PONTRYAGIN'S PRINCIPLE FOR LOCAL SOLUTIONS OF CONTROL PROBLEMS WITH MIXED CONTROL-STATE CONSTRAINTS*

E. CASAS†, J.-P. RAYMOND‡, AND H. ZIDANI‡

Abstract. This paper deals with optimal control problems of semilinear parabolic equations with pointwise state constraints and coupled integral state-control constraints. We obtain necessary optimality conditions in the form of a Pontryagin's minimum principle for local solutions in the sense of L^p, $p \leq +\infty$.

Key words. Optimal control, nonlinear boundary controls, semilinear parabolic equations, state constraints, Pontryagin's minimum principle, unbounded controls.

AMS subject classifications. 49K20, 35K20

1. Introduction. Let T be a positive number, Ω be a bounded open subset in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary Γ, and q, σ and $\bar{\sigma}$ be numbers satisfying:

$$q > N/2 + 1 \quad \text{and} \quad \sigma > \bar{\sigma} > N + 1.$$

Consider the parabolic system:

$$\frac{\partial y}{\partial t} + Ay + f(x,t,y) = 0 \text{ in } Q,\ \frac{\partial y}{\partial n_A} + g(s,t,y,v) = 0 \text{ on } \Sigma,\ y(0) = y_0 \text{ in } \Omega, \tag{1.1}$$

(where $Q := \Omega\times]0,T[$, $\Sigma := \Gamma\times]0,T[$, $T > 0$, v is a boundary control, $y_0 \in C(\overline{\Omega})$, A is a second order elliptic operator) and the following control and state constraints:

$$v \in \widetilde{V}_{ad} := \{v \in L^\sigma(\Sigma) \mid v(s,t) \in V(s,t) \quad \text{for a.e. } (s,t) \in \Sigma\},$$

$$\begin{aligned} &\int_\Sigma h_i(s,t,v(s,t))\, dsdt = 0,\ 1 \leq i \leq \ell_0, \\ &\int_\Sigma h_i(s,t,v(s,t))\, dsdt \leq 0,\ \ell_0 + 1 \leq i \leq \ell, \end{aligned} \tag{1.2}$$

$$\Phi(y) \in \mathcal{C}, \tag{1.3}$$

$$\begin{aligned} &\int_\Sigma \Psi_i(s,t,y(s,t),v(s,t))\, dsdt = 0, 1 \leq i \leq m_0, \\ &\int_\Sigma \Psi_i(s,t,y(s,t),v(s,t))\, dsdt \leq 0, m_0 + 1 \leq i \leq m. \end{aligned} \tag{1.4}$$

(V is a measurable set-valued mapping from Σ with closed and nonempty values in $\mathcal{P}(\mathbb{R}^k)$, the set of all subsets of $\mathbb{R}^k$, $h = (h_1, \ldots, h_\ell)$ is a function with values in $\mathbb{R}^\ell$,

*The research of the first author was partially supported by Dirección General de Investigación Científica y Técnica (Madrid)

†Departamento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain

‡Université Paul Sabatier, UMR CNRS MIP, UFR MIG, 31062 Toulouse Cedex 4, France

$\Psi = (\Psi_1, \dots, \Psi_m)$ is a function with values in $\mathbb{R}^m$, Φ is a continuous mapping from $C(\overline{D})$ into $C(\overline{D})$, $\mathcal{C} \subset C(\overline{D})$, $\overline{D}$ is a nonempty compact subset of $\overline{Q}$.) Let us set the following optimal control problems:

$$(P_1) \qquad \inf\{J(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), v \in \widetilde{V}_{ad}, (y,v) \text{ satisfies } (1.1), (1.2)\},$$

$$(P_2) \quad \inf\{J(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), v \in \widetilde{V}_{ad}, (y,v) \text{ satisfies } (1.1), (1.2), (1.3)\},$$

$$(P) \ \inf\{J(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), v \in \widetilde{V}_{ad}, (y,v) \text{ satisfies } (1.1), (1.2), (1.3), (1.4)\},$$

for the cost functional:

$$J(y,v) = \int_Q F(x,t,y(x,t))\,dx\,dt + \int_\Sigma G(s,t,y(s,t),v(s,t))\,ds\,dt + \int_\Omega L(x,y(x,T))dx.$$

We are mainly interested in optimality conditions for such problems, in the form of Pontryagin's principles. The existence of optimal solutions for (P) is a priori supposed.

Problems of the form (P_1) deal with pointwise control constraints and integral control constraints (often called isoperimetric constraints), but do not deal with state constraints. Problems of the form (P_2) deal with the same control constraints as in (P_1), but also with pointwise state constraints. Optimality conditions for problems with control contraints as in (P_2) are derived in [2], where a multiplier is associated to the control constraint (1.2). Such problems are also considered in [6] for semilinear elliptic equations. The novelty in [6] is that Pontryagin's principles are obtained with no multiplier associated with (1.2). In this paper we want to extend results of [6] to problems of the form (P).

Our main result, stated in Section 2, is a Pontryagin's principle for problem (P), with no Lagrange multiplier associated with the constraints (1.2). This can be used to derive Pontryagin's principle for local solutions in $L^\sigma(\Sigma)$, $\sigma < +\infty$. This result is presented in Section 2 as a corollary to our main result, Theorem 2.1. The corresponding result for local solutions in $L^\infty(\Sigma)$, in the case of bounded controls, is easily deduced from Pontryagin's principle for problems with pointwise control constraints of the form $v(s,t) \in V(s,t)$. We will also show that the full version of Pontryagin's principle is an immediate consequence of Theorem 2.1. With the full version of Pontryagin's principle we mean the classical formulation of this principle with the Lagrange multipliers of the integral control constraints included in the Hamiltonian function.

Here we deal with parabolic equations of the form (1.1), where the coefficients of the operator A are not regular, and where the nonlinear terms $f(x,t,\cdot)$ and $g(s,t,\cdot)$ are neither Lipschitz nor monotone with respect to y. When $g(s,t,\cdot,v)$ is Lipschitz and monotone such an equation is studied in [5] for bounded controls. For unbounded controls, when $g(s,t,\cdot,v)$ is neither Lipschitz nor monotone, but when the coefficients of A are time independent and regular, equation (1.1) is studied in [24, 25] by means of estimates on analytic semigroups. Here we combine these different difficulties. Equation (1.1) and the adjoint state equation are studied in Section 3.

Section 4 is devoted to the study of the metric space of the controls and to the existence of diffuse perturbations of controls. These perturbations are the key for the proof of Pontryagin's principle, which is done in Section 5.

2. Main results. We set $\overline{\Omega}_0 = \overline{\Omega} \times \{0\}$ and $\overline{\Omega}_T = \overline{\Omega} \times \{T\}$. For every $1 \leq \tau \leq \infty$, the usual norms of the spaces $L^\tau(\Omega)$, $L^\tau(\Gamma)$, $L^\tau(Q)$, $L^\tau(\Sigma)$ will be denoted by $\|.\|_{\tau,\Omega}$, $\|.\|_{\tau,\Gamma}$, $\|.\|_{\tau,Q}$, $\|.\|_{\tau,\Sigma}$. For every $t > 0$, we define the norm $\|.\|_{Q(t)}$ by $\|y\|^2_{Q(t)} := \|y\|^2_{L^2(0,t;H^1(\Omega))} + \|y\|^2_{L^\infty(0,t;L^2(\Omega))}$. The Hilbert space $W(0,T;H^1(\Omega),(H^1(\Omega))') = \{y \in L^2(0,T;H^1(\Omega)) \mid \frac{dy}{dt} \in L^2(0,T;(H^1(\Omega))')\}$, endowed with its usual norm, will be denoted by $W(0,T)$. We denote by V_{ad} the set of admissible controls:

$$V_{ad} := \{v \in \widetilde{V}_{ad} \mid v \text{ satisfies } (1.2)\}.$$

2.1. Assumptions. **(A1)** - The operator A is defined by:

$$Ay(x,t) = -\sum_{i=1}^N D_i \left(\sum_{j=1}^N (a_{ij}(x,t)D_j y(x,t)) + a_i(x,t)y(x,t) \right) + \sum_{i=1}^N (b_i(x,t)D_i y(x,t)),$$

the coefficients a_{ij} belong to $L^\infty(Q)$, a_i and b_i belong to $L^{2q}(Q)$, and

$$(2.1)\quad \Lambda|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x,t)\xi_j\xi_i \quad \text{for all } \xi \in \mathbb{R}^N, \quad \text{a.e. } (x,t) \in Q, \quad \text{with } \Lambda > 0.$$

We make the following assumptions on f, g, F, G, L, Φ, Ψ.

(A2) - For every $y \in \mathbb{R}$, $f(\cdot,y)$ is measurable on Q. For almost every $(x,t) \in Q$, $f(x,t,\cdot)$ is of class C^1 on $\mathbb{R}$. The following estimates hold

$$|f(x,t,0)| \leq M_1(x,t), \qquad C_0 \leq f'_y(x,t,y) \leq M_1(x,t)\eta(|y|),$$

where M_1 belongs to $L^q(Q)$, η is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and $C_0 \in \mathbb{R}$. (We have denoted by f'_y the partial derivative of f with respect to y, throughout the sequel we adopt the same kind of notation for other functions.)

(A3) - For every $(y,v) \in \mathbb{R}^2$, $g(\cdot,y,v)$ is measurable on Σ. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbb{R}$, $g(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. For almost every $(s,t) \in \Sigma$, $g(s,t,\cdot)$ and $g'_y(s,t,\cdot)$ are continuous on $\mathbb{R} \times \mathbb{R}$. The following estimates hold

$$|g(s,t,0,v)| \leq M_2(s,t) + \Lambda_1|v|, \qquad C_0 \leq g'_y(s,t,y,v) \leq (M_2(s,t) + \Lambda_1|v|)\eta(|y|),$$

where M_2 belongs to $L^\sigma(\Sigma)$, $\Lambda_1 > 0$, C_0 and η are as in $(A2)$.

(A4) - For every $y \in \mathbb{R}$, $L(\cdot,y)$ is measurable on Ω. For almost every $x \in \Omega$, $L(x,\cdot)$ is of class C^1 on $\mathbb{R}$. The following estimate holds

$$|L(x,y)| + |L'_y(x,y)| \leq M_3(x)\eta(|y|),$$

where $M_3 \in L^1(\Omega)$, η is as in (A2).

(A5) - For every $y \in \mathbb{R}$, $F(\cdot,y)$ is measurable on Q. For almost every $(x,t) \in Q$, $F(x,t,\cdot)$ is of class C^1 on $\mathbb{R}$. The following estimate holds

$$|F(x,t,y)| + |F'_y(x,t,y)| \leq M_4(x,t)\eta(|y|),$$

where $M_4 \in L^1(Q)$, η is as in $(A2)$.

(A6) - For every $(y,v) \in \mathbb{R}^2$, $G(\cdot,y,v)$ is measurable on Σ. For almost every $(s,t) \in \Sigma$

and every $v \in \mathbb{R}$, $G(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. For almost every $(s,t) \in \Sigma$, $G(s,t,\cdot)$ and $G'_y(s,t,\cdot)$ are continuous on $\mathbb{R} \times \mathbb{R}$. The following estimate holds

$$-M_5(s,t) - \Lambda_1 |v|^{\bar{\sigma}} \le G(s,t,0,v) \le M_5(s,t) + \Lambda_1 |v|^{\sigma},$$

$$|G'_y(s,t,y,v)| \le (M_5(s,t) + \Lambda_1 |v|^{\bar{\sigma}})\eta(|y|),$$

where $M_5 \in L^1(\Sigma)$, Λ_1 and η are as in $(A3)$.

(A7) - The function $h = (h_1, ..., h_\ell)$ is a Carathéodory function from $\Sigma \times \mathbb{R}$ into $\mathbb{R}^\ell$ satisfying

$$|h_i(s,t,v)| \le M_5(s,t) + \Lambda_1 |v|^{\bar{\sigma}}, \quad \text{for } i = 1, \dots, \ell_0,$$

$$-M_5(s,t) - \Lambda_1 |v|^{\bar{\sigma}} \le h_i(s,t,v) \le M_5(s,t) + \Lambda_1 |v|^{\sigma} \quad \text{for } i = \ell_0 + 1, \dots, \ell,$$

Λ_1 and M_5 are the same as above.

(A8) - The function $\Psi = (\Psi_1, ..., \Psi_m)$ is a Carathéodory function from $\Sigma \times \mathbb{R}^2$ into $\mathbb{R}^m$. For almost every $(s,t) \in \Sigma$ and every $v \in \mathbb{R}$, $\Psi(s,t,\cdot,v)$ is of class C^1 on $\mathbb{R}$. For almost every $(s,t) \in \Sigma$, $\Psi'_y(s,t,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}$. The following estimate holds

$$|\Psi_i(s,t,0,v)| \le M_5(s,t) + \Lambda_1 |v|^{\bar{\sigma}}, \quad \text{for } i = 1, ..., m_0,$$

$$-M_5(s,t) - \Lambda_1 |v|^{\bar{\sigma}} \le \Psi_i(s,t,0,v) \le M_5(s,t) + \Lambda_1 |v|^{\sigma}, \quad \text{for } i = m_0 + 1, ..., m,$$

$$|\Psi'_{iy}(s,t,y,v)| \le (M_5(s,t) + \Lambda_1 |v|^{\bar{\sigma}})\eta(|y|), \quad \text{for } i = 1, ..., m,$$

where Λ_1, M_5, η are as before. We also suppose that the function $\Phi : C(\overline{D}) \to C(\overline{D})$ is of class C^1, and that $\mathcal{C} \subset C(\overline{D})$ is a closed convex subset of finite codimension in $C(\overline{D})$, where $\overline{D}$ is a compact subset of $\overline{Q}$.

2.2. Statement of the main result. We define the boundary Hamiltonian function by:

$$H_\Sigma(y,v,p,\nu,\lambda) = \int_\Sigma [\nu G(s,t,y,v) - pg(s,t,y,v) + \lambda \Psi(s,t,y,v)] \, dsdt$$

for every $(y,v,p,\nu,\lambda) \in C(\overline{Q}) \times L^\sigma(\Sigma) \times L^{\sigma'}(\Sigma) \times \mathbb{R}^{1+m}$. (Here $\lambda = (\lambda^1, ..., \lambda^m)$, $\lambda\Psi(s,t,y,v) = \sum_{i=1}^m \lambda^i \Psi_i(s,t,y,v)$. Throughout the paper we adopt the same kind of notation for scalar products in $\mathbb{R}^m$.)

THEOREM 2.1. *If* $(A1)-(A8)$ *are fulfilled and if* $(\bar{y},\bar{v})$ *is a solution of* (P)*, there then exist* $\bar{p} \in L^1(0,T;W^{1,1}(\Omega))$, $\bar{\nu} \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathcal{M}(\overline{D})$ (*the space of Radon measures on* $\overline{D}$) *such that*

$$(\bar{\nu},\bar{\lambda},\bar{\mu}) \ne 0, \quad \bar{\nu} \ge 0, \; \textit{for } m_0+1 \le i \le m, \; \bar{\lambda}_i \ge 0, \; \bar{\lambda}_i \int_\Sigma \Psi_i(s,t,\bar{y},\bar{v}) \, dsdt = 0, \tag{2.2}$$

$$\langle \bar{\mu}, z - \Phi(\bar{y}) \rangle_{\overline{D}} \le 0 \quad \textit{for all } z \in \mathcal{C}, \tag{2.3}$$

$$
(2.4)\quad \begin{cases} -\dfrac{\partial \bar p}{\partial t} + A^*\bar p + f'_y(x,t,\bar y)\bar p = \bar\nu F'_y(x,t,\bar y) + [\Phi'(\bar y)^*\bar\mu]|_Q & \text{in } Q,\\[2mm] \dfrac{\partial \bar p}{\partial n_{A^*}} + g'_y(s,t,\bar y,\bar v)\bar p = \bar\nu G'_y(s,t,\bar y,\bar v) + \bar\lambda\Psi'_y(s,t,\bar y,\bar v) + [\Phi'(\bar y)^*\bar\mu]|_\Sigma & \text{on } \Sigma,\\[2mm] \bar p(T) = \bar\nu L'_y(x,\bar y(T)) + [\Phi'(\bar y)^*\bar\mu]|_{\overline\Omega_T} & \text{on } \overline\Omega, \end{cases}
$$

$$
(2.5)\qquad \bar p \in L^{\delta'}(0,T;W^{1,d'}(\Omega)) \qquad \text{for every } (\delta,d) \text{ satisfying } \frac{N}{2d}+\frac{1}{\delta}<\frac{1}{2},
$$

$$
(2.6)\qquad H_\Sigma(\bar y,\bar v,\bar p,\bar\nu,\bar\lambda) = \min_{v\in V_{ad}} H_\Sigma(\bar y,v,\bar p,\bar\nu,\bar\lambda),
$$

where $[\Phi'(\bar y)^*\bar\mu]|_Q$ *is the restriction of* $[\Phi'(\bar y)^*\bar\mu]$ *to* Q, $[\Phi'(\bar y)^*\bar\mu]|_\Sigma$ *is the restriction of* $[\Phi'(\bar y)^*\bar\mu]$ *to* Σ, *and* $[\Phi'(\bar y)^*\bar\mu]|_{\overline\Omega_T}$ *is the restriction of* $[\Phi'(\bar y)^*\bar\mu]$ *to* $\overline\Omega_T$, $[\Phi'(\bar y)^*\bar\mu]$ *is the Radon measure on* $\overline D$ *defined by* $z \longmapsto \langle\bar\mu,\Phi'(\bar y)z\rangle_{\mathcal M(\overline D)\times C(\overline D)}$ *for* $z\in C(\overline D)$, $\langle\cdot,\cdot\rangle_{\overline D}$ *denotes the duality pairing between* $\mathcal M(\overline D)$ *and* $C(\overline D)$, A^* *is the formal adjoint of* A *that is*

$$
A^*y(x,t) = -\sum_{i=1}^N D_i\left(\sum_{i=1}^N (a_{ji}(x,t)D_jy(x,t)) + b_i(x,t)y(x,t)\right) + \sum_{i=1}^N a_i(x,t)D_iy(x,t).
$$

2.3. Pontryagin's principles for local solutions. By definition a local solution in $L^\sigma(\Sigma)$ of (P) is a solution $(\bar y,\bar v)$ of the following problem

$$
(P_{\bar v,\epsilon})\qquad \begin{aligned} \inf\{J(y,v) \mid\ & y\in W(0,T)\cap C(\overline Q),\ v\in \widetilde V_{ad},\\ & (y,v) \text{ satisfies } (1.1)-(1.4), \|\bar v - v\|_{\sigma,\Sigma}\le\epsilon\}, \end{aligned}
$$

for some $\epsilon>0$. The following Pontryagin's principle for local solutions of (P) is a direct consequence of Theorem 2.1.

COROLLARY 2.2. *If* $(A1)-(A8)$ *are fulfilled and if* $(\bar y,\bar v)$ *is a solution of* $(P_{\bar v,\epsilon})$, *there then exist* $\bar p\in L^1(0,T;W^{1,1}(\Omega))$, $\bar\nu\in\mathbb R$, $\bar\lambda\in\mathbb R^m$, $\bar\mu\in\mathcal M(\overline D)$ *satisfying* (2.2)–(2.5) *along with*

$$
H_\Sigma(\bar y,\bar v,\bar p,\bar\nu,\bar\lambda) = \min_{v\in V_{ad},\|\bar v-v\|_{\sigma,\Sigma}\le\epsilon} H_\Sigma(\bar y,v,\bar p,\bar\nu,\bar\lambda).
$$

As a consequence of this corollary we can get the classical pointwise Pontryagin principle for a local solution in $L^\sigma(\Sigma)$ of the control problem

$$
(\tilde P)\qquad \inf\{J(y,v)\mid y\in W(0,T)\cap C(\overline Q), v\in\widetilde V_{ad}, (y,v) \text{ satisfies } (1.1),(1.3),(1.4)\}.
$$

COROLLARY 2.3. *If* $(A1)-(A8)$ *are fulfilled and if* $(\bar y,\bar v)$ *is a local solution of* $(\tilde P)$ *in* $L^\sigma(\Sigma)$, *there then exist* $\bar p\in L^1(0,T;W^{1,1}(\Omega))$, $\bar\nu\in\mathbb R$, $\bar\lambda\in\mathbb R^m$, $\bar\mu\in\mathcal M(\overline D)$ *satisfying* (2.2)–(2.5) *along with*

$$
\mathcal H_\Sigma(s,t,\bar y(s,t),\bar v(s,t),\bar p(s,t),\bar\nu,\bar\lambda) = \min_{\xi\in V(s,t)} \mathcal H_\Sigma(s,t,\bar y(s,t),\xi,\bar p(s,t),\bar\nu,\bar\lambda)
$$

for almost all $(s,t)\in\Sigma$, *where*

$$\mathcal{H}_\Sigma(s,t,y,\xi,p,\nu,\lambda)=\nu G(s,t,y,\xi)-pg(s,t,y,\xi)+\lambda\Psi(s,t,y,\xi).$$

Proof. The pointwise Pontryagin's principle stated in the corollary may be derived from the integral Pontryagin's principle of Corollary 2.2 by using the same construction as in ([25], proof of Theorem 2.1). The idea in the proof of [25] is to construct a pointwise perturbation v_n of $\bar v$ such that $\lim_{(s,t)\to(s_0,t_0)}v_n(s,t)=\xi$, $\lim_n\mathcal{L}^N(\{(s,t)\in\Sigma\mid v_n(s,t)\neq\bar v(s,t)\})=0$, where $\xi\in V(s_0,t_0)$, $(s_0,t_0)\in\Sigma$, $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure. We obtain the pointwise Pontryagin's principle by replacing v by v_n in the integral Pontryagin's principle of Corollary 2.2, by dividing by $\mathcal{L}^N(\{(s,t)\in\Sigma\mid v_n(s,t)\neq\bar v(s,t)\})\neq 0$, and by passing to the limit when n tends to infinity. The only difference with [25] is that v_n must satisfy $\|\bar v-v_n\|_{\sigma,\Sigma}\le\epsilon$. Due to the condition $\lim_n\mathcal{L}^N(\{(s,t)\in\Sigma\mid v_n(s,t)\neq\bar v(s,t)\})=0$, it is clear that this condition will be realized for n big enough. □

Let us observe that integral control constraints can be studied in the framework of the problem $(\tilde P)$. Indeed the mixed constraints (1.4) can include the integral control constraints. Then Corollary 2.3 provides a Pontryagin's principle for problems with integral constraints on the control and the state, even with mixed integral constraints, as well as pointwise constraints on the control and state too. With the exception of pointwise control constraints, the corresponding Lagrange multipliers are included in the Hamiltonian formulation.

3. State and adjoint equations.

3.1. State equation. Existence and regularity results for equations (1.1) and (2.2) relies on estimates in $C(\overline{Q})$ for solutions of linear equations of the form:

$$(3.1)\quad \frac{\partial y}{\partial t}+Ay+ay=\phi-\mathrm{div}\xi\quad\text{in } Q,\quad \frac{\partial y}{\partial n_A}+by=\psi\quad\text{on }\Sigma,\quad y(0)=y_0\quad\text{in }\Omega.$$

If assumption (A1) is satisfied, if $(a,\phi)\in L^q(Q)\times L^q(Q)$, $(b,\psi)\in L^{\bar\sigma}(\Sigma)\times L^{\bar\sigma}(\Sigma)$, the existence of a unique solution in $C([0,T];L^2(\Omega))\cap L^2([0,T];H^1(\Omega))$ for (3.1) is proved in ([16], Chapter 3, Theorem 5.1) when $\xi\equiv 0$. The result can be extended to (3.1) by the same method if the support of ξ is compact in Q and if ξ belongs to $L^\delta(0,T,(L^d(\Omega))^N)$ with $d>1,\delta>1,N/2d+1/\delta<1/2$. Recall that a weak solution in $L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ of (3.1) is a function $y\in L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ satisfying

$$\int_Q\left(-y\frac{\partial z}{\partial t}+\sum_{i,j}a_{ij}D_jyD_iz+\sum_i(a_iyD_iz+b_iD_iyz)+ayz\right)dxdt+\int_\Sigma byz\,dsdt$$

$$=\int_Q[\phi z+\sum_i\xi_iD_iz]dxdt+\int_\Sigma\psi z\,dsdt+\int_\Omega y(0)z(0)\,dx$$

for every $z\in C^1(\overline{Q})$ such that $z(\cdot,T)=0$ on $\overline{\Omega}$. For linear equations with Dirichlet boundary conditions

$$\frac{\partial y}{\partial t}+Ay+ay=\phi-\mathrm{div}\xi\quad\text{in } Q,\qquad y=\psi\quad\text{on }\Sigma,\qquad y(0)=y_0\quad\text{in }\Omega,$$

estimates of the form

$$\|y\|_{L^\infty(Q)} \le C(\|\phi\|_{q,Q} + \|\psi\|_{\infty,\Sigma} + \sum_i \|\xi_i\|_{d,\delta,\Omega} + \|y_0\|_{C(\overline{\Omega})}) \tag{3.2}$$

are obtained in ([16], Chapter 3, Theorem 7.1) for $d > 1, \delta > 1, N/2d + 1/\delta < 1/2$. In this estimate the constant C depends on $T, \Omega, N, \Lambda, q, \bar\sigma, \delta, d, \sum_i \|a_i^2\|_{q,Q}, \sum_i \|b_i^2\|_{q,Q}$, but also on $\|a\|_{q,Q}$. The case of Robin boundary conditions is considered in [5] to study nonlinear equations of the form (1.1) when the function $g(s,t,\cdot,v)$, in the boundary condition, is monotone and Lipschitz, and when the boundary control v is bounded ([5], Theorem 5.1). The case when the function $g(s,t,\cdot,v)$ in equation (1.1) is neither Lipschitz nor monotone (g satisfies (A3)), and when the control v belongs to $L^{\bar\sigma}(\Sigma)$, but when the coefficients of the operator A are regular and independent of the time variable, is studied in [23]. Estimates in $C(\overline{Q})$ are obtained by semigroup techniques and comparison principles [23, Proposition 3.3 and Theorem 3.1]. Here we emphasize on the fact that assumptions on the operator A are minimal (bounded leading coefficients, unbounded coefficients of order zero), that we deal with nonhomogeneous boundary conditions, and that source terms in the domain and in the boundary conditions are unbounded.

THEOREM 3.1. *Under assumptions (A1)-(A3), if $v \in L^{\bar\sigma}(\Sigma)$, then equation (1.1) admits a unique weak solution y_v in $W(0,T) \cap C(\overline{Q})$. This solution obeys*

$$||y_v||_{C(\overline{Q})} \le C_1(||v||_{\bar\sigma,\Sigma} + 1),$$

where $C_1 = C_1(T, \Omega, N, C_0, q, \bar\sigma)$. Moreover, the mapping $v \longmapsto y_v$ is continuous from $L^{\bar\sigma}(\Sigma)$ into $C(\overline{Q})$. *Proof.* The proof relioes on Theorem 3.2 (see [23]). □

THEOREM 3.2. *Suppose that (A1) is satisfied, $(a,\phi) \in L^q(Q) \times L^q(Q)$, $(b,\psi) \in L^{\bar\sigma}(\Sigma) \times L^{\bar\sigma}(\Sigma)$, and ξ belongs to $(\mathcal{D}(Q))^N$. If in addition $a \ge C_0$ a.e. in Q, and $b \ge C_0$ a.e. in Σ (for some $C_0 \in \mathbb{R}$), then the unique weak solution y of (3.1) belongs to $C(\overline{Q})$ and satisfies the following estimate*

$$\|y\|_{C(\overline{Q})} \le C_2(\|\phi\|_{q,Q} + \|\psi\|_{\bar\sigma,\Sigma} + \sum_i \|\xi_i\|_{L^\delta(0,T;L^d(\Omega))} + \|y_0\|_{C(\overline{\Omega})})$$

where $d > 1$, $\delta > 1$ satisfy $N/2d + 1/\delta < 1/2$ and the constant C_2 only depends on T, Ω, N, C_0, Λ, q, $\bar\sigma$, δ, d, $\sum_i \|a_i^2\|_{q,Q}$, $\sum_i \|b_i^2\|_{q,Q}$.

REMARK 3.3. *Notice that the constant C_2 does not depend on $\|a\|_{q,Q}$ and $\|b\|_{\bar\sigma,\Sigma}$. As in [16] (see the above estimate (3.2)), the assumption $a \ge C_0$ may be dropped out, and in this case the constant C_2 must be replaced by a constant also depending on $\|a\|_{q,Q}$. But the corresponding estimate cannot be used to treat nonlinear equations of the form (1.1).*

Proof. To prove this theorem, we only need to prove the L^∞-estimate, the rest is classical. We prove the L^∞-estimate by using the so-called truncation method as in [16, Chapter 3, proof of Theorem 7.1]. If y is a weak solution of (3.1), then we have

$$\int_\Omega [y(x,t)z(x,t) - y(x,0)z(x,0)]\, dx$$

$$+ \int_0^t \int_\Omega [-y\frac{\partial z}{\partial t} + \sum_{i,j} a_{ij} D_j y D_i z + \sum_i (a_i y D_i z + b_i D_i y z) + ayz]\, dx d\tau$$

$$+\int_0^t \int_\Gamma byz\, dsd\tau = \int_0^t \int_\Omega [\phi z + \sum_i \xi_i D_i z]\, dxd\tau + \int_0^t \int_\Gamma \psi z\, dsd\tau$$

for every $t \in [0,T]$ and every $z \in W_2^{1,1}(Q)$. We only establish the upper bound for y (the lower bound can be obtained in the same way). For $k \geq 0$ we set $y^k(x,t) = \max(y(x,t) - k, 0)$. By using Steklov averangings, as in [16, p. 183], we prove that

$$\frac{1}{2}\int_\Omega [y^k(x,t)^2 - y^k(x,0)^2]\, dx \tag{3.3}$$

$$+\int_0^t \int_\Omega [\sum_{i,j} a_{ij} D_j y^k D_i y^k + \sum_i (a_i y D_i y^k + b_i D_i y^k y^k) + a y y^k]\, dxd\tau =$$

$$+\int_0^t \int_\Gamma b y y^k\, dsd\tau = \int_0^t \int_\Omega [\phi y^k + \sum_i \xi_i D_i y^k]\, dxd\tau + \int_0^t \int_\Gamma \psi y^k\, dsd\tau$$

for every $t \in]0,T]$. Thus, it follows that

$$\frac{1}{2}\int_\Omega y^k(x,t)^2\, dx + \int_0^t\!\!\int_\Omega [\sum_{i,j} a_{ij} D_j y^k D_i y^k + (a - C_0 + \Lambda) y y^k]\, dxd\tau \tag{3.4}$$

$$+\int_0^t\!\!\int_\Gamma (b - C_0) y y^k\, dsd\tau = -\int_0^t\!\!\int_\Omega [\sum_i (a_i y D_i y^k + b_i D_i y^k y^k) + (C_0 - \Lambda) y y^k]\, dxd\tau$$

$$-\int_0^t\!\!\int_\Gamma C_0 y y^k\, dsd\tau + \int_0^t\!\!\int_\Omega [\phi y^k + \sum_i \xi_i D_i y^k]\, dxd\tau + \int_0^t \int_\Gamma \psi y^k\, dsd\tau$$

for every $k > \tilde{k} := \|y_0\|_{C(\overline{\Omega})}$. Since $a - C_0 \geq 0$ a.e. in Q, $b - C_0 \geq 0$ a.e. on Σ, and $yy^k \geq (y^k)^2$ a.e. in Q, with (2.1) we obtain:

$$\|y^k(t)\|_{2,\Omega}^2 + 2\Lambda \|y^k\|_{L^2(0,t;H^1(\Omega))}^2 \leq \tag{3.5}$$

$$-2\int_0^t\!\!\int_\Omega [\sum_i (a_i y D_i y^k + b_i D_i y^k y^k) + (C_0 - \Lambda) y y^k]\, dxd\tau$$

$$-2\int_0^t\!\!\int_\Gamma C_0 y y^k\, dsd\tau + 2\int_0^t\!\!\int_\Omega [\phi y^k + \sum_i \xi_i D_i y^k]\, dxd\tau + 2\int_0^t\!\!\int_\Gamma \psi y^k\, dsd\tau$$

for every $k > \tilde{k}$. Set $A_k(t) = \{x \in \Omega \mid y(x,t) > k\}$, $B_k(t) = \{s \in \Gamma \mid y(s,t) > k\}$, $Q_k(t) = \{(x,\tau) \in \Omega \times]0,t[\mid y(x,\tau) > k\}$, $\Sigma_k(t) = \{(s,\tau) \in \Gamma \times]0,t[\mid y(s,\tau) > k\}$. We estimate the terms in the right hand side of (3.5) by means of Young's inequality and we obtain

$$\|y^k(t)\|_{2,\Omega}^2 + \Lambda \|y^k\|_{L^2(0,t;H^1(\Omega))}^2 \leq$$

$$+\frac{3}{\Lambda}\int_0^t\!\!\int_{A_k(\tau)}[\sum_i((a_iy)^2+(b_iy^k)^2)+(C_0-\Lambda)^2y^2]\,dxd\tau+\frac{3K^2}{\Lambda}\int_0^t\!\!\int_{B_k(\tau)}C_0^2y^2\,dsd\tau$$

$$+2\int_0^t\!\!\int_{A_k(\tau)}[|\phi||y^k|+\sum_i|\xi_i||D_iy^k|]\,dxd\tau+2\int_0^t\!\!\int_{B_k(\tau)}|\psi||y^k|\,dsd\tau,$$

where $K>0$ satisfies $\|\varphi\|_{2,\Gamma}\leq K\|\varphi\|_{H^1(\Omega)}$ for all $\varphi\in H^1(\Omega)$. Since $y=y^k+k$ in $A_k(\tau)$ and $B_k(\tau)$ for a.e. τ, it follows that

$$\|y^k(t)\|_{2,\Omega}^2+\Lambda\|y^k\|^2_{L^2(0,t;H^1(\Omega))}\leq$$

$$\frac{6}{\Lambda}\int_0^t\!\!\int_{A_k(\tau)}\left[\sum_i(a_i^2+b_i^2)+(C_0-\Lambda)^2\right]((y^k)^2+k^2)\,dxd\tau$$

$$+\frac{6K^2}{\Lambda}\int_0^t\!\!\int_{B_k(\tau)}C_0^2((y^k)^2+k^2)\,dsd\tau+2\int_0^t\!\!\int_{A_k(\tau)}[|\phi||y^k|+\sum_i|\xi_i||D_iy^k|]\,dxd\tau+$$

$$2\int_0^t\!\!\int_{B_k(\tau)}|\psi||y^k|\,dsd\tau$$

for every $t\in[0,T]$ and every $k>\tilde{k}$. With Hölder's inequality we have

$$\|y^k(t)\|_{2,\Omega}^2+\Lambda\|y^k\|^2_{L^2(0,t;H^1(\Omega))}\leq \tag{3.6}$$

$$\left(K_1(|Q_k(t)|^{\frac{1}{q'}}+|Q_k(t)|)+K_2|\Sigma_k(t)|\right)k^2$$

$$+K_1(|Q_k(t)|^{\frac{2}{N+2}}+|Q_k(t)|^{\frac{1}{q'}-\frac{N}{N+2}})\|y^k\|^2_{\frac{2(N+2)}{N},\Omega\times]0,t[}+2\|\phi\|_{q,Q}|Q_k(t)|^{\frac{1}{q'}-\frac{N}{2(N+2)}}\|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,t[}$$

$$+K_2|\Sigma_k(t)|^{\frac{1}{N+1}}\|y^k\|^2_{\frac{2(N+1)}{N},\Gamma\times]0,t[}+2\|\psi\|_{\bar{\sigma},\Sigma}|\Sigma_k(t)|^{\frac{1}{\bar{\sigma}'}-\frac{N}{2(N+1)}}\|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,t[}$$

$$+\frac{\Lambda}{2}\|y^k\|^2_{L^2(0,t;H^1(\Omega))}+\frac{2K_3}{\Lambda}\left(\int_0^t|A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}}\,d\tau\right)^{\frac{\delta-2}{\delta}},$$

where

$$K_1=\frac{6}{\Lambda}\left[\sum_i(\|a_i^2\|_{q,Q}+\|b_i^2\|_{q,Q})+(C_0-\Lambda)^2\right],$$

$$K_2=\frac{6K^2}{\Lambda}C_0^2\quad\text{and}\quad K_3=\sum_i\|\xi_i\|^2_{L^\delta(0,T;L^d(\Omega))},$$

$|Q_k(t)|$ denotes the $(N+1)$-dimensional Lebesgue measure of $Q_k(t)$, $|\Sigma_k(t)|$ denotes the N-dimensional Lebesgue measure of $\Sigma_k(t)$ and $|A_k(\tau)|$ denotes the $(N+1)$-dimensional Lebesgue measure of $A_k(\tau)$. Notice that $\frac{N(d-2)}{2d} + \frac{\delta-2}{\delta} > \frac{N}{2}$. Then there exists $\tilde{r} > \frac{2\delta}{\delta-2} > 2$ such that $\frac{N(d-2)}{2d} + \frac{\delta-2}{\delta} > \frac{N}{2}\frac{\tilde{r}(\delta-2)}{2\delta} > \frac{N}{2}$. For such an $\tilde{r}$ we have $\frac{1}{\tilde{r}}\left(\frac{N}{2}\frac{\delta}{\delta-2}\frac{d-2}{d}+1\right) > \frac{N}{4}$. We define $r > 2$ by $\frac{1}{r} = \frac{1}{\tilde{r}}\frac{\delta}{\delta-2}\frac{d-2}{d} < \frac{d-2}{2d} < \frac{1}{2}$ and we obtain $\frac{N}{2r} + \frac{1}{\tilde{r}} > \frac{N}{4}$. Thus the imbedding from $L^2(0,t;H^1(\Omega)) \cap C([0,t];L^2(\Omega))$ into $L^{\tilde{r}}(0,t;L^r(\Omega))$ is continuous; see [16, p. 75]. Observe that

$$|Q_k(t)|^{\frac{2}{N+2}} + |Q_k(t)|^{\frac{1}{q'}-\frac{N}{N+2}} \le (t|\Omega|)^{\frac{2}{N+2}} + (t|\Omega|)^{\frac{1}{q'}-\frac{N}{N+2}}, \qquad |\Sigma_k(t)|^{\frac{1}{N+1}} \le (t|\Gamma|)^{\frac{1}{N+1}}.$$

Let us choose $\bar{t} > 0$ small enough to have

$$K_1((\bar{t}|\Omega|)^{\frac{2}{N+2}} + (\bar{t}|\Omega|)^{\frac{1}{q'}-\frac{N}{N+2}})\|y\|^2_{\frac{2(N+2)}{N},\Omega\times]0,\bar{t}[} + K_2(\bar{t}|\Gamma|)^{\frac{1}{N+1}}\|y\|^2_{\frac{2(N+1)}{N},\Gamma\times]0,\bar{t}[} \le \frac{1}{2}\min(1,\frac{\Lambda}{2})\|y\|^2_{Q(\bar{t})} \tag{3.7}$$

for every $y \in L^2(0,\bar{t};H^1(\Omega)) \cap C([0,\bar{t};L^2(\Omega)])$. Then from (3.6) and imbedding theorems, it follows that

$$\nu(\|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar{t}[} + \|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar{t}[} + \|y^k\|_{L^{\tilde{r}}(0,\bar{t};L^r(\Omega))}) \le K_4\left(|Q_k(\bar{t})|^{\frac{1}{2q'}} + |Q_k(\bar{t})|^{\frac{1}{2}} + |\Sigma_k(\bar{t})|^{\frac{1}{2}}\right)k$$
$$+K_4\left(|Q_k(\bar{t})|^{\frac{1}{q'}-\frac{N}{2(N+2)}} + |\Sigma_k(\bar{t})|^{\frac{1}{\tilde{\sigma}'}-\frac{N}{2(N+1)}}\right) + K_4\left(\int_0^{\bar{t}}|A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}}\,d\tau\right)^{\frac{\delta-2}{2\delta}}, \tag{3.8}$$

for $k > \tilde{k}$, where $\nu > 0$ depends on Λ, and where K_4 depends on K_1, K_2, K_3, $\|\phi\|_{q,Q}$, $\|\psi\|_{\tilde{\sigma},\Sigma}$ and Λ. Now, we set $\theta(k) = |Q_k(\bar{t})|^{\frac{N}{2(N+2)}} + |\Sigma_k(\bar{t})|^{\frac{N}{2(N+1)}} + (\int_0^{\bar{t}}|A_k(\tau)|^{\frac{\tilde{r}}{r}}d\tau)^{\frac{1}{\tilde{r}}}$. Observe that, for every $\ell \ge k \ge 0$, we have $y^k \ge \ell - k$ a.e. in $Q_\ell(\bar{t})$, a.e. on $\Sigma_\ell(\bar{t})$ and a.e. in $A_\ell(\tau)$ for a.e. $\tau \in]0,\bar{t}[$, therefore

$$(\ell-k)\theta(\ell) \le \|y^k\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar{t}[} + \|y^k\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar{t}[} + \|y^k\|_{L^{\tilde{r}}(0,\bar{t};L^r(\Omega))}. \tag{3.9}$$

Taking $k = 0$ in the above inequality, with the definition of the function θ we first obtain $\ell\theta(\ell) \le K_0$ for all $\ell \ge 0$, where $K_0 = \|y\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar{t}[} + \|y\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar{t}[} + \|y\|_{L^{\tilde{r}}(0,\bar{t};L^r(\Omega))}$. In particular, for $\ell = K_0$, this implies $\theta(K_0) \le 1$. On the other hand (3.8) and (3.9) give

$$(\ell-k)\theta(\ell) \le \frac{K_4}{\nu}\left(|Q_k(\bar{t})|^{\frac{1}{2q'}} + +|Q_k(\bar{t})|^{\frac{1}{2}} + |\Sigma_k(\bar{t})|^{\frac{1}{2}}\right.$$
$$\left.+|Q_k(\bar{t})|^{\frac{1}{q'}-\frac{N}{2(N+2)}} + |\Sigma_k(\bar{t})|^{\frac{1}{\tilde{\sigma}'}-\frac{N}{2(N+1)}}\right)k + \frac{K_4}{\nu}\left(\int_0^{\bar{t}}|A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}}d\tau\right)^{\frac{\delta-2}{2\delta}} \tag{3.10}$$

for all $\ell \geq k > \max(K_0, 1, \tilde{k})$. Set

$$\alpha_1 = \frac{N+2}{Nq'}, \quad \alpha_2 = \frac{N+1}{N\bar{\sigma}'}, \quad \alpha_3 = \tilde{r}\frac{\delta-2}{2\delta}, \quad \alpha = \min(\alpha_1, \alpha_2, \alpha_3),$$

and observe that $\alpha > 1$. Since $\theta(K_0) \leq 1$, and since θ is a nonincreasing function we also have $|Q_k(\bar{t})| \leq 1$, $|\Sigma_k(\bar{t})| \leq 1$ and $\int_0^{\bar{t}} |A_k(\tau)|^{\frac{\tilde{r}}{r}} d\tau \leq 1$ for all $k \geq K_0$. Thus it follows that

$$|Q_k(\bar{t})|^{\frac{1}{2q'}} + |\Sigma_k(\bar{t})|^{\frac{1}{2}} + |Q_k(\bar{t})|^{\frac{1}{2}}$$

$$+|Q_k(\bar{t})|^{\frac{1}{q'}-\frac{N}{2(N+2)}} + |\Sigma_k(\bar{t})|^{\frac{1}{\bar{\sigma}'}-\frac{N}{2(N+1)}} + \left(\int_0^{\bar{t}} |A_k(\tau)|^{\frac{\delta(d-2)}{d(\delta-2)}} d\tau\right)^{\frac{\delta-2}{2\delta}} \leq 3\theta(k)^\alpha$$

From (3.10), we deduce

$$(\ell - k)\theta(\ell) \leq K_5\theta(k)^\alpha k \tag{3.11}$$

for every $\ell \geq k > \max(K_0, 1, \tilde{k})$. With the same arguments as in [16, Chapter 3, p. 186], we finally obtain

$$\|y\|_{\infty,Q} \leq K_6,$$

where K_6 depends not only on $T, \Omega, N, C_0, \Lambda, q, \bar{\sigma}, \delta, d, \sum_i \|a_i^2\|_{q,Q}, \sum_i \|b_i^2\|_{q,Q}$, but also on $\|y_0\|_{C(\overline{\Omega})}$, $\|\phi\|_{q,Q}$, $\|\psi\|_{\bar{\sigma},\Sigma}$ and $\sum_i \|\xi_i\|^2_{L^\delta(0,T;L^d(\Omega))}$. Since the equation (3.1) is linear, the estimate given in Theorem 3.2 can be easily deduced from this estimate. □

REMARK 3.4. *In the proof of Theorem 3.2, the constant* K_6 *depends on* $K_0 = \|y\|_{\frac{2(N+2)}{N},\Omega\times]0,\bar{t}[} + \|y\|_{\frac{2(N+1)}{N},\Gamma\times]0,\bar{t}[} + \|y\|_{L^{\tilde{r}}(0,\bar{t};L^r(\Omega))} \leq C\|y\|_{Q(\bar{t})}$, *and on* $\tilde{k} = \|y_0\|_{C(\overline{\Omega})}$. *By using the same trick as in (3.4), we can obtain an estimate of* $\|y\|_{Q(\bar{t})}$ *depending on* $T, \Omega, N, C_0, \Lambda, q, \bar{\sigma}, \delta, d, \sum_i \|a_i^2\|_{q,Q}, \sum_i \|b_i^2\|_{q,Q}$, $\sum_i \|\xi_i\|^2_{L^\delta(0,T;L^d(\Omega))}$, $\|\phi\|_{q,Q}$, *and* $\|\psi\|_{\bar{\sigma},\Sigma}$, *but independent of* $\|a\|_{q,Q}$ *and* $\|b\|_{\bar{\sigma},\Sigma}$.

3.2. Adjoint equation. Let (a, b) be in $L^q(Q) \times L^{\bar{\sigma}}(\Sigma)$ with $a \geq C_0$ and $b \geq C_0$. We consider the terminal boundary value problem :

$$-\frac{\partial p}{\partial t} + A^* p + ap = \mu_Q \text{ in } Q, \quad \frac{\partial p}{\partial n_{A^*}} + bp = \mu_\Sigma \text{ on } \Sigma, \quad p(T) = \mu_{\overline{\Omega}_T} \text{ on } \overline{\Omega}, \tag{3.12}$$

where $\mu = \mu_Q + \mu_\Sigma + \mu_{\overline{\Omega}_T}$ is a bounded Radon measure on $\overline{Q} \setminus \overline{\Omega}_0$, μ_Q is the restriction of μ to Q, μ_Σ is the restriction of μ to Σ and $\mu_{\overline{\Omega}_T}$ is the restriction of μ to $\overline{\Omega}_T$.

DEFINITION 3.5. *A function* $p \in L^1(0, T; W^{1,1}(\Omega))$ *is a weak solution of (3.12) if*

$$ap \in L^1(Q), \quad bp \in L^1(\Sigma), \quad a_i D_i p \in L^1(Q) \quad \textit{and} \quad b_i p \in L^1(Q) \quad \textit{for } i = 1, ..., N,$$

$$\int_Q \left(p\frac{\partial y}{\partial t} + \sum_{i,j} a_{ji} D_j p D_i y + \sum_i (a_i D_i p y + b_i p D_i y) + apy \right) dxdt + \int_\Sigma bpy\, dsdt$$

$$= \int_{\overline{Q}\setminus\overline{\Omega}_0} y d\mu(x,t) \quad \textit{for every } y \in C^1(\overline{Q}) \textit{ satisfying } y(x,0) = 0 \textit{ on } \overline{\Omega}.$$

As for elliptic equations [27], it is well known that equation (3.12) may admit more than one solution. However uniqueness is guaranteed if we look for solutions of (3.12) satisfying some Green formula (such uniqueness results are proved in [1] for elliptic equations and in [5] for parabolic equations).

THEOREM 3.6. *Let μ be in $\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}_0)$ and let (a,b) be in $L^q(Q)\times L^{\bar\sigma}(\Sigma)$ satisfying $a\geq C_0$ a.e. in Q, $b\geq C_0$ a.e. Σ, for some $C_0\in\mathbb{R}$. Equation (3.12) admits a unique solution p in $L^1(0,T;W^{1,1}(\Omega))$ satisfying*

$$\int_Q p\{\frac{\partial y}{\partial t}+Ay+ay\}\,dxdt+\int_\Sigma p\{\frac{\partial y}{\partial n_A}+by\}\,dsdt=\langle y,\mu\rangle_{C_b(\overline{Q}\setminus\overline{\Omega}_0)\times\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}_0)}$$

for every $y\in\{y\in W(0,T)\cap C(\overline{Q})\mid \frac{\partial y}{\partial t}+Ay\in L^q(Q),\ \frac{\partial y}{\partial n_A}\in L^{\bar\sigma}(\Sigma),\ y(x,0)=0$ on $\overline{\Omega}\}$. Moreover p belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every $\delta>2$, $d>2$ satisfying $\frac{N}{2d}+\frac{1}{\delta}<\frac{1}{2}$ and we have:

$$\|p\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))}\leq C_4(\delta,d)\|\mu\|_{\mathcal{M}_b(\overline{Q}\setminus\overline{\Omega}_0)},$$

where $C_4(\delta,d)=C_4(T,\Omega,N,C_0,q,\bar\sigma,\delta,d,\|a_i\|_{L^{2q}(Q)},\|b_i\|_{L^{2q}(Q)})$, but C_4 is independent of a and b.

Proof. Due to Theorem 3.2, the proof of Theorem 3.3 follows the lines of the proofs of Theorem 6.3 in [5] and of Theorem 4.2 in [22]. Since we improve the results given in [5, 22], we sketch the main points of the proof. Let $(h_n)_n$ be a sequence in $C_c(Q)$ (the space of continuous functions with compact support in Q), $(k_n)_n$ be a sequence in $C_c(\Sigma)$, and $(\ell_n)_n$ be a sequence in $C(\overline{\Omega})$ such that

$$\|h_n\|_{L^1(Q)}=\|\mu_Q\|_{\mathcal{M}_b(Q)},\quad \|k_n\|_{L^1(\Sigma)}=\|\mu_\Sigma\|_{\mathcal{M}_b(\Sigma)},\ \|\ell_n\|_{L^1(\Omega)}=\|\mu_{\overline{\Omega}_T}\|_{\mathcal{M}(\overline{\Omega}_T)},$$

$$\lim_n\int_Q h_n\phi dxdt=\langle\phi,\mu_Q\rangle_{C_b(Q)\times\mathcal{M}_b(Q)},$$

$$\lim_n\int_\Sigma k_n\phi dsdt=\langle\phi,\mu_\Sigma\rangle_{C_b(\Sigma)\times\mathcal{M}_b(\Sigma)},$$

$$\lim_n\int_\Omega \ell_n\phi dx=\langle\phi,\mu_{\bar\Omega_T}\rangle_{C(\overline{\Omega}_T)\times\mathcal{M}(\overline{\Omega}_T)},$$

for every $\phi\in C(\overline{Q})$. Let $(p_n)_n$ be the sequence in $W(0,T)$ defined by

$$-\frac{\partial p_n}{\partial t}+Ap_n+ap_n=h_n\quad\text{in } Q,\quad \frac{\partial p_n}{\partial n_A}+bp_n=k_n\quad\text{on }\Sigma,\quad p_n(T)=\ell_n\quad\text{in }\Omega.$$

Due to Theorem 3.2, and by using the same arguments as in [5, 22], we can prove that there exists a constant $C_5(\delta,d)=C_5(T,\Omega,N,C_0,q,\bar\sigma,\delta,d,\|a_i\|_{L^{2q}(Q)},\|b_i\|_{L^{2q}(Q)})$ such that

$$\|p_n\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))}\leq C_5(\delta,d)\|\mu\|_{\mathcal{M}_b(\bar Q\setminus\bar\Omega_0)}$$

for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. Since $q > \frac{N}{2} + 1$ and $\bar{\sigma} > N+1$, there exist (δ_1, d_1), (δ_2, d_2), (δ_3, d_3) satisfying $\frac{N}{2d_i} + \frac{1}{\delta_i} < \frac{1}{2}$ for $i = 1, 2, 3$, such that $\delta_1' \geq q'$, $d_1'^* = \frac{Nd_1'}{N-d_1'} \geq q'$, $\delta_2' \geq \bar{\sigma}'$, $\frac{(N-1)d_2 d_2'}{(N-1)d_2 - d_2'} \geq \bar{\sigma}'$, $\delta_3' \geq (2q)'$ and $d_3' \geq (2q)'$. Therefore

$$\|p_n\|_{L^{q'}(Q)} \leq C\|p_n\|_{L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))} \leq CC_5(\delta_1, d_1)\|\mu\|_{\mathcal{M}_b(\bar{Q}\setminus\bar{\Omega}_0)},$$

$$\|p_n\|_{L^{\bar{\sigma}'}(\Sigma)} \leq C\|p_n\|_{L^{\delta_2'}(0,T;W^{1,d_2'}(\Omega))} \leq CC_5(\delta_2, d_2)\|\mu\|_{\mathcal{M}_b(\bar{Q}\setminus\bar{\Omega}_0)},$$

$$\|p_n\|_{L^{(2q)'}(0,T;W^{1,(2q)'}(\Omega))} \leq C\|p_n\|_{L^{\delta_3'}(0,T;W^{1,d_3'}(\Omega))} \leq CC_5(\delta_3, d_3)\|\mu\|_{\mathcal{M}_b(\bar{Q}\setminus\bar{\Omega}_0)}.$$

Then, there exist a subsequence, still indexed by n, and p such that $(p_n)_n$ converges to p for the weak-star topology of $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. By passing to the limit in the variational formulation satisfied by $(p_n)_n$, we prove that p is a solution of (3.12). The uniqueness can be proved as in [1, 5]. □

4. Technical results.

4.1. Metric space of controls. To apply the Ekeland variational principle, we have to define a metric space of controls in such a way that the mapping $v \longmapsto y_v$ is continuous from this metric space to $C(\overline{Q})$. Due to Theorem 3.1, this continuity condition will be realized if convergence in the metric space of controls implies convergence in $L^{\bar{\sigma}}(\Sigma)$. In the case when boundary controls are bounded, convergence in (V_{ad}, d) (where d is the so-called Ekeland's distance) implies convergence in $L^{\bar{\sigma}}(\Sigma)$. This condition is no longer true for unbounded controls; see [14, p. 227]. To overcome this difficulty, we proceed as in [6] and we define a new metric space in the following way. Let $\tilde{v}$ be in V_{ad} (in Section 5, $\tilde{v}$ will be an optimal boundary control that we want to characterize). For $0 < M < \infty$, we define the set:

$$V_{ad}(\tilde{v}, M) = \{v \in V_{ad} \mid \|v - \tilde{v}\|_{\sigma,\Sigma} \leq M\}.$$

We endow the set $V_{ad}(\tilde{v}, M)$ with the Ekeland metric:

$$d(v_1, v_2) = \mathcal{L}^N(\{(s,t) \in \Sigma \mid v_1(s,t) \neq v_2(s,t)\}).$$

Proposition 4.1. *Let $\tilde{v}$ be in V_{ad}. Let $M > 0$ and $\{(v_n)_n, v\} \subset V(\tilde{v}, M)$. If $(v_n)_n$ tends to v in $(V(\tilde{v}, M), d)$, then $(v_n)_n$ tends to v in $L^{\bar{\sigma}}(\Sigma)$.*

Proof. Since $1 \leq \bar{\sigma} < \sigma$, the proof is immediate if we notice that we have

$$\int_\Sigma |v - v_n|^{\bar{\sigma}}\, ds \leq \|v - v_n\|^{\bar{\sigma}}_{\sigma,\Sigma}(d(v_n, v))^{\frac{\sigma-\bar{\sigma}}{\sigma}} \leq (2M)^{\bar{\sigma}}(d(v_n, v))^{\frac{\sigma-\bar{\sigma}}{\sigma}}.$$

□

Proposition 4.2. *For every $M > 0$, we have:*

i) $(V_{ad}(\tilde{v}, M), d)$ is a complete metric space;

ii) the mapping which associates y_v with v is continuous from $(V_{ad}(\tilde{v}, M), d)$ into $C(\overline{Q})$;

iii) the mappings $v \to J(y_v, v)$ and $v \to \int_\Sigma \Psi_i(s, t, y_v, v)\, dsdt$ are continuous (respectively, lower semicontinuous) on $(V_{ad}(\tilde{v}, M), d)$ for $1 \leq i \leq m_0$ (repectively $m_0 + 1 \leq i \leq m$).

Proof. Claims (i) and (ii) are proved in [6], for control problems of elliptic equations; this proof can be repeated here with the obvious modifications. Contrary to [5], [25], the mapping $v \to J(y_v, v)$ is not necessarily continuous on the space of "truncated controls" endowed with the Ekeland metric. We can only prove a lower semicontinuity result. This result is stated in [6, Proposition 3.1] under the additional assumption that $G(s,t,y,\cdot)$ is convex. In fact we can prove the same result without this convexity assumption. Let $(v_n)_n$ be a sequence converging to v in $(V_{ad}(\tilde{v}, M), d)$. From Proposition 4.1 and Theorem 3.1 we know that $(v_n)_n$ converges to v in $L^{\bar{\sigma}}(\Sigma)$ and $(y_{v_n})_n$ converges to y_v uniformly on $\overline{Q}$. With assumption (A6), with Fatou's Lemma and with Lebesgue's dominated convergence Theorem we have

$$\text{liminf}_n \int_\Sigma G(s,t,0,v_n)\,dsdt \geq \int_\Sigma G(s,t,0,v)\,dsdt,$$

$$\text{lim}_n \int_\Sigma \int_0^1 G'_y(s,t,\theta y_{v_n}, v_n) y_{v_n} d\theta\,dsdt = \int_\Sigma \int_0^1 G'_y(s,t,\theta y_v, v) y_v d\theta\,dsdt.$$

Therefore we obtain

$$\text{liminf}_n \int_\Sigma G(s,t,y_{v_n},v_n)\,dsdt = \text{liminf}_n \int_\Sigma G(s,t,0,v_n)\,dsdt$$

$$+\text{lim}_n \int_\Sigma \int_0^1 G'_y(s,t,\theta y_{v_n}, v_n) y_{v_n} d\theta\,dsdt$$

$$\geq \int_\Sigma G(s,t,0,v)\,dsdt + \int_\Sigma \int_0^1 G'_y(s,t,\theta y_v, v) y_v d\theta\,dsdt = \int_\Sigma G(s,t,y_v,v)\,dsdt.$$

Following the same ideas, we can prove the continuity (for $1 \leq i \leq m_0$) or the lower semicontinuity (for $m_0 + 1 \leq i \leq m$) of $v \to \int_\Sigma \Psi_i(s,t,y_v,v)\,dsdt$. □

4.2. Existence of diffuse perturbations. Let $\tilde{v}$ be an admissible control, and let v_1 and v_2 be in $V_{ad}(\tilde{v}, M)$. A diffuse perturbation of v_1 by v_2 is a family of functions $(v_\rho)_{\rho>0}$ defined by

$$v_\rho(s,t) = \begin{cases} v_1(s,t) & \text{on } \Sigma \setminus E_\rho, \\ v_2(s,t) & \text{on } E_\rho, \end{cases}$$

where E_ρ is a measurable subset of Σ satisfying some conditions. Such perturbations are used to derive Pontryagin's principles from the Ekeland variational principle. In the case of bounded controls (when $V_{ad}(\tilde{v}, M) \equiv V_{ad}$) the use of this kind of perturbations goes back to Yao [29] and Li [17] (see also [21], [15]). Some variants have been developed in [5] for bounded controls, and in [25] for unbounded controls. In [6] we have investigated the case of unbounded controls with integral control constraints. Here we prove that the diffuse perturbations defined in [25] may be extended to derive a Pontryagin's principle for problems with integral coupled control-state constraints. Before proving the existence of such diffuse perturbations let us state an auxiliary lemma analogous to Lemma 3.2 of [6].

LEMMA 4.3. *Let ρ be such that $0 < \rho < 1$. For every $v_1, v_2, v_3 \in V_{ad}$ and $y \in C(\overline{Q})$, there exists a sequence of measurable sets $(E_\rho^n)_n$ in Σ such that*

$$\mathcal{L}^N(E_\rho^n) = \rho \mathcal{L}^N(\Sigma), \tag{4.1}$$

$$\int_{E_\rho^n} |v_i - v_3|^\sigma \, dsdt = \rho \int_\Sigma |v_i - v_3|^\sigma \, dsdt \quad \textit{for } i = 1, 2, \tag{4.2}$$

$$\int_{E_\rho^n} h(s,t,v_i) \, dsdt = \rho \int_\Sigma h(s,t,v_i) \, dsdt \quad \textit{for } i = 1, 2, \tag{4.3}$$

$$\frac{1}{\rho}\chi_{E_\rho^n} \rightharpoonup 1 \quad \textit{weakly-star in } L^\infty(\Sigma) \textit{ when } n \textit{ tends to infinity}, \tag{4.4}$$

where $\chi_{E_\rho^n}$ *is the characteristic function of* E_ρ^n.

Proof. We follows the ideas of [25, Lemma 4.1]. Let us take a sequence $(\varphi_n)_n$ dense in $L^1(\Sigma)$. For $n \geq 1$ we define $f^n \in (L^1(\Sigma))^{n+2\ell+3}$ by

$$f^n = (1, \varphi_1, \ldots, \varphi_n, |v_1 - v_3|^\sigma, |v_2 - v_3|^\sigma, h(\cdot,\cdot,v_1), h(\cdot,\cdot,v_2), \Psi(\cdot,\cdot,y,v_2) - \Psi(\cdot,\cdot,y,v_1)).$$

Thanks to Lyapunov's Convexity Theorem, for every $n \geq 1$ and every $\rho \in (0,1)$, there exists a measurable subset $E_\rho^n \subset \Sigma$ satisfying

$$\int_{E_\rho^n} f^n \, dsdt = \rho \int_\Sigma f^n \, dsdt.$$

As in [25], it is easy to prove that (4.1)–(4.4) hold for the sequence $(E_\rho^n)_n$. □

THEOREM 4.4. *Let* ρ *be such that* $0 < \rho < 1$. *For every* $v_1, v_2, v_3 \in V_{ad}$, *there exists a measurable subset* $E_\rho \subset \Sigma$ *such that:*

$$\mathcal{L}^N(E_\rho) = \rho \mathcal{L}^N(\Sigma), \tag{4.5}$$

$$\begin{aligned} &\int_{\Sigma \setminus E_\rho} |v_1 - v_3|^\sigma \, dsdt + \int_{E_\rho} |v_2 - v_3|^\sigma \, dsdt = \\ &(1-\rho)\int_\Sigma |v_1 - v_3|^\sigma \, dsdt + \rho \int_\Sigma |v_2 - v_3|^\sigma \, dsdt, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\int_{\Sigma \setminus E_\rho} h(s,t,v_1) \, dsdt + \int_{E_\rho} h(s,t,v_2) \, dsdt = \\ &(1-\rho)\int_\Sigma h(s,t,v_1) \, dsdt + \rho \int_\Sigma h(s,t,v_2) \, dsdt, \end{aligned} \tag{4.7}$$

$$y_\rho = y_1 + \rho z + r_\rho, \qquad \textit{with} \ \lim_{\rho \to 0} \frac{1}{\rho} ||r_\rho||_{C(\bar{Q})} = 0, \tag{4.8}$$

$$J(y_\rho, v_\rho) = J(y_1, v_1) + \rho[J'_y(y_1, v_1)z + J(y_1, v_2) - J(y_1, v_1)] + o(\rho), \tag{4.9}$$

$$\int_\Sigma \Psi(s,t,y_\rho,v_\rho) \, dsdt = \tag{4.10}$$

$$\int_\Sigma\Big(\Psi(s,t,y_1,v_1)+\rho[\Psi'_y(s,t,y_1,v_1)z+\Psi(s,t,y_1,v_2)-\Psi(s,t,y_1,v_1)]\Big)\,dsdt+o(\rho),$$

where v_ρ is the control defined by:

$$v_\rho(s,t)=\begin{cases} v_1(s,t) & \text{on } \Sigma\setminus E_\rho,\\ v_2(s,t) & \text{on } E_\rho,\end{cases} \tag{4.11}$$

y_ρ, y_1 are the solutions of (1.1) corresponding respectively to v_ρ and to v_1, z is the weak solution of

$$\begin{cases} \dfrac{\partial z}{\partial t}+Az+f'_y(x,t,y_1)z=0 & \text{in } Q,\\[2mm] \dfrac{\partial z}{\partial n_A}+g'_y(s,t,y_1,v_1)z=g(s,t,y_1,v_1)-g(s,t,y_1,v_2) & \text{on } \Sigma,\\[2mm] z(0)=0 & \text{in } \Omega.\end{cases} \tag{4.12}$$

Proof. Using Lemma 4.3, the proof is similar to the one of Theorem 4.1 in [25] and the one of Theorem 3.4 in [5]. The relation (4.10), which does not appear in our previous papers, is deduced with the help of (4.4) and (4.8). □

5. Proof of Pontryagin's principle.

5.1. Penalized problem. Following [19], [20], since $C(\overline{D})$ is separable, there exists a norm $|\cdot|_{C(\overline{D})}$, which is equivalent to the usual norm $\|\cdot\|_{C(\overline{D})}$ such that $(C(\overline{D}),|\cdot|_{C(\overline{D})})$ is strictly convex, and $\mathcal{M}(\overline{D})$, endowed with the dual norm of $|\cdot|_{C(\overline{D})}$ (denoted by $|\cdot|_{\mathcal{M}(\overline{D})}$), is also strictly convex; see [9], Corollary 2 p. 148, or Corollary 2 p. 167. We define the distance function to $\mathcal{C}$ (for the new norm $|\cdot|_{C(\overline{D})}$) by

$$d_{\mathcal{C}}(\varphi)=\inf_{z\in\mathcal{C}}|\varphi-z|_{C(\overline{D})}.$$

Since $\mathcal{C}$ is convex, then $d_{\mathcal{C}}$ is convex and Lipschitz of rank 1, and we have

$$\limsup_{\substack{\rho\searrow 0,\\ \varphi'\to\varphi}}\frac{d_{\mathcal{C}}(\varphi'+\rho z)-d_{\mathcal{C}}(\varphi')}{\rho}=\max\{\langle\xi,z\rangle_{\mathcal{M}(\overline{D})\times C(\overline{D})}\mid \xi\in\partial d_{\mathcal{C}}(\varphi)\} \tag{5.1}$$

for every $\varphi, z\in C(\overline{D})$, where $\partial d_{\mathcal{C}}$ is the subdifferential in the sense of convex analysis (see [7]). Therefore, for a given $\varphi\in C(\overline{D})$ we have

$$\langle\xi,z-\varphi\rangle_{\mathcal{M}(\overline{D})\times C(\overline{D})}+d_{\mathcal{C}}(\varphi)\le d_{\mathcal{C}}(z)\quad \forall\xi\in\partial d_{\mathcal{C}}(\varphi)\ \text{ and } \forall z\in C(\overline{D}), \tag{5.2}$$

$$|\xi|_{\mathcal{M}(\overline{D})}\le 1\quad\text{for every } \xi\in\partial d_{\mathcal{C}}(\varphi).$$

Moreover it is proved in [20, Lemma 3.4] that, since $\mathcal{C}$ is a closed convex subset of $C(\overline{D})$, for every $\varphi\notin\mathcal{C}$, and every $\xi\in\partial d_{\mathcal{C}}(\varphi)$, then $|\xi|_{\mathcal{M}(\overline{D})}=1$. Since $\partial d_{\mathcal{C}}(\varphi)$ is convex in $\mathcal{M}(\overline{D})$ and $(\mathcal{M}(\overline{D}),|\cdot|_{\mathcal{M}(\overline{D})})$ is strictly convex, if $\varphi\notin\mathcal{C}$, then $\partial d_{\mathcal{C}}(\varphi)$ is a singleton and $d_{\mathcal{C}}$ is Gâteaux-differentiable at φ.

Let $(\bar{y}, \bar{v})$ be an optimal solution of (P). Consider the penalized functional:

$$J_k(y,v) = \left\{ \left[\left(J(y,v) - J(\bar{y},\bar{v}) + \frac{1}{k^2} \right)^+ \right]^2 + (d_{\mathcal{C}}(\Phi(y)))^2 \right.$$

$$\left. + \sum_{i=1}^{m_0} \left[\int_\Sigma \Psi_i(s,t,y,v)\, dsdt \right]^2 + \sum_{i=m_0+1}^{m} \left[\left(\int_\Sigma \Psi_i(s,t,y,v)\, dsdt \right)^+ \right]^2 \right\}^{\frac{1}{2}}.$$

We easily verify that $(\bar{y}, \bar{v})$ is a $\frac{1}{k^2}$-solution of the penalized problem

$$(P_k^M) \qquad \inf\{J_k(y,v) \mid y \in W(0,T) \cap C(\overline{Q}), v \in V_{ad}(\bar{v}, M),\ (y,v) \text{ satisfies } (1.1)\},$$

for every $M > 0$ and every $k > 0$. For every $k > 0$, we set $M_k = k^{(\frac{1}{2\bar{\sigma}} - \frac{1}{2\sigma})}$ and we denote by (P^k) the penalized problem $(P_k^{M_k})$.

5.2. Proof of Theorem 2.1. Step 1. For every $k \geq 1$, the metric space $(V_{ad}(\bar{v}, M_k), d)$ is complete; see Proposition 4.2. Let us prove that the functional $v \longmapsto J_k(y_v, v)$ is lower semicontinuous on this metric space. Since the mappings $v \to J(y_v, v)$ and $v \to \int_\Sigma \Psi_i(s,t,y_v,v)\, dsdt$ $(m_0 + 1 \leq i \leq m)$ are lower semicontinuous on $(V_{ad}(\tilde{v}, M_k), d)$, it is clear that $v \to \left(J(y_v,v) - J(\bar{y},\bar{v}) + \frac{1}{k^2}\right)^+$ and $v \to \left(\int_\Sigma \Psi_i(s,t,y_v,v)\, dsdt\right)^+$ $(m_0 + 1 \leq i \leq m)$ are also lower semicontinuous on $(V_{ad}(\tilde{v}, M_k), d)$ because $r \to r^+$ is a nondecreasing continuous mapping from $\mathbb{R}$ into $\mathbb{R}^+$. On the other hand, the mappings $v \to \int_\Sigma \Psi_i(s,t,y_v,v)\, dsdt$ $(1 \leq i \leq m_0)$ are continuous on $(V_{ad}(\tilde{v}, M_k), d)$. Since the mappings $r \to r^2$ and $r \to r^{\frac{1}{2}}$ are nondecreasing and continuous from $\mathbb{R}^+$ into $\mathbb{R}^+$, then $v \to J_k(y_v, v)$ is lower semicontinuous. Thanks to Ekeland's variational principle, for every $k \geq 1$, there exists $v_k \in V_{ad}(\bar{v}, M_k)$ such that

$$(5.3) \quad d(v_k, \bar{v}) \leq \frac{1}{k} \text{ and } J_k(y_k, v_k) \leq J_k(y_v, v) + \frac{1}{k} d(v_k, v) \text{ for every } v \in V_{ad}(\bar{v}, M_k)$$

(y_k and y_v are the solutions of (1.1) corresponding respectively to v_k and v). Let v_0 be in V_{ad}. Let k_0 be large enough so that v_0 belong to $V_{ad}(\bar{v}, M_k)$ for every $k \geq k_0$. Let us remark that, thanks to the choice of M_k, $(v_k)_k$ tends to $\bar{v}$ in $L^{\bar{\sigma}}(\Sigma)$. Let us check this. Denote by Σ_k the set of points $(s,t) \in \Sigma$ where $v_k(s,t) \neq \bar{v}(s,t)$. From (5.3) we know that $\mathcal{L}^N(\Sigma_k) \leq 1/k$. Then

$$(5.4) \quad \begin{aligned} \textstyle\int_\Sigma |\bar{v} - v_k|^{\bar{\sigma}} dsdt = \int_{\Sigma_k} |\bar{v} - v_k|^{\bar{\sigma}} dsdt &\leq \|\bar{v} - v_k\|_{\sigma,\Sigma}^{\bar{\sigma}} \mathcal{L}^N(\Sigma_k)^{1 - \frac{\bar{\sigma}}{\sigma}} \\ &\leq M_k^{\bar{\sigma}} k^{\frac{\bar{\sigma}}{\sigma} - 1} = k^{\frac{1}{2}(\frac{\bar{\sigma}}{\sigma} - 1)} \longrightarrow 0 \ \text{ when } k \to +\infty. \end{aligned}$$

Step 2. Theorem 3.1 gives the existence of measurable sets $E_\rho^k \subset \Sigma$, such that $\mathcal{L}^N(E_\rho^k) = \rho \mathcal{L}^N(\Sigma)$,

$$(5.5) \quad \begin{aligned} &\int_{\Sigma \setminus E_\rho^k} |v_k - \bar{v}|^\sigma\, dsdt + \int_{E_\rho^k} |v_0 - \bar{v}|^\sigma\, dsdt = \\ &(1 - \rho) \int_\Sigma |v_k - \bar{v}|^\sigma\, dsdt + \rho \int_\Sigma |v_0 - \bar{v}|^\sigma\, dsdt, \end{aligned}$$

$$\int_{\Sigma\setminus E_\rho^k} h(s,t,v_k)\,dsdt + \int_{E_\rho^k} h(s,t,v_0)\,dsdt = \tag{5.6}$$
$$(1-\rho)\int_\Sigma h(s,t,v_k)\,dsdt + \rho\int_\Sigma h(s,t,v_0)\,dsdt,$$

$$\int_\Sigma(\Psi(s,t,y_\rho^k,v_\rho^k) - \Psi(s,t,y_k,v_k))\,dsdt = \tag{5.7}$$
$$\rho\int_\Sigma(\Psi_y'(s,t,y_k,v_k)z_k + \Psi(s,t,y_k,v_0) - \Psi(s,t,y_k,v_k))\,dsdt + o(\rho),$$

$$y_\rho^k = y_k + \rho z_k + r_\rho^k, \quad \lim_{\rho\to 0}\frac{1}{\rho}\|r_\rho^k\|_{C(\overline{Q})} = 0, \tag{5.8}$$

$$J(y_\rho^k, v_\rho^k) = J(y_k,v_k) + \rho\Delta J_k + o(\rho), \tag{5.9}$$

where v_ρ^k is defined by

$$v_\rho^k(s,t) = \begin{cases} v_k(s,t) & \text{on } \Sigma\setminus E_\rho^k, \\ v_0(s,t) & \text{on } E_\rho^k, \end{cases} \tag{5.10}$$

y_ρ^k is the state corresponding to v_ρ^k, z_k is the weak solution of

$$\begin{cases} \dfrac{\partial z_k}{\partial t} + Az_k + f_y'(x,t,y_k)z_k = 0 & \text{in } Q, \\ \dfrac{\partial z_k}{\partial n_A} + g_y'(s,t,y_k,v_k)z_k = g(s,t,y_k,v_k) - g(s,t,y_k,v_0) & \text{on } \Sigma, \\ z_k(0) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\Delta J_k = \int_Q F_y'(x,t,y_k(x,t))z_k(x,t)\,dxdt + \int_\Sigma G_y'(s,t,y_k(s,t),v_k(s,t))z_k(s,t)\,dsdt$$

$$+\int_\Sigma[G(s,t,y_k(s,t),v_0(s,t)) - G(s,t,y_k(s,t),v_k(s,t))]\,dsdt + \int_\Omega L_y'(x,y_k(T))z_k(T)\,dx.$$

On the other hand, for every $k > k_0$ and every $0 < \rho < 1$, due to (5.5) and (5.6), v_ρ^k belongs to $V_{ad}(\bar{v}, M_k)$. If we set $v = v_\rho^k$ in (5.3), it follows that:

$$\lim_{\rho\to 0}\frac{J_k(y_k,v_k) - J_k(y_\rho^k,v_\rho^k)}{\rho} \le \frac{1}{k}\mathcal{L}^N(\Sigma). \tag{5.11}$$

Taking (5.1), (5.7), (5.9) and the definition of J_k into account, we obtain

$$-\nu_k\Delta J_k - \lambda_k\int_\Sigma\left[\Psi(s,t,y_k,v_0) - \Psi(s,t,y_k,v_k) + \Psi_y'(s,t,y_k,v_k)z_k\right]\,dsdt\text{ff} \tag{5.12}$$
$$\langle\mu_k, \Phi'(y_k)z_k\rangle_{\overline{D}} \le \tfrac{1}{k}\mathcal{L}^N(\Sigma),$$

where

$$\lambda_k^i = \frac{\int_\Sigma \Psi_i(s,t,y_k,v_k)\,dsdt}{J_k(y_k,v_k)} \quad \text{for } 1\le i\le m_0,$$

$$\lambda_k^i = \frac{\left(\int_\Sigma \Psi_i(s,t,y_k,v_k)\,dsdt\right)^+}{J_k(y_k,v_k)} \quad \text{for } m_0+1 \le i \le m,$$

$$\nu_k = \frac{(J(y_k,v_k) - J(\bar y,\bar v) + \frac{1}{k^2})^+}{J_k(y_k,v_k)}, \quad \mu_k = \begin{cases} \dfrac{d_{\mathcal C}(\Phi(y_k))\nabla d_{\mathcal C}(\Phi(y_k))}{J_k(y_k,v_k)} & \text{if } \Phi(y_k)\notin \mathcal C,\\ 0 & \text{otherwise.}\end{cases}$$

For every $k>0$, we consider the weak solution p_k of

$$(5.13)\begin{cases} -\dfrac{\partial p_k}{\partial t} + A^* p_k + f'_y(x,t,y_k)p_k = \nu_k F'_y(x,t,y_k) + [\Phi'(y_k)^*\mu_k]|_Q & \text{in } Q,\\ \dfrac{\partial p_k}{\partial n_{A^*}} + g'_y(s,t,y_k,v_k)p_k = \nu_k G'_y(s,t,y_k,v_k) + \lambda_k \Psi'_y(s,t,y_k,v_k) + \\ \qquad\qquad [\Phi'(y_k)^*\mu_k]|_\Sigma & \text{on } \Sigma,\\ p_k(T) = \nu_k L'_y(x,y_k(T)) + [\Phi'(y_k)^*\mu_k]|_{\overline{\Omega}_T} & \text{on } \overline{\Omega},\end{cases}$$

where $[\Phi'(y_k)^*\mu_k]|_Q$, $[\Phi'(y_k)^*\mu_k]|_\Sigma$ and $[\Phi'(y_k)^*\mu_k]|_{\bar\Omega_T}$ have the same meaning as in Theorem 2.1. By using the Green formula of Theorem 3.6, we obtain

$$\nu_k \int_Q F'_y(x,t,y_k)z_k\,dxdt + \nu_k \int_\Sigma G'_y(s,t,y_k,v_k)z_k\,dsdt + \nu_k \int_\Omega L'_y(x,y_k(T))z_k(T)\,dx$$

$$+\lambda_k \int_\Sigma \Psi'_y(s,t,y_k,v_k)z_k\,dsdt + \langle \mu_k, \Phi'(y_k)z_k\rangle_{\overline{D}}$$

$$= \int_Q p_k\Big(\frac{\partial z_k}{\partial t} + Az_k + f'_y(x,t,y_k)z_k\Big)\,dxdt + \int_\Sigma p_k\Big(\frac{\partial z_k}{\partial n_A} + g'_y(s,t,y_k,v_k)z_k\Big)\,dsdt$$

$$= \int_\Sigma p_k[g(s,t,y_k,v_k) - g(s,t,y_k,v_0)]\,dsdt.$$

With this equality, (5.12) and the definition of ΔJ_k, we have

$$(5.14)\qquad \int_\Sigma [\nu_k G(s,t,y_k,v_k) + \lambda_k \Psi(s,t,y_k,v_k) - p_k g(s,t,y_k,v_k)]\,dsdt$$

$$\le \int_\Sigma [\nu_k G(s,t,y_k,v_0) + \lambda_k \Psi(s,t,y_k,v_0) - p_k g(s,t,y_k,v_0)]\,dsdt + \frac{1}{k}\mathcal{L}^N(\Sigma)$$

for every $k \ge k_0$.

Step 3. Notice that $\nu_k^2 + \sum_i (\lambda_k^i)^2 + |\mu_k|^2_{\mathcal M(\overline{D})} = 1$. Then there exist an element $(\bar\nu,\bar\lambda,\bar\mu)$ in $\mathbb{R}^{1+m}\times\mathcal M(\overline{D})$, with $\bar\nu\ge 0$ and $\bar\lambda_i \ge 0$ for $m_0+1\le i\le m$, and a subsequence, still denoted by $(\nu_k,\lambda_k,\mu_k)_k$, such that

$$(\nu_k,\lambda_k) \longrightarrow (\bar\nu,\bar\lambda) \text{ in } \mathbb{R}^{1+m}, \ \mu_k \rightharpoonup \bar\mu \text{ weak* in } \mathcal M(\overline{D}).$$

From Theorem 3.6, we obtain the estimate:

$$\|p_k\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \le C_4(\delta,d)\left\{\|F'_y(\cdot,y_k)\|_{1,Q} + \|G'_y(\cdot,y_k,v_k)\|_{1,\Sigma} + \right.$$

$$\|L'_y(\cdot, y_k(T))\|_{1,\Omega} + |\lambda_k| \|\Psi'_y(.,y_k,v_k)\|_{1,\Sigma} + |\mu_k|_{\mathcal{M}(\overline{D})} \|\Phi'_y(y_k)\|_{\mathcal{L}(C(\overline{D});C(\overline{D}))} \Big\},$$

for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$, where $\mathcal{L}(C(\overline{D}); C(\overline{D}))$ denotes the space of linear continuous mappings from $C(\overline{D})$ to $C(\overline{D})$.
Since the sequences $(\nu_k)_k, (\lambda_k)_k, (\mu_k)_k, (y_k)_k$, and $(v_k)_k$ are bounded respectively in $\mathbb{R}$, $\mathbb{R}^m$, $\mathcal{M}(\overline{D})$, $C(\overline{Q})$, and in $L^{\bar\sigma}(\Sigma)$, the sequence $(p_k)_k$ is bounded in $L^{\delta'}(0,T; W^{1,d'}(\Omega))$. Then there exist $\bar p \in L^{\delta'}(0,T; W^{1,d'}(\Omega))$ and a subsequence, still denoted by $(p_k)_k$, such that $(p_k)_k$ weakly converges to $\bar p$ in $L^{\delta'}(0,T; W^{1,d'}(\Omega))$ for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. By using the same arguments as in [25], we can prove that $\bar p$ is the weak solution of (2.4).

Step 4. Recall that $(v_k)_k$ tends to $\bar v$ in $L^{\bar\sigma}(\Sigma)$ (see (5.4)).

By passing to the limit when k tends to infinity in (5.14), with Fatou's Lemma (applied to the sequence of functions $(\nu_k G(\cdot, 0, v_k(\cdot)), \lambda_k \Psi(\cdot, 0, v_k(\cdot)))_k)$ and the convergence results stated in Step 2, we obtain

$$H_\Sigma(\bar y, \bar v, \bar p, \bar\nu, \bar\lambda) \le H_\Sigma(\bar y, v_0, \bar p, \bar\nu, \bar\lambda), \tag{5.15}$$

for every $v_0 \in V_{ad}$. On the other hand, from definitions of μ_k and λ_k, and from (5.2), we deduce

$$\lambda_k^i \int_\Sigma \Psi_i(s,t,y_k,v_k)\, dsdt = 0, \ m_0 + 1 \le i \le m,$$

$$\langle \mu_k, z - \Phi(y_k) \rangle_{\mathcal{M}(\overline{D}) \times C(\overline{D})} \le 0 \ \text{ for all } z \in \mathcal{C}.$$

We obtain (2.2) and (2.3) by passing to the limit in these expressions. Since $\mathcal{C}$ is of finite codimension, by using the same arguments as in [26], we prove that $(\bar\nu, \bar\lambda, \bar\mu)$ is non zero. □

REFERENCES

[1] J. J. Alibert, J. P. Raymond, Boundary Control of Semilinear Elliptic Equations with Discontinuous Leading Coefficients and Unbounded Controls, Numer. Funct. Anal. and Optim., Vol. 18 (1997), 235-250.

[2] N. Basile, M. Mininni, An Extension of the Maximum Principle for a Class of Optimal Control Problems in Infinite-Dimensional Spaces, SIAM J. Cont. and Optim., Vol. 28 (1990), p. 1113-1135.

[3] V. Barbu, "Analysis and Control of Nonlinear Infinite Dimensional Systems", Academic Press, New York, 1993.

[4] J. F. Bonnans, E. Casas, An Extension of Pontryagin's Principle for State Constrained Optimal Control of Semilinear Elliptic Equations and Variational Inequalities, SIAM J. Cont. and Optim., Vol. 33 (1995), p. 274-298.

[5] E. Casas, Pontryagin's Principle for State-Constrained Boundary Control Problems of Semilinear Parabolic Equations, SIAM J. Cont. and Optim., Vol. 35 (1997), p. 1263-1296.

[6] E. Casas, J.-P. Raymond, H. Zidani, Optimal Control Problem Governed by Semilinear Elliptic Equations with Integral Control Constraints and Pointwise State Constraints, in International Conference on Control and Estimations of Distributed Parameter Systems, W. Desch, F. Kappel, K. Kunisch Eds., Birkhaüser-Verlag, 1998, p. 85-100.

[7] F. H. Clarke, Optimization and Nonsmooth Analysis, John Wiley, New York, 1983.

[8] E. DiBenedetto, "Degenerate Parabolic Equations", Springer-Verlag, New York, 1993.

[9] J. Diestel, "Geometry of Banach Spaces- Selected Topics", Lecture Notes in Mathematics No 485, Springer-Verlag/Berlin/Heidelberg/New York, 1975.

[10] I. Ekeland, Nonconvex Minimization Problems, Bull. Amer. Math. Soc. N.S., Vol. 1 (1979), p. 443-474.

[11] H. O. Fattorini, A Unified Theory of Necessary Conditions for Nonlinear Nonconvex Control Systems, Appl. Math. Optim., Vol. 15(1987), p. 141-185.

[12] H. O. Fattorini, T. Murphy, Optimal Control Problems for Nonlinear Parabolic Boundary Control Systems: The Dirichlet Boundary Condition, Diff. and Integ. Equat., Vol 7 (1994), p. 1367-1388.

[13] H. O. Fattorini, T. Murphy, Optimal Problems for Nonlinear Parabolic Boundary Control Systems, SIAM J. Control and Optim., Vol. 32 (1994), p. 1577-1596.

[14] H. O. Fattorini, S. Sritharan, Necessary and Sufficient Conditions for Optimal Controls in Viscous Flow Problems, Proceedings of the Royal Society of Edinburgh, 124A (1994), p. 211-251.

[15] B. Hu, J. Yong, Pontryagin Maximum Principle for Semilinear and Quasilinear Parabolic Equations with Pointwise State Constraints, SIAM J. Cont. and Optim. Vol. 33 (1995), p. 1857-1880.

[16] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, R. I., 1968.

[17] X. J. Li, Vector Measure and the Necessary Conditions for the Optimal Control Problems of Linear Systems, Proceedings of the Third IFAC Symposium on the Control of Distributed Parameter Systems, Toulouse, France 1982.

[18] X. J. Li, S. N. Chow, Maximum Principle of Optimal Control for Functional Differential Systems, J. Optim. Theory and Applic., Vol. 54 (1987), p. 335-360.

[19] X. J. Li, Y. Yao, Maximum Principle of Distributed Parameter Systems with Time Lags, Proceedings on the Conference on Control Theory of Distributed Parameter Systems and Applications, F. Kappel and K. Kunish Eds., Springer-Verlag, New York, 1985, p. 410-427.

[20] X. J. Li, J. Yong, Necessary Conditions for Optimal Control of Distributed Parameter Systems, SIAM J. Control and Optim., Vol. 29 (1991), p. 895-908.

[21] X. J. Li, J. Yong, "Optimal Control Theory for Infinite Dimensional Systems", Birkhäuser, Boston/Basel/Berlin, 1995.

[22] J. P. Raymond, Nonlinear Boundary Control of Semilinear Parabolic Equations with Pointwise State Constraints, Disc. and Cont. Dyn. Syst., Vol. 3 (1997), 341-370.

[23] J. P. Raymond, H. Zidani, Hamiltonian Pontryagin's Principles for Control Problems Governed by Semilinear Parabolic Equations, to appear in Appl. Math. and Optim..

[24] J. P. Raymond, H. Zidani, Optimal Control Problem Governed by a Semilinear Parabolic Equation, in System Modelling and Optimization, J. Dolezal and J. Fidler Eds., Chapman and Hall, 1996, p. 211-217.

[25] J. P. Raymond, H. Zidani, Pontryagin's Principles for State-Constrained Control Problems Governed by Semilinear Parabolic Equations with unbounded Controls, to appear in SIAM J. Cont. and Optim..

[26] J. P. Raymond, H. Zidani, Pontryagin's Principle for Time Optimal Problems, to appear in J. O. T. A..

[27] J. Serrin, Pathological Solutions of Elliptic Differential Equations, Ann. Scuola Norm. Sup. Pisa, Vol. 18 (1964), 385-387.

[28] F. Tröltzsch, Optimality Conditions for Parabolic Control Problems and Applications, Teubner-Texte zür Mathematik, 62, B.G. Teubner Verlagsgesellshaft, Leipzig, 1984.

[29] Y. Yao, Vector Measure and Maximum Principle of Distributed Parameter Systems, Sci. Sinica Ser., Vol. 26 (1983), p. 102-112.

[30] Y. Yao, Maximum principle of Semi-linear Distributed Systems, Proceedings of the Third IFAC Symposium on the Control of Distributed Parameter Systems, Toulouse, France 1982.

[31] J. Yong, Pontryagin Maximum Principle for Semilinear Second Order Elliptic Partial Differential Equations and Variational Inequalities with State Constraints, Diff. and Integ. Equat., Vol. 5 (1992), p. 1307-1334.