=\sigma=\infty$), we can see that there exists $K_3=K_3(N,\Omega,T,k_1,k_2)$ verifying

$$\|\tilde{y}_1\|_{\infty,Q}\le K_3\tilde{m}.$$

Then

$$(-K_3-1)\tilde{m}\le y(x,t) \qquad \text{for a.e. } (x,t)\in Q.$$

In the same way, we can prove that

$$y(x,t)\le (K_3+1)\tilde{m} \qquad \text{for a.e. } (x,t)\in Q.$$

Therefore

$$\|y\|_{\infty,Q}\le (K_3+1)\|y_0\|_{\infty,\Omega}.$$

Thus (10) is established.

e) Now, to prove that $y\in C(\overline{Q}_{\varepsilon,T})$, we set $y=\bar{y}+z$ where $\bar{y}$ is the weak solution of (9) corresponding to $(\phi=0,\psi=0,y_0)$ and z is the weak solution of (9) corresponding to $(\phi,\psi,y_0=0)$. We consider a sequence $(\phi_n)_n$ of regular functions converging to ϕ in $L^\mu(0,T;L^m(\Omega))$ and a sequence $(\psi_n)_n$ of regular functions with compact support in Σ, converging to ψ in $L^\sigma(0,T;L^\nu(\Gamma))$. From a well-known regularity result ([22], Theorem 4.5.3), the solution y_n of (9) corresponding to $(\phi_n, \psi_n, 0)$, belongs to $C(\overline{Q})$. ¿From (10) we easily see that $(y_n)_n$ converges to z in $L^\infty(Q)$ and that $(y_n)_n$ is a Cauchy sequence in $C(\overline{Q})$. Thus z belongs to $C(\overline{Q})$ and satisfies the estimate (10).

f) To prove that $\bar{y}$, the weak solution of (9) corresponding to $(\phi = 0, \psi = 0, y_0)$, belongs to $C(\overline{Q}_{\varepsilon,T})$, we first consider the case when $y_0 \in C(\overline{\Omega})$. In this case we can construct a sequence $(y_{0n})_n$ in $C^{2,\beta}(\overline{\Omega})$ and a sequence $(\psi_n)_n$ in $C^{2,\beta}(\overline{\Sigma})$, such that the pair (y_{0n}, ψ_n) satisfies the compatibility condition of order zero stated in ([22], p. 320), such that $(y_{0n})_n$ converges to y_0 in $C(\overline{\Omega})$ and $(\psi_n)_n$ converges to zero in $L^r(0,T; L^\nu(\Gamma))$. The solution y_n corresponding to $(\phi = 0, \psi_n, y_{0n})$ belongs to $C(\overline{Q})$. Moreover thanks to (10), $(y_n)_n$ is a Cauchy sequence in $C(\overline{Q})$. We can easily verify that $(y_n)_n$ converges in $C(\overline{Q})$ to $\bar{y}$.

g) If $y_0 \in L^\infty(\Omega)$, thanks to ([26], Lemma 1), $\bar{y}(\varepsilon) \in C(\overline{\Omega})$ for every $\varepsilon > 0$. Therefore it follows from (f) that $\bar{y} \in C(\overline{Q}_{\varepsilon,T})$. □

Proposition 3.2 *(Comparison principle). Let a, ϕ be in $L^q(Q)$, let b, ψ be in $L^r(\Sigma)$, let y_0 be in $L^2(\Omega)$. We suppose that*

$$a(x,t) \geq C_0 \text{ in } Q \quad \text{and} \quad b(s,t) \geq C_0 \text{ on } \Sigma. \tag{13}$$

Let y be the weak solution in $C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ of

$$\frac{\partial y}{\partial t} + Ay + ay = \phi \quad \text{in } Q, \qquad \frac{\partial y}{\partial n_A} + by = \psi \quad \text{on } \Sigma, \qquad y(0) = y_0 \quad \text{in } \Omega. \tag{14}$$

If $\phi \leq 0$, $g \leq 0$ and $y_0 \leq 0$, then $y \leq 0$ a.e. in Q, $y|_\Sigma$ (the trace of y on Σ) satisfies $y|_\Sigma \leq 0$ a.e. on Σ and $y(\cdot, T) \leq 0$ a.e in Ω.

Remark 3.3. The equation (14) admits a unique weak solution in $C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ (in the sense of [22]). The proof is given in ([22], Theorem 5.1, Chapter 3).

Remark 3.4. In the case when A is the Laplace operator, when $a = 0$ and b is nonnegative and bounded, Proposition 3.2 is the comparison principle proved in ([29], Appendix Theorem A.1). Even in this case it is noticed in ([29], p. 113) that a complete proof of comparison theorem for weak solutions is not easy to found in classical literature on parabolic equations. The proof of the comparison principle stated in ([29], Theorem A1, p. 113 and Theorem 2.4 p. 101) is based on an identity satisfied by y^+, the nonnegative part of y, where y is the solution of the heat equation with a nonhomogeneous Neumann boundary conditions ([29], Propositon 2.3). The result of ([29], Proposition 2.3) is still valid if we replace the equation $\frac{\partial y}{\partial t} - \Delta y = 0$, $\frac{\partial y}{\partial n} = h$, $y(0) = y_0$, (with $h \in L^2(\Sigma)$, $y_0 \in L^2(\Omega)$) by the equation $\frac{\partial y}{\partial t} + Ay + ay = \phi$, $\frac{\partial y}{\partial n_A} + by = \psi$, $y(0) = y_0$. More precisely we can establish the following Lemma.

Lemma 3.2 *Let a, ϕ be in $L^q(Q)$. Let b, ψ be in $L^r(\Sigma)$ and let y_0 be in $L^2(\Omega)$. Let y be the weak solution in $C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ of (14). Then $y^+(x,t) = \max(0, y(x,t))$ satisfies the following identity:*

$$\frac{1}{2}\int_\Omega (y^+(x,T)^2 + y^+(x,0)^2)\, dx + \sum_{i,j=1}^{N} \int_Q a_{ij}(x) D_j y^+(x,t) D_i y^+(x,t)\, dxdt +$$

$$+ \int_Q a(x,t) y^+(x,t)^2\, dxdt + \int_\Sigma b(s,t) y^+(s,t)^2\, dsdt =$$

$$= \int_\Omega y_0(x)y^+(x,0)\,dx + \int_\Sigma \psi(s,t)y^+(s,t)\,dsdt + \int_Q \phi(x,t)y^+(x,t)\,dxdt. \tag{15}$$

Proof of Lemma 3.2. Since our equation is more general than the one given in [29], we slightly modify the proof given in [29] and we recall the main steps for the convenience of the reader.
Let us remark that the weak solution y of (14) does not necessarily belong to $W(0,T)$ but belongs to $V_2^{1,\frac{1}{2}}(Q)$ (see [29], Theorem 5.1, Chapter 3), this regularity result is essential in the proof. Since $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is the weak solution of (14) then, for every $z \in W^{1,2}(Q)$, we have

$$-\int_Q y(x,t)\frac{\partial z}{\partial t}(x,t)\,dxdt + \int_\Omega y(x,T)z(x,T)\,dx + \sum_{i,j=1}^N \int_Q a_{ij}(x)D_jy(x,t)D_iz(x,t)\,dxdt +$$

$$+\int_Q a(x,t)y(x,t)z(x,t)\,dxdt + \int_\Sigma b(s,t)y(s,t)z(s,t)\,dsdt =$$

$$= \int_\Omega y_0(x)z(x,0)\,dx + \int_\Sigma \psi(s,t)z(s,t)\,dsdt + \int_Q \phi(x,t)z(x,t)\,dxdt. \tag{16}$$

Let T_1 be a positive number such that $T_1 > T$. We consider the function $\tilde{y}^+$ defined by

$$\tilde{y}^+(x,t) = \begin{cases} y^+(x,t) & \text{on } Q, \\ y^+(x,T-(t-T)) & \text{on } \Omega \times [T,T_1]. \end{cases}$$

Let y_δ^+ be the average of $\tilde{y}^+$ defined by $y_\delta^+(\cdot,t) = \delta^{-1}\int_t^{t+\delta} \tilde{y}^+(\cdot,\tau)d\tau$. ¿From Lemma 4.7 in ([22] p. 85), it follows that $y_\delta^+ \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and that y_δ^+ converges to y^+ in $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ as δ tends to zero. Moreover for a.e. $t \in [0,T]$ we have

$$\frac{\partial y_\delta^+}{\partial t}(\cdot,t) = D_\delta \tilde{y}^+(\cdot,t) := \delta^{-1}[\tilde{y}^+(\cdot,t+\delta) - \tilde{y}^+(\cdot,t)],$$

therefore $\frac{\partial y_\delta^+}{\partial t} \in L^2(Q)$. If we set $z = y_\delta^+(\cdot,t)$ in (16), we get:

$$-\int_0^T y(x,t)D_\delta\tilde{y}^+(x,t)\,dxdt + \int_\Omega y(x,T)y_\delta^+(x,T)\,dx + \sum_{i,j=1}^N \int_Q a_{ij}(x)D_jy(x,t)D_iy_\delta^+(x,t)\,dxdt +$$

$$+\int_Q a(x,t)y(x,t)y_\delta^+(x,t)\,dxdt = \int_\Sigma b(s,t)y(s,t)y_\delta^+(s,t)\,dsdt =$$

$$= \int_\Omega y_0(x)y_\delta^+(x,0)\,dx + \int_\Sigma \psi(s,t)y_\delta^+(s,t)\,dsdt + \int_Q \phi(x,t)y_\delta^+(x,t)\,dxdt.$$

¿From convergence results mentioned above, we can easily verify that

$$\sum_{i,j=1}^N \int_Q a_{ij}D_jy(D_iy_\delta^+ - D_iy^+)\,dxdt - \int_Q \phi(y_\delta^+ - y^+)\,dxdt - \int_\Sigma \psi(y_\delta^+ - y^+)\,dsdt$$

$$+\int_\Omega y(T)(y_\delta^+(T) - y^+(T))\,dx - \int_\Omega y(0)(y_\delta^+(0) - y^+(0))\,dx \longrightarrow 0$$

as δ tends to zero. Moreover, from Hölder inequality and from estimates given in ([22] p. 75 and p. 78), we get

$$|\int_Q a(x,t)y(x,t)(y_\delta^+(x,t)-y^+(x,t))\,dxdt| \le \|a\|_{\frac{N}{2}+1,Q}\|y\|_{\frac{2(N+2)}{N},Q}\|y_\delta^+-y^+\|_{\frac{2(N+2)}{N},Q}$$

$$\le K\|a\|_{\frac{N}{2}+1,Q}\|y\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))}\|y_\delta^+-y^+\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))} \longrightarrow 0$$

and

$$|\int_\Sigma b(s,t)y(s,t)(y_\delta^+(s,t)-y^+(s,t))\,dsdt| \le \|b\|_{N+1,\Sigma}\|y\|_{\frac{2(N+1)}{N},\Sigma}\|y_\delta^+-y^+\|_{\frac{2(N+1)}{N},\Sigma}$$

$$\le K_1\|b\|_{N+1,\Sigma}\|y\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))}\|y_\delta^+-y^+\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))} \longrightarrow 0$$

as δ tends to zero. To complete the proof, it remains to pass to the limit in

$$-\int_Q y(x,t)\frac{\partial y_\delta^+}{\partial t}(x,t)dxdt = -\int_Q y(x,t)D_\delta\tilde{y}^+(x,t)dxdt$$

$$= -\int_0^{T-\delta}\int_\Omega y(x,t)D_\delta y^+(x,t)\,dxdt - \int_{T-\delta}^T\int_\Omega y(x,t)D_\delta\tilde{y}^+(x,t)\,dxdt.$$

Since $y \in V_2^{1,\frac{1}{2}}(Q)$, the first term $-\int_0^{T-\delta}\int_\Omega y(x,t)D_\delta y^+(x,t)\,dxdt$ converges to $\frac{1}{2}\int_\Omega[-y^+(x,T)^2 + y^+(x,0)^2]dx$ (see [29] p. 99). The term $\int_{T-\delta}^T\int_\Omega y(x,t)D_\delta\tilde{y}^+(x,t)\,dxdt$ tends to 0. □

In all the sequel, if $a \in L^q(Q)$ and $b \in L^r(\Sigma)$, we shall say that $((a_{ij})_{i,j}, a, b)$ verifies the ellipticity condition (Em_0) if

$$\int_\Omega \sum_{i,j} a_{ij}(x)D_j\varphi(x)D_i\varphi(x)\,dx + \int_\Omega a(x,t)\varphi(x)^2\,dx + \int_\Gamma b(s,t)\varphi(s)^2\,ds \ge \frac{m_0}{2}\|\varphi\|^2_{H^1(\Omega)} \tag{17}$$

for a.e. $t \in [0,T]$ and for every $\varphi \in H^1(\Omega)$, where m_0 is the constant in (1).

Proof of Proposition 3.2.

a) We first consider the case when $((a_{ij})_{i,j}, a, b)$ verifies the ellipticity condition (Em_0). Thanks to Lemma 3.2, since $y_0 \le 0$, we have:

$$0 \le \frac{1}{2}\int_\Omega (y^+(x,T)^2 + y^+(x,0)^2)\,dx + \sum_{i,j=1}^N \int_Q a_{ij}(x)D_jy^+(x,t)D_iy^+(x,t)\,dx\,dt$$

$$+\int_Q a(x,t)y^+(x,t)^2\,dx\,dt + \int_\Sigma b(s,t)y^+(s,t)^2\,ds\,dt$$

$$\le \int_Q \phi(x,t)y^+(x,t)\,dx\,dt + \int_\Sigma \psi(s,t)y^+(s,t)\,ds\,dt.$$

Taking into account the ellipticity condition (17), we get:

$$0 \le \frac{1}{2}\int_\Omega (y^+(x,T)^2 + y^+(x,0)^2)\,dx + \frac{m_0}{2}||y^+||^2_{L^2(0,T;H^1(\Omega))}$$

$$\leq \int_{Q^+} \phi(x,t) y^+(x,t)\, dx\, dt + \int_{\Sigma^+} \psi(s,t) y^+(s,t)\, ds\, dt,$$

where

$$Q^+ = \{(x,t) \in Q \mid y(x,t) > 0\ \}, \qquad \Sigma^+ = \{(s,t) \in \Sigma^+ \mid y(s,t) > 0\ \}.$$

Since $\phi \leq 0$ and $\psi \leq 0$, it results that meas $Q^+ = 0$ and meas $\Sigma^+ = 0$. In other words $y^+ = 0$ a.e. in Q, $(y|_\Sigma)^+ = 0$ a.e. on Σ and $y(T)^+ = 0$ a.e. in Ω.

b) Now we consider the case when the condition (17) is not necessarily satisfied. We set $z(x,t) = e^{-\theta t} y(x,t)$, where $\theta \in \mathbb{R}$ will be precisely defined later. We remark that z is the weak solution of the equation:

$$\frac{\partial z}{\partial t} + Az + (a+\theta)z = \phi e^{-\theta t} \quad \text{in } Q, \qquad \frac{\partial z}{\partial n_A} + bz = \psi e^{-\theta t} \quad \text{on } \Sigma, \qquad z(0) = y_0 \quad \text{in } \Omega. \tag{18}$$

Let us prove that $\theta > 0$ can be chosen in such a way that $((a_{ij})_{i,j}, a+\theta, b)$ satisfies the ellipticity condition (Em_0). Let $\tilde{C} = \min(0, C_0)$ and let φ be in $H^1(\Omega)$; then we have:

$$\sum_{i,j} \int_\Omega a_{ij}(x) D_j\varphi(x) D_i\varphi(x)\, dx + \int_\Omega (a(x,t)+\theta)\varphi(x)^2\, dx + \int_\Gamma b(s,t)\varphi(s)^2\, ds$$

$$\geq m_0 \|D\varphi\|_{2,\Omega}^2 + (\tilde{C}+\theta)\|\varphi\|_{2,\Omega}^2 + \tilde{C}\|\varphi\|_{2,\Gamma}^2.$$

Thanks to Lemma 3.3 (see below), we get:

$$\sum_{i,j} \int_\Omega a_{ij}(x) D_j\varphi(x) D_i\varphi(x)\, dx + \int_\Omega (a(x,t)+\theta)\varphi(x)^2\, dx + \int_\Gamma b(s,t)\varphi(s)^2\, ds$$

$$\geq m_0 \|D\varphi\|_{2,\Omega}^2 + (\tilde{C}+\theta)\|\varphi\|_{2,\Omega}^2 + \frac{C_\Omega \tilde{C}}{\varepsilon}\|\varphi\|_{2,\Omega}^2 + \tilde{C} C_\Omega \varepsilon \|D\varphi\|_{2,\Omega}^2.$$

If $\tilde{C} = 0$, we recall that $((a_{ij})_{i,j}, a+m_0, b)$ satisfies (17). If $\tilde{C} < 0$, we set:

$$\varepsilon = \frac{-m_0}{2C_\Omega \tilde{C}} \qquad \theta = \frac{m_0}{2} - \tilde{C} - \frac{C_\Omega \tilde{C}}{\varepsilon} = \frac{m_0}{2} - \tilde{C} + \frac{2\tilde{C}^2 C_\Omega^2}{m_0}.$$

With such values for ε and θ we have:

$$\sum_{i,j} \int_\Omega a_{ij}(x) D_j\varphi(x) D_i\varphi(x)\, dx + \int_\Omega (a(x,t)+\theta)\varphi(x)^2\, dx + \int_\Gamma b(s,t)\varphi(s)^2\, ds \geq \frac{m_0}{2}\|\varphi\|_{H^1(\Omega)}^2.$$

Therefore (with (a)), the weak solution z of (18) in $C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ satisfies $z(x,T) \leq 0$ for a.e. $x \in \Omega$, $z(x,t) \leq 0$ for a.e. $(x,t) \in Q$ and $z|_\Sigma(s,t) \leq 0$ for a.e. $(s,t) \in \Sigma$. Since $y(x,t) = e^{\theta t} z(x,t)$, the proof is complete. □

Lemma 3.3 *There exists $C_\Omega > 0$, such that for every $\varphi \in H^1(\Omega)$ and for every $\varepsilon > 0$, we have:*

$$\|\varphi\|_{2,\Gamma}^2 \leq \frac{C_\Omega}{\varepsilon}\|\varphi\|_{2,\Omega}^2 + C_\Omega \varepsilon \|D\varphi\|_{2,\Omega}^2.$$

Proof of Lemma 3.3. The result is classical for $\varepsilon = 1$. To prove Lemma 3.3 we rewrite the proof as in the case $\varepsilon = 1$ but with a slight modification at the end of the proof. Since Ω is regular there exist a finite number of regular open bounded subsets $\{\mathcal{O}_i\}_{i=0}^I$ and regular inversible mappings ϕ_i from $\mathcal{O}_i$ onto $B = B(0_{\mathbb{R}^N}, 1)$ (the unit ball in $\mathbb{R}^N$) such that:

$$\bar{\mathcal{O}}_0 \subset \Omega, \qquad \Omega \subset \bigcup_{i=0}^I \mathcal{O}_i,$$

for $1 \leq i \leq I$,

$$\phi_i(\mathcal{O}_i \cap \Omega) = B \cap \mathbb{R}^N_+ = \{\chi = (\chi', \chi_N) \in \mathbb{R}^N \mid |\chi| < 1, \chi_N > 0\},$$

$$\phi_i(\mathcal{O}_i \cap \Gamma) = \{\chi = (\chi', \chi_N) \in \mathbb{R}^N \mid |\chi'| < 1, \chi_N = 0\},$$

and the mappings ϕ_i^{-1} are regular. Moreover there exist a partition of the unity $\{\alpha_i\}_{i=0}^I$ subordinate to the covering $\{\mathcal{O}_i\}_{i=0}^I$ and a constant K such that

$$\|\varphi\|_{2,\Gamma} \leq K \sum_{i=1}^I \|P((\alpha_i \varphi) o \phi_i^{-1})\|_{L^2(\mathbb{R}^{N-1})}$$

where $P((\alpha_i\varphi)o\phi_i^{-1})$ is the extension of $(\alpha_i\varphi)o\phi_i^{-1}$ by 0 to $\mathbb{R}^{N-1} \setminus \{\chi' \in \mathbb{R}^{N-1}, \ \ |\chi'| < 1\}$. Thus to prove the lemma it is sufficient to show that

$$\int_{\mathbb{R}^{N-1}} |\varphi(\chi', 0)|^2 d\chi' \leq \int_{\mathbb{R}^N_+} (\varepsilon |\frac{\partial \varphi}{\partial \chi_N}(\chi)|^2 + \frac{1}{\varepsilon}|\varphi(\chi)|^2) d\chi \tag{19}$$

for every $\varphi \in \mathcal{D}(\overline{\mathbb{R}^N_+})$. Since by a direct calculation we have

$$|\varphi(\chi', 0)|^2 = -2 \int_0^\infty \varphi(\chi', \chi_N) \frac{\partial \varphi}{\partial \chi_N}(\chi', \chi_N) d\chi_N,$$

the lemma follows from the Young's inequality. $\square$

Proposition 3.3 *Let a be in $L^q(Q)$, let b be in $L^r(\Sigma)$ verifying $a(x,t) \geq C_0$ in Q and $b(s,t) \geq C_0$ on Σ. There exists a positive constant $C_5 = C_5(N, q, r, \Omega, T, C_0)$ (independent of a and b), such that for every $\phi \in L^q(Q)$, every $\psi \in L^r(\Sigma)$ and every $y_0 \in L^\infty(\Omega)$, the weak solution $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ of the equation (14) belongs to $L^\infty(Q) \cap C(\overline{Q}_{\varepsilon,T})$ (for every $\varepsilon > 0$) and satisfies the estimate:*

$$\|y\|_{\infty,Q} + \|y\|_{\infty,\Sigma} \leq C_5(\|\phi\|_{q,Q} + \|\psi\|_{r,\Sigma} + \|y_0\|_{\infty,\Omega}). \tag{20}$$

Moreover, this solution belongs to $W(0,T)$.

Proof of Proposition 3.3. We set $\phi^+ = \max(0, \phi)$, $\phi^- = -\min(0, \phi)$, $\psi^+ = \max(0, \psi)$, $\psi^- = -\min(0, \psi)$, $y_0^+ = \max(0, y_0)$, $y_0^- = -\min(0, y_0)$. We denote by y_1 (respectively y_2) the solution of (14) corresponding to ϕ^+, ψ^+, y_0^+ (respectively ϕ^-, ψ^-, y_0^-). From Proposition 3.2 we deduce that $y_1 \geq 0$ in Q, $y_1|_\Sigma \geq 0$ on Σ, $y_2 \geq 0$ in Q and $y_2|_\Sigma \geq 0$. Moreover y (the solution of (14) corresponding to (ϕ, ψ, y_0)) is equal to $y_1 - y_2$. Thus to prove Proposition 3.3 we have only to establish the estimate (20) for y_1 (the estimate for y_2 is obtained in the same way).
As in the proof of Proposition 3.2, we can find $\theta \in \mathbb{R}^+$ such that $((a_{ij})_{i,j}, C_0 + \theta, C_0)$ verifies the

ellipticity condition (Em_0). In this case, (4) is satisfied for $k_1 = C_0 + \theta,\ k_2 = C_0$. We denote by z the weak solution of the equation:

$$\frac{\partial z}{\partial t} + Az + (C_0+\theta)z = e^{-\theta t}\phi^+ \quad \text{in } Q, \qquad \frac{\partial z}{\partial n_A} + C_0 z = e^{-\theta t}\psi^+ \quad \text{on } \Sigma, \qquad z(0) = y_0^+ \quad \text{in } \Omega.$$

¿From Proposition 3.2, it follows $z \geq 0$ on Q and $z|_\Sigma \geq 0$ in Σ. Moreover thanks to (Em_0) and to Proposition 3.1, we get $z \in L^\infty(Q)$, $z|_\Sigma \in L^\infty(\Sigma)$ and z satisfies the estimate

$$\|z\|_{\infty,Q} + \|z\|_{\infty,\Sigma} \leq K(\|\phi^+\|_{q,Q} + \|\psi^+\|_{r,\Sigma} + \|y_0^+\|_{\infty,\Omega}).$$

We set $\varphi = e^{-\theta t}y_1 - z$; then φ satisfies:

$$\frac{\partial \varphi}{\partial t} + A\varphi + (a+\theta)\varphi = (C_0 - a)z \quad \text{in } Q, \qquad \frac{\partial \varphi}{\partial n_A} + b\varphi = (C_0 - b)z \quad \text{on } \Sigma, \qquad \varphi(\cdot,0) = 0 \quad \text{in } \Omega.$$

Since $C_0 - a(\cdot) \leq 0$, $C_0 - b(\cdot) \leq 0$ and $z \geq 0$, from Proposition 3.2 it follows $\varphi \leq 0$ a.e. in Q and $\varphi|_\Sigma \leq 0$ a.e. on Σ. Therefore we have

$$0 \leq y_1 \leq e^{\theta t}z \quad \text{a.e. in } Q \ \text{ and } \ 0 \leq (y_1)|_\Sigma \leq e^{\theta t}z|_\Sigma \quad \text{a.e. on } \Sigma.$$

Thus $y_1 \in L^\infty(Q)$ and

$$\|y_1\|_{\infty,Q} + \|y_1\|_{\infty,\Sigma} \leq K(\|\phi^+\|_{q,Q} + \|\psi^+\|_{r,\Sigma} + \|y_0^+\|_{\infty,\Omega}). \tag{21}$$

Since y_1 verifies the equation

$$\frac{\partial \xi}{\partial t} + A\xi = \phi_1 \quad \text{in } Q, \qquad \frac{\partial \xi}{\partial n_A} = \psi_1 \quad \text{on } \Sigma, \qquad \xi(0) = y_0 \quad \text{in } \Omega,$$

with $\phi_1(x,t) = \phi^+(x,t) - a(x,t)y_1(x,t) \in L^q(Q)$, $\psi_1(s,t) = \psi^+(s,t) - b(s,t)y_1(s,t) \in L^r(\Sigma)$, thanks to Proposition 3.1, we get $y_1 \in C(\overline{Q}_{\varepsilon,T})$. Since $y_1 \in L^\infty(Q)$ and $y_1|_\Sigma \in L^\infty(\Sigma)$, it is clear that $y \in W(0,T)$. □

Remark 3.5. If we proceed as above, we can show that if $y_0 \in C(\overline{\Omega})$ then the solution of (14) belongs to $C(\overline{Q})$.

Proposition 3.4 *Let a be in $L^q(Q)$, let b be in $L^r(\Sigma)$ verifying $a(x,t) \geq C_0$ in Q and $b(s,t) \geq C_0$ on Σ and let ε be such that $0 < \varepsilon < T$. There exists a positive constant $C_6(\varepsilon) = C_6(\varepsilon, N, q, r, \Omega, T, C_0)$ (independent of a and b), such that for every $\phi \in L^q(Q)$, every $\psi \in L^r(\Sigma)$ and every $y_0 \in L^\infty(\Omega)$, the weak solution $y \in W(0,T)$ of the equation (14) belongs to $C(\overline{Q}_{\varepsilon,T})$ and satisfies the estimate:*

$$\|y\|_{C(\overline{Q}_{\varepsilon,T})} \leq C_6(\varepsilon)(\|\phi\|_{q,Q} + \|\psi\|_{r,\Sigma} + \|y_0\|_{2,\Omega}).$$

Proof of Proposition 3.4. Thanks to Proposition 3.3 we have only to prove that the weak solution z of the equation:

$$\frac{\partial z}{\partial t} + Az + az = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n_A} + bz = 0 \quad \text{on } \Sigma, \qquad z(0) = y_0 \quad \text{in } \Omega,$$

satisfies the estimate

$$\|z(t)\|_{\infty,\Omega} \leq K(\varepsilon)\|y_0\|_{2,\Omega}$$

for every $t \in [\varepsilon, T]$. We proceed as in the proof of Proposition 3.3; we denote by $\bar{z}$ the weak solution of

$$\frac{\partial \bar{z}}{\partial t} + A\bar{z} + (C_0 + \theta)\bar{z} = 0 \quad \text{in } Q, \qquad \frac{\partial \bar{z}}{\partial n_A} + C_0\bar{z} = 0 \quad \text{on } \Sigma, \qquad \bar{z}(0) = y_0^+ \quad \text{in } \Omega,$$

where $\theta \geq 0$ is such that $((a_{ij})_{i,j}, C_0 + \theta, C_0)$ satisfies the ellipticity condition (Em_0). If we take $\lambda = \infty, l = 2$ in (6) it follows that

$$\|\bar{z}(t)\|_{\infty,\Omega} \leq C_2(\varepsilon^{-\frac{N}{4}})\|y_0\|_{2,\Omega}$$

for every $\varepsilon \leq t \leq T$. Now, we consider the weak solution z_1 of

$$\frac{\partial z_1}{\partial t} + Az_1 + az_1 = 0 \quad \text{in } Q, \qquad \frac{\partial z_1}{\partial n_A} + bz_1 = 0 \quad \text{on } \Sigma, \qquad z_1(0) = y_0^+ \quad \text{in } \Omega.$$

As in the proof of Proposition 3.3, we get

$$0 \leq z_1(x,t) \leq e^{\theta t}\bar{z}(x,t) \leq C_2 e^{\theta T}(\varepsilon^{-\frac{N}{4}})\|y_0\|_{2,\Omega}$$

for a.e. $(x,t) \in Q$. We obtain a similar estimate for the weak solution z_2 of

$$\frac{\partial z_2}{\partial t} + Az_2 + az_2 = 0 \quad \text{in } Q, \qquad \frac{\partial z_2}{\partial n_A} + bz_2 = 0 \quad \text{on } \Sigma, \qquad z_2(0) = y_0^- \quad \text{in } \Omega,$$

and finally we get

$$\|z(t)\|_{\infty,\Omega} \leq K(\varepsilon)\|y_0\|_{2,\Omega} \quad \forall t \in [\varepsilon, T]. \qquad \square$$

3.2 State equation.

Theorem 3.1 *If the assumptions* (A1) *and* (A2) *are satisfied, if* $u \in L^q(Q)$, $v \in L^r(\Sigma)$ *and* $w \in L^\infty(\Omega)$, *then the equation (2) admits a unique weak solution* y *in* $W(0,T) \cap L^\infty(Q)$, *this solution belongs to* $C(\overline{Q}_{\varepsilon,T})$ *for every* $\varepsilon > 0$ *and satisfies the estimates*

$$||y||_{\infty,Q} + ||y||_{\infty,\Sigma} \leq C_7(||u||_{q,Q} + ||v||_{r,\Sigma} + ||w||_{\infty,\Omega} + 1),$$

$$||y||_{C(\overline{Q}_{\varepsilon,T})} \leq C_8(\varepsilon)(||u||_{q,Q} + ||v||_{r,\Sigma} + ||w||_{2,\Omega} + 1),$$

where $C_7 = C_7(N,q,r,\Omega,T,C_0)$ *and* $C_8(\varepsilon) = C_8(\varepsilon,N,q,r,\Omega,T,C_0)$ *are positive constants.*

Proof of Theorem 3.1.

a) Let us prove the uniqueness result. Let y_1 and y_2 be two weak solutions of (2) in $W(0,T) \cap L^\infty(Q)$. If we set $z = y_1 - y_2$, then z satisfies:

$$\frac{\partial z}{\partial t} + Az + az = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n_A} + bz = 0 \quad \text{on } \Sigma, \qquad z(0) = 0 \quad \text{in } \Omega,$$

with $a(x,t) = \int_0^1 f'_y(x,t,y_2 + \theta z, u)d\theta \geq C_0$ in Q and with $b(s,t) = \int_0^1 g'_y(s,t,y_2 + \theta z, v)d\theta \geq C_0$ on Σ. Moreover, since y_1, y_2 belong to $W(0,T) \cap L^\infty(Q)$, thanks to (A1), (A2) we easily see that $a \in L^q(Q)$ and $b \in L^r(\Sigma)$. From Proposition 3.2 it follows $z = 0$, that is $y_1 = y_2$.

b) To prove existence of solution for (2), we cannot directly use the Faedo-Galerkin method as in ([23], Chapter 2, Theorem 1.2). Indeed, if we look for a solution y in $W(0,T)$, the estimates on f and g are not sufficient to prove that the following integrals are well defined

$$\int_Q f(x,t,y(x,t),u(x,t))z(x,t)\,dxdt \quad \text{and} \quad \int_\Sigma g(s,t,y(s,t),v(s,t))z(s,t)\,dsdt$$

when $z \in L^2(0,T;H^1(\Omega))$. To overcome these difficulties, we have to consider equations with truncated and regularized functions. The details of the proof are given in Section 5. $\square$

4 Proof of Theorem 2.1.

4.1 Hamiltonian variation and cost functional variation.

In this section, we derive a basic equality on the cost functional which will be used to prove the Pontryagin principles. Thanks to a Taylor formula we define $\tilde{F}_y(x,t,y_1,y_2,u)$ by:

$$
\begin{aligned}
F(x,t,y_1,u) - F(x,t,y_2,u) &= \left(\int_0^1 F'_y(x,t,y_2+\theta(y_1-y_2),u)\,d\theta\right)(y_1-y_2) \\
&= \tilde{F}_y(x,t,y_1,y_2,u)(y_1-y_2),
\end{aligned}
$$

for every $x \in \Omega$, $t \in]0,T[$, $u \in \mathbb{R}$, $y_1 \in \mathbb{R}$ and $y_2 \in \mathbb{R}$. We define $\tilde{G}_y(s,t,y_1,y_2,v)$, $\tilde{f}_y(x,t,y_1,y_2,u)$, $\tilde{g}_y(s,t,y_1,y_2,v)$ and $\tilde{\ell}_y(x,y_1,y_2,w)$ in a similar manner.

Let (u_1,v_1,w_1), (u_2,v_2,w_2) be two triplets of admissible controls. For $i=1,2$, we denote by y_i the weak solution of (2) corresponding to (u_i,v_i,w_i). We define the intermediate adjoint state p_{12} as the weak solution of the equation:

$$
\begin{aligned}
&-\frac{\partial p}{\partial t} + Ap + \tilde{f}_y(x,t,y_1,y_2,u_1)p = \tilde{F}_y(x,t,y_1,y_2,u_1) && \text{in } Q, \\
&\frac{\partial p}{\partial n_A} + \tilde{g}_y(s,t,y_1,y_2,v_1)p = \tilde{G}_y(s,t,y_1,y_2,v_1) && \text{on } \Sigma, \\
&p(T) = \tilde{\ell}_y(x,y_1(T),y_2(T),w_1) && \text{in } \Omega.
\end{aligned}
\tag{22}
$$

Let us notice that if $u_1 = u_2$, $v_1 = v_2$ and $w_1 = w_2$ then $y_1 = y_2$ and $p_{12} = p_1$ is the so-called adjoint state associated with (u_1,v_1,w_1).

Proposition 4.1 *The equation* (22) *admits a unique weak solution* p_{12} *in* $W(0,T) \cap C(\overline{Q}_{0,T-\varepsilon}) \cap L^\infty(Q)$*. Moreover,* p_{12} *satisfies the estimates:*

$$
\|p_{12}\|_{\infty,Q} \le C_5(\|\tilde{F}_y(\cdot,y_1,y_2,u_1)\|_{q,Q} + \|\tilde{G}_y(\cdot,y_1,y_2,v_1)\|_{r,\Sigma} + \|\tilde{\ell}_y(\cdot,y_1(T),y_2(T),w_1)\|_{\infty,\Omega}),
$$

$$
\|p_{12}\|_{C(\overline{Q}_{0,T-\varepsilon})} \le C_6(\varepsilon)(\|\tilde{F}_y(\cdot,y_1,y_2,u_1)\|_{q,Q} + \|\tilde{G}_y(\cdot,y_1,y_2,v_1)\|_{r,\Sigma} + \|\tilde{\ell}_y(\cdot,y_1(T),y_2(T),w_1)\|_{2,\Omega}),
$$

where $C_5 = C_5(N,q,r,\Omega,T,C_0)$ *and* $C_6(\varepsilon) = C_6(\varepsilon,N,q,r,\Omega,T,C_0)$ *are the same constants as in Propositions* 3.3 *and* 3.4.

Proof of Proposition 4.1. ¿From assumptions (A1)-(A2), we first remark that

$$
\tilde{f}_y(x,t,y_1,y_2,u_1) \ge C_0 \ \text{ a.e. in } Q, \quad \tilde{g}_y(s,t,y_1,y_2,v_1) \ge C_0 \ \text{ a.e. on } \Sigma
$$

and $\tilde{f}_y(\cdot,y_1,y_2,u_1) \in L^q(Q)$, $\tilde{g}_y(\cdot,y_1,y_2,v_1) \in L^r(\Sigma)$. In the same way as in Proposition 3.4 we can prove that the equation (22) admits a unique weak solution p_{12} in $W(0,T) \cap C(\overline{Q}_{0,T-\varepsilon}) \cap L^\infty(Q)$ and that p_{12} satisfies the estimates stated in the proposition (Thanks to (A3)-(A5) the right sides of the above estimates are finite). □

Proposition 4.2 *If* (u_1,v_1,w_1) *and* (u_2,v_2,w_2) *are two triplets of admissible controls, if* y_1 *(respectively* y_2*) is the weak solution of* (2) *corresponding to* (u_1,v_1,w_1) *(respectively* (u_2,v_2,w_2)*), then we have:*

$$
J(y_1,u_1,v_1,w_1) - J(y_2,u_2,v_2,w_2) = \int_Q \{H_Q(x,t,y_2,u_1,p_{12}) - H_Q(x,t,y_2,u_2,p_{12})\}\,dxdt
$$

$$+\int_{\Sigma}\{H_{\Sigma}(s,t,y_2,v_1,p_{12})-H_{\Sigma}(s,t,y_2,v_2,p_{12})\}\,dsdt$$

$$+\int_{\Omega}\{H_{\Omega}(x,y_2(T),w_1,p_{12}(0))-H_{\Omega}(x,y_2(T),w_2,p_{12}(0))\}\,dx$$

where p_{12} is the intermediate adjoint state given by (22).

Remark 4.1. If $v_2=v_1$ and $w_2=w_1$ then

$$J(y_1,u_1,v_1,w_1)-J(y_2,u_2,v_1,w_1)=\int_Q\{H_Q(x,t,y_2,u_1,p_{12})-H_Q(x,t,y_2,u_2,p_{12})\}\,dx\,dt$$

where y_2 is the weak solution of (2) corresponding to (u_2,v_1,w_1). In a similar manner, if $u_2=u_1$ and $w_2=w_1$ we have

$$J(y_1,u_1,v_1,w_1)-J(y_2,u_1,v_2,w_1)=\int_{\Sigma}\{H_{\Sigma}(s,t,y_2,v_1,p_{12})-H_{\Sigma}(s,t,y_2,v_2,p_{12})\}\,ds\,dt,$$

if $u_2=u_1$ and $v_2=v_1$ we have

$$J(y_1,u_1,v_1,w_1)-J(y_2,u_1,v_1,w_2)=\int_{\Omega}\{H_{\Omega}(x,y_2(T),w_1,p_{12}(0))-H_{\Omega}(x,y_2(T),w_2,p_{12}(0))\}\,dx.$$

This is the reason why we can obtain three decoupled Pontryagin minimum principles.

Proof of Proposition 4.2. The proof can be obtained by a straightforward calculation

$$J(y_1,u_1,v_1,w_1)-J(y_2,u_2,v_2,w_2)=$$

$$=\int_Q\{F(x,t,y_1,u_1)-F(x,t,y_2,u_1)\}\,dxdt+\int_Q\{F(x,t,y_2,u_1)-F(x,t,y_2,u_2)\}\,dxdt+$$

$$+\int_{\Sigma}\{G(s,t,y_1,v_1)-G(s,t,y_2,v_1)\}\,dsdt+\int_{\Sigma}\{G(s,t,y_2,v_1)-G(s,t,y_2,v_2)\}\,dsdt+$$

$$+\int_{\Omega}\{\ell(x,y_1(T),w_1)-\ell(x,y_2(T),w_1)\}\,dx+\int_{\Omega}\{\ell(x,y_2(T),w_1)-\ell(x,y_2(T),w_2)\}\,dx.$$

On the other hand, we have

$$\int_Q\{F(x,t,y_1,u_1)-F(x,t,y_2,u_1)\}\,dxdt=\int_Q\tilde{F}_y(x,t,y_1,y_2,u_1)(y_1-y_2)\,dxdt$$

$$=\int_0^T\langle-\frac{\partial p_{12}}{\partial t}+Ap_{12}+\tilde{f}_y(\cdot,t,y_1,y_2,u_1)p_{12},y_1(t)-y_2(t)\rangle_{(H^1(\Omega))'\times H^1(\Omega)}\,dt$$

where p_{12} is the intermediate adjoint state. By using the Green formula, we obtain

$$\int_Q\{F(x,t,y_1,u_1)-F(x,t,y_2,u_1)\}\,dxdt=\int_Q p_{12}(x,t)\{f(x,t,y_1,u_1)-f(x,t,y_2,u_1)\}\,dxdt$$

$$+\int_0^T\left\{\langle p_{12},\frac{\partial(y_1-y_2)}{\partial t}+A(y_1-y_2)\rangle_{H^1(\Omega)\times(H^1(\Omega))'}-\langle\frac{\partial p_{12}(t)}{\partial n_A},y_1(t)-y_2(t)\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)}\right\}dt$$

$$+\int_0^T\langle\frac{\partial(y_1-y_2)}{\partial n_A}(t),p_{12}(t)\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)}\,dt-\int_{\Omega}\{p_{12}(T)(y_1(T)-y_2(T))+p_{12}(0)(y_1(0)-y_2(0))\}\,dx.$$

Since y_1 and y_2 are the weak solutions of (2) corresponding respectively to (u_1, v_1, w_1) and to (u_2, v_2, w_2) and since p_{12} is the weak solution of (22), we get

$$\int_Q \{F(x,t,y_1,u_1) - F(x,t,y_2,u_1)\}\,dxdt = \int_Q p_{12}(x,t)\{f(x,t,y_2,u_2) - f(x,t,y_2,u_1)\}\,dxdt$$

$$- \int_\Sigma \{G(s,t,y_1,v_1) - G(s,t,y_2,v_1)\}\,dsdt - \int_\Omega \{\ell(x,y_1(T),w_1) - \ell(x,y_2(T),w_1)\}\,dx$$

$$+ \int_\Sigma p_{12}(s,t)\{g(s,t,y_2,v_2) - g(s,t,y_2,v_1)\}\,dsdt + \int_\Omega p_{12}(0)(w_1 - w_2)\,dx.$$

This equality, with the first one, proves the Proposition. □

4.2 Convergence results.

Let (u^*, v^*, w^*) be a triplet of admissible controls and let y^* be the associated state. We consider sequences $\{u_k, v_k, w_k\}_k$ of admissible controls such that:

$$\lim_k \|u_k - u^*\|_{q,Q} = 0, \quad \lim_k \|v_k - v^*\|_{r,\Sigma} = 0,$$

$$\lim_k \|w_k - w^*\|_{2,\Omega} = 0 \quad \text{and} \quad \sup_k \|w_k\|_{\infty,\Omega} \le C_{10}.$$

For every $k \ge 1$ we denote by y_k (respectively y^k) the weak solution of (2) corresponding to (u_k, v_k, w^*) (respectively (u^*, v^*, w_k)) and by p_k (respectively p^k) the weak solution of (22) corresponding to $y_2 = y^*$ and $y_1 = y_k$ (respectively $y_2 = y^*$, $y_1 = y^k$). Let us notice that the sequences $\{||u_k||_{q,Q}\}_k$, $\{||v_k||_{r,\Sigma}\}_k$ and $\{||w_k||_{\infty,\Omega}\}_k$ are uniformly bounded. Thus from Theorem 3.1 (respectively Proposition 4.1), it follows that $(y_k)_k$ and $(y^k)_k$ (respectively $(p_k)_k$ and $(p^k)_k$) are uniformly bounded in $L^\infty(Q)$.

We study the convergence of $(y_k)_k$, $(y^k)_k$, $(p_k)_k$ and $(p^k)_k$ in the following proposition.

Proposition 4.3 *a) If $(y_k)_k$ and $(p_k)_k$ are defined as above then $(y_k)_k$ converges to y^* in $L^\infty(Q)$, $(p_k)_k$ converges to p^* in $L^\infty(Q)$ and $(p_k|_\Sigma)_k$ converges to $p^*|_\Sigma$ in $L^\infty(\Sigma)$, where p^* is the weak solution of the equation:*

$$\begin{aligned}
&-\frac{\partial p}{\partial t} + Ap + f'_y(x,t,y^*,u^*)p = F'_y(x,t,y^*,u^*) && \text{in } Q,\\
&\frac{\partial p}{\partial n_A} + g'_y(s,t,y^*,v^*)p = G'_y(s,t,y^*,v^*) && \text{on } \Sigma,\\
&p(\cdot,T) = \ell'_y(x,y^*(x,T),w^*(x)) && \text{in } \overline{\Omega}.
\end{aligned} \tag{23}$$

b) If $(y^k)_k$ and $(p^k)_k$ are defined as above then $(y^k(T))_k$ converges to $y^*(T)$ in $C(\overline{\Omega})$ and $(p^k(0))_k$ converges to $p^*(0)$ in $C(\overline{\Omega})$.

Remark 4.2. With Theorem 3.1 and Proposition 4.1 we have $\{(y_k)_k, (y^k)_k, y^*\} \subset C(\overline{\Omega}\times]0,T]) \cap L^\infty(Q)$, $\{(p_k)_k, (p^k)_k, p^*\} \subset C(\overline{\Omega}\times[0,T[) \cap L^\infty(Q)$ and the traces of $(y_k)_k, (y^k)_k, y^*, (p_k)_k, (p^k)_k$ and p^* belong to $L^\infty(\Sigma)$.

Proof of Proposition 4.3.

i) For every $k \geq 1$, $z = y_k - y^*$ is the weak solution of the equation:

$$\begin{aligned}
&\frac{\partial z}{\partial t} + Az + a_k z = f(x,t,y^*,u^*) - f(x,t,y^*,u_k) && \text{in } Q,\\
&\frac{\partial z}{\partial n_A} + b_k z = g(s,t,y^*,v^*) - g(s,t,y^*,v_k) && \text{on } \Sigma,\\
&z(0) = 0 && \text{in } \Omega,
\end{aligned}$$

where $a_k(x,t) = \int_0^1 f'_y(x,t,y^* + \theta(y_k - y^*), u_k)\,d\theta \geq C_0$, $b_k(s,t) = \int_0^1 g'_y(s,t,y^* + \theta(y_k - y^*), v_k)\,d\theta \geq C_0$. Thus, from Proposition 3.3 we get

$$\|y_k - y^*\|_{\infty,Q} \leq C_5\{\|f(y^*,u^*) - f(y^*,u_k)\|_{q,Q} + \|g(y^*,v^*) - g(y^*,v_k)\|_{r,\Sigma}\}$$

where $f(y^*,u^*)$ denotes the function $(x,t) \longmapsto f(x,t,y^*(x,t),u^*(x,t))$ and $g(y^*,v^*)$ denotes the function $(s,t) \longmapsto g(s,t,y^*(s,t),v^*(s,t))$; we use the same kind of notation for others functions throughout the proof. Since $(u_k)_k$ converges to u^* in $L^q(Q)$ and $(v_k)_k$ converges to v^* in $L^r(\Sigma)$, with estimates in (A1) and (A2) we can prove by classical arguments that

$$\lim_{k\to\infty} \|f(y^*,u^*) - f(y^*,u_k)\|_{q,Q} = 0, \qquad \lim_{k\to\infty} \|g(y^*,v^*) - g(y^*,v_k)\|_{r,\Sigma} = 0.$$

Therefore, $(y_k)_k$ converges to y^* in $L^\infty(Q)$. As previously, $p = p_k - p^*$ is the weak solution of

$$\begin{aligned}
&-\frac{\partial p}{\partial t} + Ap + \tilde{f}_y(y_k,y^*,u^*)p = \tilde{F}_y(y_k,y^*,u^*) - F'_y(y^*,u^*) + (f'_y(y^*,u^*) - \tilde{f}_y(y_k,y^*,u^*))p^*\\
&\frac{\partial p}{\partial n_A} + \tilde{g}_y(y_k,y^*,v^*)p = \tilde{G}_y(y_k,y^*,v^*) - G'_y(y^*,v^*) + (g'_y(y^*,v^*) - \tilde{g}_y(y_k,y^*,v^*))p^*\\
&p(T) = \tilde{\ell}_y(y_k(T),y^*(T),w^*) - \ell'_y(y^*(T),w^*).
\end{aligned}$$

To complete the proof, it remains to show that $\|p_k - p^*\|_{\infty,\Sigma} + \|p_k - p^*\|_{\infty,Q} \to 0$ as $k \to \infty$. From Proposition 4.1, we get

$$\begin{aligned}
\|p_k - p^*\|_{\infty,\Sigma} + \|p_k - p^*\|_{\infty,Q} \leq\\
C_5\Big\{\|\tilde{F}_y(y_k,y^*,u^*) - F'_y(y^*,u^*)\|_{q,Q} + \|\tilde{G}_y(y_k,y^*,v^*) - G'_y(y^*,v^*)\|_{r,\Sigma} +\\
+\|\tilde{f}_y(y_k,y^*,u^*) - f'_y(y^*,u^*)\|_{q,Q}\|p^*\|_{\infty,Q} + \|\tilde{g}_y(y_k,y^*,v^*) - g'_y(y^*,v^*)\|_{r,\Sigma}\|p^*\|_{\infty,\Sigma} +\\
+\|\tilde{\ell}_y(y_k(T),y^*(T),w^*) - \ell'_y(y^*(T),w^*)\|_{\infty,\Omega}\Big\}.
\end{aligned}$$

Since $(y_k)_k$ converges to y^* in $L^\infty(Q)$ and since $\{(y_k)_k, y^*\} \subset C(\overline{\Omega}\times]0,T])$ then the right-hand side of the above inequality converges to 0 as k tends to infinity and part (a) of Proposition 4.3 is proved.

ii) Now we prove part (b). We first remark that $z = y^k - y^*$ is the solution of the equation

$$\frac{\partial z}{\partial t} + Az + a^k z = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n_A} + b^k z = 0 \quad \text{on } \Sigma, \qquad z(0) = w_k - w^* \quad \text{in } \Omega,$$

(where $a^k(x,t) = \int_0^1 f'_y(x,t,y^* + \theta(y^k - y^*), u^*)\,d\theta \geq C_0$, $b^k(s,t) = \int_0^1 g'_y(s,t,y^* + \theta(y^k - y^*), v^*)\,d\theta \geq C_0$). Moreover $p = p^k - p^*$ is the weak solution of the equation

$$-\frac{\partial p}{\partial t} + Ap + \tilde{f}_y(y^k,y^*,u^*)p = \tilde{F}_y(y^k,y^*,u^*) - F'_y(y^*,u^*) + (f'_y(y^*,u^*) - \tilde{f}_y(y^k,y^*,u^*))p^*$$

$$\frac{\partial p}{\partial n_A} + \tilde{g}_y(y^k, y^*, v^*)p = \tilde{G}_y(y^k, y^*, v^*) - G'_y(y^*, v^*) + (g'_y(y^*, v^*) - \tilde{g}_y(y^k, y^*, v^*))p^*$$

$$p(T) = \tilde{\ell}_y(y^k(T), y^*(T), w^*) - \ell'_y(y^*(T), w^*).$$

¿From Proposition 3.4, it follows that

$$||y^k - y^*||_{C(\overline{Q}_{\varepsilon,T})} \le C_6(\varepsilon)\|w_k - w^*\|_{2,\Omega}$$

for every $\varepsilon > 0$ and from Proposition 4.1

$$\|p^k(0) - p^*(0)\|_{C(\overline{\Omega})} \le$$

$$\le C_6(\varepsilon)\Big\{\|\tilde{F}_y(y^k, y^*, u^*) - F'_y(y^*, u^*)\|_{q,Q} + \|\tilde{G}_y(y^k, y^*, v^*) - G'_y(y^*, v^*)\|_{r,\Sigma}$$

$$+\|\tilde{f}_y(y^k, y^*, u^*) - f'_y(y^*, u^*)\|_{q,Q}\|p^*\|_{\infty,Q} + \|\tilde{g}_y(y^k, y^*, v^*) - g'_y(y^*, v^*)\|_{r,\Sigma}\|p^*\|_{\infty,\Sigma}$$

$$+\|\tilde{\ell}_y(y^k(T), y^*(T), w^*) - \ell'_y(y^*(T), w^*)\|_{2,\Omega}\Big\}. \qquad (24)$$

Thus $(y^k)_k$ converges to y^* almost everywhere in Q, $(y^k|_\Sigma)_k$ converges to $y^*|_\Sigma$ almost everywhere in Σ and $y^k(T)$ converges to $y^*(T)$ in $C(\overline{\Omega})$. Therefore, the right-hand side in (24) converges to 0. This completes the proof. □

For any $(x_0, t_0) \in \overline{Q}$, we consider a sequence of balls $(S_k(x_0, t_0))_k$ centered at (x_0, t_0) such that $\lim_k \mathcal{L}^{N+1} S_k(x_0, t_0) = 0$ (where $\mathcal{L}^{N+1}$ is the $(N+1)-$dimensional Lebesgue measure in $\mathbb{R}^{N+1}$). For any $(s_0, t_0) \in \Sigma$, we consider the sequence $B_k(s_0, t_0) = S_k(s_0, t_0) \cap \Sigma$. For any $x_0 \in \Omega$, we consider a sequence of balls $(W_k(x_0))_k$, centered at x_0, such that $\lim_k \mathcal{L}^N W_k(x_0) = 0$.

Let u be in K_U, v be in K_V and w be in K_W, we define the sequence $(u_k)_k$, $(v_k)_k$ and $(w_k)_k$ of admissible controls by:

$$u_k(x,t) = \begin{cases} u^*(x,t) & \text{on } Q \setminus S_k(x_0, t_0), \\ u & \text{on } S_k(x_0, t_0), \end{cases} \qquad (25)$$

$$v_k(s,t) = \begin{cases} v^*(s,t) & \text{on } \Sigma \setminus B_k(s_0, t_0), \\ v & \text{on } B_k(s_0, t_0), \end{cases} \qquad (26)$$

$$w_k(x) = \begin{cases} w^*(x) & \text{on } \Omega \setminus W_k(x_0), \\ w & \text{on } W_k(x_0). \end{cases} \qquad (27)$$

In the literature on optimal control the sequences defined in (25), (26), (27) are called spike perturbations. Other kind of perturbations (more sophisticated) are also used to prove Pontryagin's principles [10], [18], [14], [28]. These different methods can also be used here. In this case the Pontryagin's principles are first obtained in integral form, spike perturbations must be next used to recover pointwise Pontryagin's principles. Here the method of spike perturbations seems to be the most direct one.

Proposition 4.4 *Let (u^*, v^*, w^*) be a triplet of admissible controls, let u be in K_U, v be in K_V and w be in K_W.*

a) There exists a subset $Q(u^*, u) \subset Q$ (only depending on u^* and u) satisfying $\mathcal{L}^{N+1}(Q(u^*, u)) = \mathcal{L}^{N+1}(Q)$ and such that for every $(x_0, t_0) \in Q(u^*, u)$, we have

$$\lim_k \frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}[J(y_k, u_k, v^*, w^*) - J(y^*, u^*, v^*, w^*)] =$$

$$H_Q(x_0, t_0, y^*(x_0, t_0), u, p^*(x_0, t_0)) - H_Q(x_0, t_0, y^*(x_0, t_0), u^*(x_0, t_0), p^*(x_0, t_0)),$$

where $(u_k)_k$ is the sequence defined in (25), y_k is the weak solution of (2) corresponding to (u_k, v^*, w^*).

b) There exists a subset $\Sigma(v^*, v) \subset \Sigma$ (only depending on v^* and v) satisfying $\mathcal{L}^N(\Sigma(v^*, v)) = \mathcal{L}^N(\Sigma)$ and such that for every $(s_0, t_0) \in \Sigma(v^*, v)$, we have

$$\lim_k \frac{1}{\mathcal{L}^N(B_k(s_0,t_0))}[J(y_k, u^*, v_k, w^*) - J(y^*, u^*, v^*, w^*)] =$$

$$H_\Sigma(s_0, t_0, y^*(s_0, t_0), v, p^*(s_0, t_0)) - H_\Sigma(s_0, t_0, y^*(s_0, t_0), v^*(s_0, t_0), p^*(s_0, t_0)),$$

where $(v_k)_k$ is the sequence defined in (26), y_k is the weak solution of (2) corresponding to (u^*, v_k, w^*).

c) There exists a subset $\Omega(w^*, w) \subset \Omega$ (only depending on w^* and w) satisfying $\mathcal{L}^N(\Omega(w^*, w)) = \mathcal{L}^N(\Omega)$ and such that for every $x_0 \in \Omega(w^*, w)$, we have

$$\lim_k \frac{1}{\mathcal{L}^N(W_k(x_0))}[J(y^k, u^*, v^*, w_k) - J(y^*, u^*, v^*, w^*)] =$$

$$H_\Omega(x_0, y^*(x_0, T), w, p^*(x_0, 0)) - H_\Omega(x_0, y^*(x_0, T), w^*(x_0), p^*(x_0, 0)),$$

where $(w_k)_k$ is the sequence defined in (27), y^k is the weak solution of (2) corresponding to (u^*, v^*, w_k).

Proof of Proposition 4.4.

i) If $u \in K_U$, we denote by $Q(u^*, u)$ the intersection of the sets of Lebesgue points in Q for the functions:

$$(x,t) \longmapsto f(x,t,y^*(x,t),u), \qquad (x,t) \longmapsto H_Q(x,t,y^*(x,t),u^*(x,t),p^*(x,t)),$$

$$(x,t) \longmapsto f(x,t,y^*(x,t),u^*(x,t)), \qquad (x,t) \longmapsto H_Q(x,t,y^*(x,t),u,p^*(x,t)).$$

Taking into account assumptions (A1) and (A2), we have $\mathcal{L}^{N+1}(Q(u^*, u)) = \mathcal{L}^{N+1}(Q)$. Let (x_0, t_0) be in $Q(u^*, u)$. With Proposition 4.2, we have

$$J(y_k, u_k, v_k, w_k) - J(y^*, u^*, v^*, w^*) = \int_Q \{H_Q(x,t,y^*,u_k,p_k) - H_Q(x,t,y^*,u^*,p_k)\}\,dx\,dt$$

$$= \int_{S_k(x_0,t_0)} \{H_Q(x,t,y^*,u,p_k) - H_Q(x,t,y^*,u^*,p_k)\}\,dx\,dt$$

$$= \int_{S_k(x_0,t_0)} \{H_Q(x,t,y^*,u,p^*) - H_Q(x,t,y^*,u^*,p^*)\}\,dx\,dt$$

$$+ \int_{S_k(x_0,t_0)} (p^* - p_k) f(x,t,y^*,u)\,dx\,dt + \int_{S_k(x_0,t_0)} (p_k - p^*) f(x,t,y^*,u^*)\,dx\,dt,$$

where p_k is the solution of (22) corresponding to $(y_2 = y_k, y_1 = y^*)$. Then

$$\lim_k \frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}[J(y_k,u_k,v_k,w_k) - J(y^*,u^*,v^*,w^*)] =$$
$$= \lim_k \frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}\int_{S_k(x_0,t_0)} \{H_Q(x,t,y^*,u,p^*) - H_Q(x,t,y^*,u^*,p^*)\}\,dx\,dt$$
$$+ \lim_k \frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}\{\int_{S_k(x_0,t_0)} (p^* - p_k) f(x,t,y^*(x,t),u)\,dx\,dt$$
$$+ \int_{S_k(x_0,t_0)} (p_k - p^*) f(x,t,y^*(x,t),u^*(x,t))\,dx\,dt\}.$$

We already know that $(p_k)_k$ converges to p^* in $L^\infty(Q)$ (see Proposition 4.3). Since $(x_0,t_0) \in Q(u^*,u)$, we have

$$\frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}|\int_{S_k(x_0,t_0)} (p^* - p_k) f(x,t,y^*,u)\,dx\,dt| \leq$$
$$\frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}\int_{S_k(x_0,t_0)} |f(x,t,y^*,u)|\,dx\,dt||(p^* - p_k)||_{\infty,Q} \longrightarrow 0 \qquad as\ k \longrightarrow \infty,$$

and

$$\frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}|\int_{S_k(x_0,t_0)} (p_k - p^*) f(x,t,y^*,u^*)\,dx\,dt| \leq$$
$$\frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}\int_{S_k(x_0,t_0)} |f(x,t,y^*,u^*)|\,dx\,dt||(p^* - p_k)||_{\infty,Q} \longrightarrow 0 \qquad as\ k \longrightarrow \infty.$$

Therefore it follows

$$\lim_k \frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}[J(y_k,u_k,v_k,w_k) - J(y^*,u^*,v^*,w^*)] =$$
$$= \lim_k \frac{1}{\mathcal{L}^{N+1}(S_k(x_0,t_0))}\int_{S_k(x_0,t_0)} \{H_Q(x,t,y^*,u,p^*) - H_Q(x,t,y^*,u^*,p^*)\}\,dx\,dt$$
$$= H_Q(x_0,t_0,y^*(x_0,t_0),u,p^*(x_0,t_0)) - H_Q(x_0,t_0,y^*(x_0,t_0),u^*(x_0,t_0),p^*(x_0,t_0)).$$

Assertions (b) can be proved in the same way.

ii) Let $\Omega(w^*,w)$ be the intersection of sets of Lebesgue points in Ω for the functions:

$$x \longmapsto w^*(x), \quad x \longmapsto H_\Omega(x,y^*(x,T),w,p^*(x,0)), \quad x \longmapsto H_\Omega(x,y^*(x,T),w^*(x),p^*(x,0)).$$

Let x_0 be in $\Omega(w^*,w)$. From Proposition 4.2, we deduce:

$$J(y^k,u^*,v^*,w_k) - J(y^*,u^*,v^*,w^*) = \int_\Omega \{H_\Omega(x,y^*(T),w_k,p^k(0)) - H_\Omega(x,y^*(T),w^*,p^k(0))\}dx$$
$$= \int_{W_k(x_0)} \{H_\Omega(x,y^*(T),w,p^k(0)) - H_\Omega(x,y^*(T),w^*,p^k(0))\}dx$$
$$= \int_{W_k(x_0)} \{H_\Omega(x,y^*(T),w,p^*(0)) - H_\Omega(x,y^*(T),w^*,p^*(0))\}dx$$
$$+ \int_{W_k(x_0)} (p^*(x,0) - p^k(x,0))(w^*(x) - w)dx,$$

where p^k is the solution of (22) corresponding to $(y_2 = y^k, y_1 = y^*)$. By using the same arguments as above and taking into account Proposition 4.3, we end the proof. □

4.3 Proof of Theorem 2.1

Let $(\bar{y}, \bar{u}, \bar{v}, \bar{w})$ be a solution of (P) and let $\bar{p}$ be its associated adjoint state. Thanks to Proposition 4.4, for every $u \in K_U$, every $v \in K_V$ and every $w \in K_W$ we have:

$$H_Q(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{p}(x, t)) \leq H_Q(x, t, \bar{y}(x, t), u, \bar{p}(x.t))$$

for every $(x, t) \in Q(\bar{u}, u)$,

$$H_\Sigma(s, t, \bar{y}(s, t), \bar{v}(s, t), \bar{p}(s, t)) \leq H_\Sigma(s, t, \bar{y}(s, t), v, \bar{p}(s, t))$$

for every $(s, t) \in \Sigma(\bar{v}, v)$ and

$$H_\Omega(x, \bar{y}(x, T), \bar{w}(x), \bar{p}(x, 0)) \leq H_\Omega(x, \bar{y}(x, T), w, \bar{p}(x, 0))$$

for every $x \in \Omega(\bar{w}, w)$.

i) Let $(u_k)_{k\geq 1}$ be a countable dense subset of K_U. With the above inequality for H_Q, we get

$$H_Q(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{p}(x, t)) \leq H_Q(x, t, \bar{y}(x, t), u_k, \bar{p}(x, t))$$

for every $(x, t) \in Q(\bar{u}, u_k)$. Let us set $Q_0 = \bigcap_{k\geq 1} Q(\bar{u}, u_k)$, we have $\mathcal{L}^{N+1}(Q_0) = \mathcal{L}^{N+1}(Q)$ and

$$H_Q(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{p}(x, t)) \leq H_Q(x, t, \bar{y}(x, t), u_k, \bar{p}(x, t))$$

for every $(x, t) \in Q_0$ and for every $k \geq 1$. Since the Hamiltonian H_Q is continuous with respect to the control variable u, we have

$$H_Q(x, t, \bar{y}(x, t), \bar{u}(x, t), \bar{p}(x, t)) = \min_{u\in K_U} H_Q(x, t, \bar{y}(x, t), u, \bar{p}(x, t))$$

for every $(x, t) \in Q_0$.

ii) The boundary Pontryagin principle and the Pontryagin principle for the initial condition can be shown in a similar manner.

5 Appendix.

Let f^* and g^* be two functions defined respectively on $\Omega\times]0, T[\times\mathbb{R}$ and on $\Gamma\times]0, T[\times\mathbb{R}$. We suppose that f^*, g^* verify the following assumptions.

(A1') - For every $y \in \mathbb{R}$, $f^*(\cdot, y)$ is measurable on Q. For almost every $(x, t) \in Q$, $f^*(x, t, \cdot)$ is of class C^1 on $\mathbb{R}$ and we have the following estimates

$$|f^*(x, t, 0)| \leq M_1^*(x, t), \qquad C_0 \leq {f_y^*}'(x, t, y) \leq M_1^*(x, t),$$

where M_1^* belongs to $L^{2q}(Q)$, C_0 is as in $(A1)$.

(A2') - For every $y \in \mathbb{R}$, $g^*(\cdot, y)$ is measurable on Σ. For almost every $(s, t) \in \Sigma$, $g^*(s, t, \cdot)$ is of class C^1 on $\mathbb{R}$ and we have the following estimates

$$|g^*(s, t, 0)| \leq M_2^*(s, t), \qquad C_0 \leq {g_y^*}'(s, t, y) \leq M_2^*(s, t),$$

where M_2^* belongs to $L^{2r}(\Sigma)$, C_0 is as in $(A1)$.

Lemma 5.1 *Let y_0 be in $L^\infty(\Omega)$. Under the assumptions $(A1') - (A2')$, the equation*

$$\frac{\partial y}{\partial t} + Ay + f^*(x,t,y) = 0 \quad in\ Q, \qquad \frac{\partial y}{\partial n_A} + g^*(s,t,y) = 0 \quad on\ \Sigma, \qquad y(0) = y_0 \quad in\ \Omega, \tag{28}$$

admits a unique weak solution y in $W(0,T)$, this solution belongs to $L^\infty(Q)$ and satisfies the estimate

$$\|y\|_{\infty,Q} + \|y\|_{\infty,\Sigma} + \|y(T)\|_{\infty,\Omega} + \|y\|_{L^2(0,T;H^1(\Omega))} \leq$$
$$\leq C_{11}(\|f^*(\cdot,0)\|_{q,Q} + \|g^*(\cdot,0)\|_{r,\Sigma} + \|y_0\|_{\infty,\Omega}),$$

where $C_{11} = C_{11}(N,q,r,\Omega,T,C_0)$.

Proof of Lemma 5.1. We can suppose, without loss of generality, that $((a_{ij})_{i,j}, C_0, C_0)$ verifies the ellipticity condition (Em_0) (see part (i) below in the proof of Theorem 3.1). In this case, the proof of existence result is similar to the one given in ([23], Chapter 2, Theorem 1.2). We denote by y the weak solution of (28). We remark that y is also the weak solution of the following equation:

$$\frac{\partial z}{\partial t} + Az + a^*z = -f^*(x,t,0) \ \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} + b^*z = -g^*(s,t,0) \ \text{ on } \Sigma, \qquad z(0) = y_0 \ \text{ in } \Omega,$$

where $a^*(x,t) = \int_0^1 f_y^{*\prime}(x,t,\theta y)\,d\theta \geq C_0$ and $b^*(s,t) = \int_0^1 g_y^{*\prime}(s,t,\theta y)\,d\theta \geq C_0$. ¿From Proposition 3.3 and Proposition 3.4, we deduce $y \in C(\overline{Q}_{\varepsilon,T}) \cap L^\infty(Q)$, $y|_\Sigma \in L^\infty(\Sigma)$ and

$$\|y\|_{\infty,Q} + \|y\|_{\infty,\Sigma} + \|y(T)\|_{\infty,\Omega} \leq K(\|f^*(\cdot,0)\|_{q,Q} + \|g^*(\cdot,0)\|_{r,\Sigma} + \|y_0\|_{\infty,\Omega}).$$

The estimate in $L^2(0,T;H^1(\Omega))$ is immediate. □

Proof of existence result of Theorem 3.1.

i) If $((a_{ij})_{i,j}, C_0, C_0)$ does not verify the ellipticity condition (Em_0), we set $z(x,t) = e^{-\theta t}y(x,t)$, where θ is chosen so that $((a_{ij})_{i,j}, C_0+\theta, C_0)$ satisfies (Em_0). We remark that y is a weak solution of (2) if and only if z is a weak solution of

$$\frac{\partial z}{\partial t} + Az + \tilde{f}(x,t,z,u) = 0 \ \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} + \tilde{g}(s,t,z,v) = 0 \ \text{ on } \Sigma, \qquad z(0) = y_0 \ \text{ in } \Omega,$$

with

$$\tilde{f}(x,t,z,u) = e^{-\theta t}f(x,t,e^{\theta t}z,u) + \theta z, \qquad \tilde{g}(s,t,z,v) = e^{-\theta t}g(s,t,e^{\theta t}z,v).$$

It is clear that

$$\tilde{f}'_z(x,t,z,u) \geq C_0 + \theta, \quad \tilde{g}'_z(s,t,z,v) \geq C_0.$$

Thus, without loss of generality, we can suppose that $((a_{ij})_{i,j}, C_0, C_0)$ verifies the ellipticity condition (Em_0).

ii) To use the previous lemma, we consider an equation with truncated and regularized functions. For this we set:

$$f_k(x,t,y,u) = \begin{cases} f(x,t,k,u) + f'_y(x,t,k,u)(y-k) & \text{if } y > k, \\ f(x,t,y,u) & \text{if } |y| \leq k, \\ f(x,t,-k,u) + f'_y(x,t,-k,u)(y+k) & \text{if } y < -k, \end{cases}$$

$$g_k(s,t,y,v) = \begin{cases} g(s,t,k,v) + g'_y(s,t,k,v)(y-k) & \text{if } y > k, \\ g(s,t,y,v) & \text{if } |y| \le k, \\ g(s,t,-k,v) + g'_y(s,t,-k,v)(y+k) & \text{if } y < -k. \end{cases}$$

Let u be in $L^q(Q)$, v in $L^r(\Sigma)$ and w in $L^\infty(\Omega)$. We want to prove that the equation

$$\frac{\partial y}{\partial t} + Ay + f_k(x,t,y,u) = 0 \quad \text{in } Q, \qquad \frac{\partial y}{\partial n_A} + g_k(s,t,y,v) = 0 \quad \text{on } \Sigma, \qquad y(0) = w \quad \text{in } \Omega, \tag{29}$$

admits a weak solution in $W(0,T) \cap L^\infty(Q)$. Lemma 5.1 cannot be directly applied to equation (29), first we must regularize f_k and g_k. For every $k \ge 1$, we consider the sequences of functions $(f_k^n)_{n\ge1}$ and $(g_k^n)_{n\ge1}$ defined by

$$f_k^n(x,t,y) = \theta_n * f_k(\cdot, y, u(\cdot))(x,t),$$

$$g_k^n(s,t,y) = \tilde{\theta}_n * g_k(\cdot, y, v(\cdot))(s,t),$$

where $(\theta_n)_n$ is a sequence of nonnegative regularizing kernels in $\mathbb{R}^N \times \mathbb{R}$ and $(\tilde{\theta}_n)_n$ is a sequence of nonnegative regularizing kernels in $\Gamma \times \mathbb{R}$ ($f_k(\cdot, y, u(\cdot))$ and $g_k(\cdot, y, v(\cdot))$ are extended by zero outside Q and Σ). We consider the equation

$$\frac{\partial y}{\partial t} + Ay + f_k^n(x,t,y) = 0 \quad \text{in } Q, \qquad \frac{\partial y}{\partial n_A} + g_k^n(s,t,y) = 0 \quad \text{on } \Sigma, \qquad y(0) = w \quad \text{in } \Omega. \tag{30}$$

Since f_k^n verifies (A1') and g_k^n verifies (A2') for every $k \ge 1$ and every $n \ge 1$, from Lemma 5.1 equation (30) admits a unique weak solution in $W(0,T)$ denoted by y_k^n. Moreover,

$$\begin{aligned} &\|y_k^n\|_{\infty,Q} + \|y_k^n\|_{\infty,\Sigma} + \|y_k^n(T)\|_{\infty,\Omega} + \|y_k^n\|_{L^2(0,T;H^1(\Omega))} \le \\ &\qquad \le K(\|f_k^n(\cdot,0)\|_{q,Q} + \|g_k^n(\cdot,0)\|_{r,\Sigma} + \|w\|_{\infty,\Omega}), \\ &\qquad \le K_1(\|f(\cdot,0,u(\cdot))\|_{q,Q} + \|g(\cdot,0,v(\cdot))\|_{r,\Sigma} + \|w\|_{\infty,\Omega} + 1). \end{aligned} \tag{31}$$

For every $k \ge 1$ and every $n \ge 1$, we define the nonlinear operator $\mathcal{A}_k^n$ from $L^2(0,T;H^1(\Omega))$ into $L^2(0,T;(H^1(\Omega))')$ such that, for every $z, \varphi \in L^2(0,T;H^1(\Omega))$, we have

$$\langle \mathcal{A}_k^n(z), \varphi \rangle = \sum_{i,j} \int_Q a_{ij}(x) D_j z(x,t) D_i \varphi(x,t)\, dxdt +$$

$$+ \int_Q f_k^n(x,t,z(x,t))\varphi(x,t)\, dxdt + \int_\Sigma g_k^n(s,t,z(s,t))\varphi(s,t)\, dsdt$$

where $\langle \cdot, \cdot \rangle$ is the duality product between $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;(H^1(\Omega))')$. ¿From (Em_0), we see that for every $k \ge 1$ and every $n \ge 1$, $\mathcal{A}_k^n$ is well defined and is monotone. Equation (30) can be written in the form:

$$\frac{dy}{dt} + \mathcal{A}_k^n(y) = 0 \quad \text{in } L^2(0,T;(H^1(\Omega))'), \qquad y(0) = w \quad \text{in } \Omega,$$

and we have

$$\|\frac{dy_k^n}{dt}\|_{L^2(0,T;(H^1(\Omega))')} = \|\mathcal{A}_k^n(y_k^n)\|_{L^2(0,T;(H^1(\Omega))')} \le$$

$$\leq K_2 \left\{ \| \int_0^1 (f_k^n)'_y(\cdot, \theta y_k^n)\, d\theta \|_{2,Q} \|y_k^n\|_{\infty,Q} + \|f_k^n(\cdot, 0)\|_{2,Q} + \|g_k^n(\cdot, 0)\|_{2,\Sigma} \right.$$

$$\left. + \| \int_0^1 (g_k^n)'_y(\cdot, \theta y_k^n)\, d\theta \|_{2,\Sigma} \|y_k^n\|_{\infty,\Sigma} + \|y_k^n\|_{L^2(0,T;H^1(\Omega))} \right\}$$

$$\leq K_1 K_2 \left\{ \| \int_0^1 f'_{ky}(\cdot, \theta y_k^n, u)\, d\theta \|_{2,Q} + \| \int_0^1 g'_{ky}(\cdot, \theta y_k^n, v)\, d\theta \|_{2,\Sigma} + \|f(\cdot, 0, u)\|_{2,Q} + \|g(\cdot, 0, v)\|_{2,\Sigma} \right\}$$

$$\times \left\{ \|f(\cdot, 0, u)\|_{q,Q} + \|g(\cdot, 0, v)\|_{r,\Sigma} + \|w\|_{\infty,\Omega} + 1 \right\}.$$

¿From the above estimates, it follows that, for every $k \geq 1$, the sequences $\{\mathcal{A}_k^n(y_k^n)\}_n$, $\{y_k^n\}_n$, $\{y_k^n|_\Sigma\}_n$ and $\{y_k^n(T)\}_n$ are bounded respectively in $L^2(0, T; (H^1(\Omega))')$, in $W(0, T) \cap L^\infty(Q)$, in $L^\infty(\Sigma)$ and in $L^2(\Omega)$. For every $k \geq 1$, there then exist subsequences, still indexed by n to simplify the notation, $y_k \in W(0, T) \cap L^\infty(Q)$ and $\chi_k \in L^2(0, T; (H^1(\Omega))')$ such that

$$y_k^n \rightharpoonup y_k \quad \text{in } W(0, T), \qquad y_k^n \rightharpoonup y_k \quad \text{weakly star in } L^\infty(Q),$$

$$y_k^n|_\Sigma \rightharpoonup y_k|_\Sigma \quad \text{weakly star in } L^\infty(\Sigma),$$

$$\mathcal{A}_k^n(y_k^n) \rightharpoonup \chi_k \quad \text{weakly in } L^2(0, T; (H^1(\Omega))'), \quad y_k^n(T) \rightharpoonup y_k(T) \quad \text{weakly in } L^2(\Omega).$$

Recall that y_k^n is the weak solution of (30), hence for every $\varphi \in L^2(0, T; H^1(\Omega))$ we have

$$\langle \frac{dy_k^n}{dt} + \mathcal{A}_k^n(y_k^n), \varphi \rangle_{L^2(0,T;(H^1(\Omega))') \times L^2(0,T;H^1(\Omega))} = 0. \tag{32}$$

Thus, by passing to the limit in this equality when n tends to infinity, we get

$$\frac{dy_k}{dt} = -\chi_k \qquad \text{in } L^2(0, T; (H^1(\Omega))').$$

To prove that y_k is the weak solution of (29), it is sufficient to prove that χ_k coincides with $\mathcal{A}_k(y_k)$, where $\mathcal{A}_k$ is the operator from $L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$ into $L^2(0, T; (H^1(\Omega))')$ defined by

$$\langle \mathcal{A}_k(z), \varphi \rangle = \sum_{i,j} \int_Q a_{ij}(x) D_j z(x, t) D_i \varphi(x, t)\, dxdt +$$

$$+ \int_Q f_k(x, t, z(x, t), u(x, t)) \varphi(x, t)\, dxdt + \int_\Sigma g_k(s, t, z(s, t), v(s, t)) \varphi(s, t)\, dsdt$$

for every $z \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$ and every $\varphi \in L^2(0, T; H^1(\Omega))$. Let us notice that, when n tends to infinity, $\{f_k^n(\cdot, z)\}_n$ converges to $f_k(\cdot, z)$ in $L^q(Q)$ and $\{g_k^n(\cdot, z)\}_n$ converges to $g_k(\cdot, z)$ in $L^r(\Sigma)$. Therefore, for every $z \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$, $\{\mathcal{A}_k^n(z)\}_n$ strongly converges to $\mathcal{A}_k(z)$ in $L^2(0, T; (H^1(\Omega))')$. Taking into account the monotony of $\mathcal{A}_k^n$, for $z \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$ we have

$$\langle \mathcal{A}_k^n(z) - \mathcal{A}_k^n(y_k^n), z - y_k^n \rangle \geq 0.$$

With this inequality and with (32), we get

$$\frac{1}{2}\|w\|_{2,\Omega}^2 - \frac{1}{2}\|y_k^n(T)\|_{2,\Omega}^2 - \langle \mathcal{A}_k^n(y_k^n), z \rangle + \langle \mathcal{A}_k^n(z), z - y_k^n \rangle \geq 0.$$

By passing to the limit when n tends to infinity, since $\liminf_n \|y_k^n(T)\|_{2,\Omega}^2 \geq \|y_k(T)\|_{2,\Omega}^2$, it results that:

$$\frac{1}{2}\|w\|_{2,\Omega}^2 - \frac{1}{2}\|y_k(T)\|_{2,\Omega}^2 - \langle \chi_k, z \rangle + \langle \mathcal{A}_k(z), z - y_k \rangle \geq 0.$$

Thus, we have:

$$\langle \mathcal{A}_k(z) - \chi_k, z - y_k \rangle \geq 0 \qquad \text{for every } z \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q). \tag{33}$$

Let ψ be in $L^2(0,T;H^1(\Omega)) \cap L^\infty(Q)$ and let $\lambda > 0$. We put $z = y_k - \lambda\psi$ in (33):

$$\langle \mathcal{A}_k(y_k - \lambda\psi) - \chi_k, \psi \rangle \leq 0.$$

If λ tends to 0, we get

$$\langle \mathcal{A}_k(y_k) - \chi_k, \psi \rangle = 0 \qquad \text{for every } \psi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q).$$

Finally, by denseness arguments we obtain

$$\mathcal{A}_k(y_k) = \chi_k \qquad \text{in } L^2(0,T;(H^1(\Omega))').$$

Therefore we have proved that y_k is the weak solution in $W(0,T) \cap L^\infty(Q)$ of (29). Moreover y_k verifies the estimate (31). It is clear that if $k \geq K_1(\|f(\cdot,0,u)\|_{q,Q} + \|g(\cdot,0,v)\|_{r,\Sigma} + \|w\|_{\infty,\Omega} + 1)$, then $f_k(x,t,y_k,u) = f(x,t,y_k,u)$, $g_k(s,t,y_k,v) = g(s,t,y_k,v)$ and y_k is also the weak solution of (2).

ii) It is easy to see that the weak solution of (2) is also the weak solution of the following linear equation

$$\frac{\partial z}{\partial t} + Az + az = -f(x,t,0,u) \ \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} + bz = -g(s,t,0,v) \ \text{ on } \Sigma, \qquad z(0) = y_0 \ \text{ in } \Omega,$$

where $a(x,t) = \int_0^1 f'_y(x,t,\theta y,u)\,d\theta \geq C_0$ and $b(s,t) = \int_0^1 g'_y(s,t,\theta y,v)\,d\theta \geq C_0$. ¿From Proposition 3.4 and taking in account the assumptions (A1)-(A2), the regularity and the estimate in $C(\overline{Q}_{\varepsilon,T})$ are immediate. □

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