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Error Estimates for a Stochastic Impulse Control Problem

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Abstract. We obtain error bounds for monotone approximation schemes of a stochastic impulse control problem. This is an extension of the theory for error estimates for the Hamilton–Jacobi–Bellman equation. We obtain almost the same estimate on the rate of convergence as in the equation without impulsions [2], [3].

1. Introduction

The aim of this paper is to give error bounds for approximation schemes of the impulse control problem. More precisely we consider the following equation:

$$\max\left\{\sup_{\alpha_i\in\mathcal{A}} L^{\alpha_i}(x,\mathcal{D}u);\, u(x)-\mathcal{M}u(x)\right\}=0, \qquad x\in\mathbb{R}^N, \tag{P}$$

where

$$L^{\alpha_i}(x,\mathcal{D}u(x)) = L^{\alpha_i}(x,u(x),Du(x),D^2u(x)),$$
$$L^{\alpha_i}(x,r,p,X) = -\mathrm{tr}[a^{\alpha_i}(x)X] - b^{\alpha_i}(x)p + c^{\alpha_i}(x)r - f^{\alpha_i}(x),$$

and

$$\begin{cases} \mathcal{M}u(x) := k + \inf_{\xi \in \mathbb{R}_+^N} \{u(x+\xi) + c(\xi)\}, \\ k > 0, \qquad c\colon \mathbb{R}_+^N \to \mathbb{R}_+, \\ c(0) = 0, \qquad c(\xi_1 + \xi_2) \le c(\xi_1) + c(\xi_2). \end{cases} \tag{1}$$

Here $\mathcal{A} = \{\alpha_1, \ldots, \alpha_M\}$ denotes the set of controls, assumed to be finite. The coefficients $(a^{\alpha_i}, b^{\alpha_i}, c^{\alpha_i}, f^{\alpha_i})$ are, for each $\alpha_i \in \mathcal{A}$, bounded and Lipschitz functions $\mathbb{R}^N \to \mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, where $\mathcal{S}^N$ denotes the set of $N \times N$ symmetric matrices. Under classical assumptions, (P) has a unique bounded viscosity solution, denoted u. The regularity of u depends on the properties of the coefficients a, b, c, f. We refer to [14] and [15], for the existence, uniqueness and regularity of u.

Then we consider monotone approximation schemes of (P), of the following form:

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_h(x)\} = 0, \qquad x \in \mathbb{R}^N, \tag{S}$$

where $S\colon \mathbb{R}_+^N \times \mathbb{R}^N \times \mathbb{R} \times C_b(\mathbb{R}^N) \to \mathbb{R}$ is a consistent, monotonic and uniformly continuous approximation of $\sup_{\alpha_i \in \mathcal{A}} L^{\alpha_i}(x, \mathcal{D}u(x))$ (see Section 2). We denote $u_h \in C_b(\mathbb{R}^N)$ as the solution of (S), which is the approximation of u, and $h \in \mathbb{R}^N$ as the mesh size. This abstract notation was introduced by Barles and Souganidis [4] to display clearly the monotonicity of the scheme: $S(h, x, r, v)$ is nondecreasing in r and nonincreasing in v. Typical approximation schemes that we consider are Classical Finite Differences [21], Generalized Finite Differences [6], [7] and Markov Chain Approximations [21].

Results on convergence rates for monotone approximation schemes of the corresponding equation without impulses are known; i.e. for the following equation:

$$\sup_{\alpha_i \in \mathcal{A}} L^{\alpha_i}(x, \mathcal{D}u(x)) = 0, \qquad x \in \mathbb{R}^N, \tag{2a}$$

and the related scheme

$$S(h, x, u_h(x), u_h) = 0, \qquad x \in \mathbb{R}^N. \tag{2b}$$

Error estimates for this equation have been obtained by Krylov [19], [20], and these results were extended by Barles and Jakobsen [2], [3]. Moreover, results on convergence rate for monotone approximation schemes of a particular Isaac equation have been obtained by the authors [5] and by Jakobsen [16], [17].

Using the method introduced by Ishii [15], to prove the existence of a unique viscosity solution of (P), we approach (P) by a sequence of cascade problems (P_n), $n \ge 1$,

$$\max\left\{\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)); u(x) - \mathcal{M}u_{n-1}(x)\right\} = 0, \qquad x \in \mathbb{R}^N, \tag{P_n}$$

where u_0 is the solution of (2a). Let u_n be the viscosity solution of (P_n). In the same way we approach (S) by a sequence of cascade schemes (S_n), $n \ge 1$,

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_{h(n-1)}(x)\} = 0, \qquad x \in \mathbb{R}^N, \tag{S_n}$$

where u_{h0} is the solution of (2b). Let u_{hn} denote the solution of (S_n).

Using the methods introduced by Barles and Jakobsen [3], upper and lower bounds of $u_n - u_{hn}$, for all $n < +\infty$, are obtained. The upper estimate of $u_n - u_{hn}$ is easier to obtain than the lower. The proof involves a "Krylov regularization" of (P_n), i.e. the perturbed equation

$$\max\left\{\sup_{\alpha_i,|e|\le\varepsilon} L^{\alpha_i}(x+e), \mathcal{D}u_n^\varepsilon(x)); u_n^\varepsilon(x) - \mathcal{M}u_{n-1}(x)\right\} = 0,$$

and its viscosity solution u_n^ε. A regularization of u_n^ε by convolution gives an approximate smooth sub-solution of (P_n), denoted $u_{n\varepsilon}$, which is also an approximate sub-solution of (S_n). So, by using the consistency property, we obtain the upper bound of $u_n - u_{hn}$, after choosing an optimal parameter of regularization. Then we consider $u - u_h$ and we do the following decomposition:

$$\begin{aligned}\sup_x(u(x) - u_h(x)) \le{}& \sup_x(u(x) - u_n(x)) + \sup_x(u_n(x) - u_{hn}(x))\\ &+ \sup_x(u_{hn}(x) - u_h(x)),\end{aligned}$$

for all n in $\mathbb{N}$. Choosing the optimal n, we obtain the result. In particular, we have that $n \sim |\ln h|$.

To obtain the lower estimate, we start by giving a lower bound of $u_n - u_{hn}$, for $n \in \mathbb{N}$. We introduce the following switching system approximating (P_n):

$$\max\left\{L^{\alpha_i}(x, \mathcal{D}v_i^n(x)); v_i^n(x) - \min_{j\ne i}\{v_j^n(x) + \ell\}; v_i^n(x) - \mathcal{M}u_{n-1}(x)\right\} = 0, \quad (3)$$

for $x \in \mathbb{R}^N$, and $i \in \mathcal{I} = \{1, \dots, M\}$, $\ell \ge 0$. For literature on the switching systems, see [8], [11], [12] and [13]. We consider the viscosity solution $v^n = (v_1^n, \dots, v_M^n)$ of this system, and give an estimate of the rate of convergence of v^n to u_n. Then we consider a perturbed system

$$\begin{aligned}\max\Bigg\{&\inf_{|e|\le\varepsilon} L^{\alpha_i}(x+e, \mathcal{D}w_i^{n\varepsilon}(x)); w_i^{n\varepsilon}(x) - \min_{j\ne i}\{w_j^{n\varepsilon}(x) + \ell\};\\ &w_i^{n\varepsilon}(x) - \mathcal{M}u_{n-1}(x)\Bigg\} = 0, \qquad (4)\end{aligned}$$

for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$, and its viscosity solution $w^{n\varepsilon} = (w_1^{n\varepsilon}, \dots, w_M^{n\varepsilon})$. We regularize $w^{n\varepsilon}$ by convolution obtaining $w_{n\varepsilon}$, and this function allows us to build a local super-solution of (P_n). Then, by applying the consistency and the monotonicity of the scheme, we obtain the lower bound of $u_n - u_{hn}$. Finally, since

$$\begin{aligned}\sup_x(u_h(x) - u(x)) \le{}& \sup_x(u_h(x) - u_{hn}(x)) + \sup_x(u_{hn}(x) - u_n(x))\\ &+ \sup_x(u_n(x) - u(x)),\end{aligned}$$

choosing the optimal n, we obtain the result. With our result, we can prove an upper bound of $|h|^{1/2}|\ln h|$ and a lower bound of $|h|^{1/5}|\ln h|$ for the Classical Finite Differences scheme and for the Generalized Finite Differences scheme. Observe that, in the case without impulsions, the results of [2] and [3] give an upper bound of $|h|^{1/2}$ and a lower bound of $|h|^{1/5}$ for the Classical Finite Differences scheme and for the Generalized Finite Differences scheme. Therefore for impulse control problems we obtain very close estimates.

The paper is organized as follows: Section 2 introduces notations and states the main result. Section 3 introduces the cascade approximations of (P) and (S). Section 4 obtains the upper bound of $u_n - u_{hn}$, for all $n < +\infty$, whereas Section 5 gives the lower bound of $u_n - u_{hn}$, for all $n < +\infty$. Section 6 is devoted to the proof of the main theorem. Finally the Appendices give some auxiliary theorems which are used throughout the paper.

2. Notations and Main Result

We start by introducing some notations used in the article. By $|\cdot|$ we mean the standard Euclidean norm in any $\mathbb{R}^M$ type space. In particular, if $X \in \mathcal{S}^N$, then $|X|^2 = \mathrm{tr}(XX^\top)$, where $X^\top$ is the transpose of X, i.e. $|X|$ is the Frobenius norm. If g is a bounded function from $\mathbb{R}^N$ into either $\mathbb{R}$, $\mathbb{R}^M$ or the space of $N \times P$ matrices, we set $|g|_0 := \sup_{x\in\mathbb{R}^N} |g(x)|$. If g is also Lipschitz continuous, we set

$$[g]_1 := \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|}, \qquad |g|_1 := |g|_0 + [g]_1.$$

We denote by $\leq$ the componentwise ordering in $\mathbb{R}^N$, and by $\preceq$ the ordering in the sense of positive semidefinite matrices in $\mathcal{S}(N)$. The space $C_b(\mathbb{R}^N)$ (resp. $C_{b,l}(\mathbb{R}^N)$) denotes the space of continuous and bounded functions (resp. bounded and Lipschitz functions) from $\mathbb{R}^N$ to $\mathbb{R}$.

Given $g \in C_{b,l}(\mathbb{R}^N)^M$, $M \geq 1$, we denote by L_g an *upper bound* of the Lipschitz constant of g, $L_g \geq \max_{i=1,\dots,M}[g_i]_1$.

We use a sequence of mollifiers $(\rho_\varepsilon)_\varepsilon$ defined as follows:

$$\rho_\varepsilon(x) = \varepsilon^{-N}\rho(x/\varepsilon), \tag{5}$$

where $\rho \in C^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}\rho = 1$, $\mathrm{supp}\{\rho\} \subseteq \bar{B}(0,1)$ and $\rho \geq 0$. We define the mollification of $g \in C_b(\mathbb{R}^N)$ as follows:

$$g_\varepsilon(x) := \int_{\mathbb{R}^N} g(x-e)\rho_\varepsilon(e)\,\mathrm{d}e. \tag{6}$$

If g is Lipschitz continuous, then

$$|g(x) - g_\varepsilon(x)| \leq L_g\varepsilon. \tag{7}$$

If $g \in C_b(\mathbb{R}^N)$ (resp. $C_{b,l}(\mathbb{R}^N)$), then we have

$$|D^i g_\varepsilon(x)| \leq C\varepsilon^{-i}|g|_0 \quad (\text{resp. } C\varepsilon^{1-i}|g|_1), \qquad \forall i = 1,\dots,n. \tag{8}$$

From [15] we have the following properties on $\mathcal{M}$, defined in (1).

Proposition 2.1. *Let $u, v\colon \mathbb{R}^N \to \mathbb{R}$. We have*:

(1) *If $u \le v$ in $\mathbb{R}$, then $\mathcal{M}u \le \mathcal{M}v$ in $\mathbb{R}^N$.*
(2) $\mathcal{M}(tu + (1-t)v) \ge t\mathcal{M}u + (1-t)\mathcal{M}v$; $t \in [0, 1]$.
(3) $\mathcal{M}(u + c) = \mathcal{M}u + c$ *for* $c \in \mathbb{R}$.
(4) $|\mathcal{M}u - \mathcal{M}v|_0 \le |u - v|_0$ *for all* $u, v \in C(\mathbb{R}^N)$.

The assumptions we use on equation (P) are as follows:

(A1) For all $\alpha_i \in \mathcal{A}$, we have $a^{\alpha_i} = \frac{1}{2}\sigma^{\alpha_i}\sigma^{\alpha_i T}$, where σ^{α_i} is an $N \times P$ measurable function of x. There exists a constant K such that, for all $\alpha_i \in \mathcal{A}$,

$$c^{\alpha_i} \ge 1 \quad \text{and} \quad |\sigma^{\alpha_i}|_1 + |b^{\alpha_i}|_1 + |c^{\alpha_i}|_1 + |f^{\alpha_i}|_1 \le K.$$

(A2) $1 > \sup_{\alpha_i}\{[\sigma^{\alpha_i}]_1^2 + [b^{\alpha}]_1\}$.

Assumption (A1) ensures that all equations we use are well-posed; assumption (A2) ensures that all solutions are Lipschitz and bounded. Without assumption (A2), we have that all solutions are Hölder and bounded. We conjecture that the techniques used in this paper can be extended to this case. In assumption (A1) we have assumed $c^{\alpha_i} \ge 1$ for simplicity of future computations. All our results can be extended to the general case $c^{\alpha_i} \ge \lambda$, where $\lambda > 0$. In this case, in assumption (A2) and in all estimates of Lipschitz constants obtained in Appendix A, we have to write $\min(\lambda, 1)$ instead of 1.

The result of Theorem 4.2 of [15] gives the existence of a viscosity solution of (P). Moreover, generalizing, in the obvious way, the proof of Theorem 3.5 of [1], which involves only the first-order impulse control problem, we obtain the uniqueness of this viscosity solution. We can then give the following proposition.

Proposition 2.2. *Under assumptions* (A1) *and* (A2), (P) *has a unique viscosity solution u in $C_{b,l}(\mathbb{R}^N)$. In particular, we have*

$$|u|_0 \le \sup_{\alpha_i} |f^{\alpha_i}|_0.$$

Let $C \ge 0$ be a constant, and consider the following equation:

$$\max\left\{\sup_{\alpha_i} L_C^{\alpha_i}(x, \mathcal{D}u);\, u(x) - \mathcal{M}u(x)\right\} = 0, \qquad x \in \mathbb{R}^N, \tag{P_C}$$

where $L_C^{\alpha_i}(x, r, p, X) = L^{\alpha_i}(x, r, p, X) - Cc^{\alpha_i}(x)$. We then have the following lemma, which is given without proof.

Lemma 2.3. *Under assumptions* (A1) *and* (A2), *u is a viscosity solution of (P) if and only if $u + C$ is a viscosity solution of (P_C).*

Remark 2.4. In what follows we assume that $f^{\alpha_i}(x) \ge 0$, for all x and α_i, since that slightly simplifies the proofs; however, using Lemma 2.3, all our results are easily extended to the case when f is not nonnegative.

We now state assumptions on the discrete scheme (S), which is an approximation of the equation (P):

(S1) Monotonicity: $S(h, x, r + m, u + m) \geq m + S(h, x, r, v)$ for all $h \in \mathbb{R}_+^N$, $r \in \mathbb{R}$, $m \geq 0$, $x \in \mathbb{R}^N$ and u, v in $C_b(\mathbb{R}^N)$ such that $u \leq v$ in $\mathbb{R}^N$.

(S2) Regularity: for all $h \in \mathbb{R}_+^N$ and $\phi \in C_b(\mathbb{R}^N)$, $x \mapsto S(h, x, \phi(x), \phi)$ is bounded and continuous; $r \mapsto S(h, x, r, \phi)$ is uniformly continuous for bounded r, uniformly with respect to $x \in \mathbb{R}^N$.

(S3) There exist $n, k_i > 0, i \in J \subseteq \{1, \ldots, n\}$, and a constant $K_c > 0$ such that for all $h \in \mathbb{R}_+^N$ and x in $\mathbb{R}^N$, and for every smooth $\phi \in C^n(\mathbb{R}^N)$ such that $|D^i\phi|_0$ is bounded, for every $i \in J$, the following holds:

$$\left| \sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}\phi) - S(h, x, \phi(x), \phi) \right| \leq K_c Q(\phi),$$

where $Q(\phi) := \sum_{i \in J} |D^i\phi|_0 |h|^{k_i}$.

(S4) Suppose now that the scheme $S(h, x, u_h(x), u_h)$ can be written in the following form:

$$\sup_{\alpha_i} S^{\alpha_i}(h, x, u_h(x), u_h), \tag{9}$$

as is the case for the Finite Differences scheme and the Generalized Finite Differences scheme.

(i) Let $C \geq 0$ be a constant. If v is solution of
$\max\{\sup_{\alpha_i} S^{\alpha_i}(h, x, v(x), v); v(x) - \mathcal{M}v(x)\} = 0$,
then $v + C$ is solution of
$\max\{\sup_{\alpha_i}(S^{\alpha_i}(h, x, v(x), v) - Cc^{\alpha_i}(x)); v(x) - \mathcal{M}v(x)\} = 0$.

(ii) If v is solution of

$$\max\left\{\sup_{\alpha_i} S^{\alpha_i}(h, x, v(x), v); v(x) - \mathcal{M}v(x)\right\} = 0, \tag{10}$$

then νv is solution of

$$\max\left\{\sup_{\alpha_i}(S^{\alpha_i}(h, x, \nu v(x), \nu v) + (\nu - 1) f^{\alpha_i}(x)); \nu v(x) - \nu\mathcal{M}v(x)\right\} = 0, \tag{11}$$

where $\nu \in (0, 1)$, and f^{α_i} is defined after equation (P).

Remark 2.5. Assumption (S4(i)) leads us to have 0 as the lower bound for every solution of the cascade schemes. If we do not assume (S4(i)), we obtain a negative constant as the lower bound; all our results can be extended to this case, but computations become more complicated. Assumption (S4(ii)) is useful to prove the uniqueness of the solution of (S).

Example 2.6. An example of a numerical scheme which satisfies these assumptions is the standard Finite Differences scheme when $N = 1$, defined as

$$S(h, x, r, \phi) := \sup_{\alpha_i \in \mathcal{A}} \left\{ -a^{\alpha_i}(x) \left[\frac{\phi(x+h) - 2r + \phi(x-h)}{h^2} \right] \right.$$
$$- b_+^{\alpha_i}(x) \left[\frac{\phi(x+h) - r}{h} \right] + b_-^{\alpha_i}(x) \left[\frac{\phi(x-h) - r}{h} \right]$$
$$\left. + c^{\alpha_i}(x) r - f^{\alpha_i}(x) \right\}, \tag{12}$$

where $b_+^{\alpha_i}(x) := \max(b^{\alpha_i}(x), 0)$, $b_-^{\alpha_i}(x) := \max(-b^{\alpha_i}(x), 0)$. Clearly assumptions (S1), (S2) and (S4) are satisfied. For the consistency hypothesis (S3), we obtain, from a Taylor expansion,

$$Q(\phi) = |D^2\phi| h + |D^4\phi| h^2, \tag{13}$$

i.e. $n = 4$, $J = \{2, 4\}$, $k_2 = 1$ and $k_4 = 2$.

We set, J being defined in (S3),

$$\bar{\gamma} := \min_{i \in J} \left\{ \frac{k_i}{i} \right\}, \qquad \underline{\gamma} := \min_{i \in J} \left\{ \frac{k_i}{3i - 2} \right\}. \tag{14}$$

We explain briefly how we obtain our main result. In the following we build sequences P_n and S_n, $n \geq 0$, of equations of type (P_n) and (S_n) respectively, which approximate (P) and (S). Then we have that the sequence of viscosity solutions u_n of (P_n), $n \geq 0$, converges to u, and the sequence of solution u_{hn} of (S_n), $n \geq 0$, converges to u_h. We will give these rates of convergence. Finally, for each n we give an upper and a lower bound of $u_n - u_{hn}$, and we use these bounds to obtain (15).

We now state our main result.

Theorem 2.7. *Assume that* (A1), (A2) *and* (S1), (S4) *hold, and let* $u \in C_{b,l}(\mathbb{R}^N)$ *be the unique viscosity solution of* (P). *Then* (S) *has a unique solution* $u_h \in C_b(\mathbb{R}^N)$. *Moreover, for* h *small enough, the following two bounds hold*:

$$-C|h|^{\underline{\gamma}} |\ln h| \leq u - u_h \leq C|h|^{\bar{\gamma}} |\ln h|, \tag{15}$$

where C *is a bounded constant, which depends on* K *defined in* (A1), *on* k, *and on the rates of convergence of* u_n *and* u_{hn}.

Consider now the Finite Differences scheme given in Example 2.6. We have the following result:

Corollary 2.8. *Let* u *be the solution of* (P), *for* $N = 1$, *and let* u_h *be the solution of* (S), *with* S *defined as in* (12). *The following two bounds hold*:

$$-C|h|^{1/5} |\ln h| \leq u - u_h \leq C|h|^{1/2} |\ln h|, \tag{16}$$

where C *is a bounded constant, which depends on* K *defined in* $(A1)$, *and on the rates of convergence of* u_n *and* u_{hn}.

Proof. Applying (13), we obtain $\underline{\gamma} = \frac{1}{5}$ and $\bar{\gamma} = \frac{1}{2}$. Then we can use the precedent theorem to obtain the result. □

Remark 2.9. Corollary 2.8 can be extended to the Finite Differences scheme in dimension $N > 1$ [21], and to the Generalized Finite Differences scheme in dimension $N \geq 1$ [6], [7]. The bounds that we obtain are the same as (16), where now h is the vector of space steps along each component of x.

3. The Cascade Approximations

In this section we present the approximations of (P) and (S), and we study their main properties.

3.1. *Cascade for the HJB equation*

We approach equation (P) by a sequence of obstacle problems, and use the same methods as in the proof of Theorem 4.2 of [15], to prove that the solutions of the sequence of equations converge to the solution of (P). By Remark 2.4, we have that $u \equiv 0$ is a viscosity sub-solution of (P). Consider the following problem:

$$\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)) = 0, \qquad x \in \mathbb{R}^N. \tag{P_0}$$

Under assumptions (A1) and (A2), this equation has a unique viscosity solution u_0 in $C_{b,l}(\mathbb{R}^N)$. Since $u \equiv 0$ is a viscosity sub-solution of (P_0), the comparison principle (see Theorem 3.3 of [15]) implies $0 \leq u_0$. Consider the following problem:

$$\max\left\{\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)); u(x) - \mathcal{M}u_0(x)\right\} = 0, \qquad x \in \mathbb{R}^N. \tag{P_1}$$

Since $\mathcal{M}u_0$ is uniformly continuous, under assumptions (A1) and (A2), there exists a unique viscosity solution u_1 of (P_1) in $C_{b,l}(\mathbb{R}^N)$. Similarly, for $n = 2, 3, \ldots$, let $u_n \in C_{b,l}(\mathbb{R}^N)$ be the unique viscosity solution of

$$\max\left\{\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}u(x)); u(x) - \mathcal{M}u_{n-1}(x)\right\} = 0, \qquad x \in \mathbb{R}^N. \tag{P_n}$$

It is easy to check that u_1 is a viscosity sub-solution of (P_0). By the comparison principle, $u_1 \leq u_0$. Moreover, $u \equiv 0$ is a sub-solution of (P_1) in $\mathbb{R}^N$, and then $0 \leq u_1 \leq u_0$ in $\mathbb{R}^N$. By point (1) of Proposition 2.1, $\mathcal{M}u_1 \leq \mathcal{M}u_0$, then we can say that u_2 is a viscosity sub-solution of (P_1), and also $u_2 \leq u_1$ in $\mathbb{R}^N$. By induction over n, we obtain

$$0 \leq \cdots \leq u_n \leq \cdots \leq u_2 \leq u_1 \leq u_0. \tag{17}$$

We can see that if $|u_0|_0 \leq k$, then u is a viscosity solution of (P) and then we refer to [2] and [3] for error estimates. Suppose now that $|u_0|_0 > k$, and let $\mu \in (0, 1)$ such that $\mu |u_0|_0 < k$.

Theorem 3.1. *We have that, for all n,*

$$u_n - u_{n+1} \le (1-\mu)^n |u_0|_0. \tag{18}$$

Proof. Let $n \in \mathbb{N}$ and $\theta_n \in (0,1]$ be such that

$$u_n - u_{n+1} \le \theta_n u_n, \quad \text{in } \mathbb{R}^N. \tag{19}$$

By (17), this holds at least for $\theta_n = 1$. Rewriting (19) as $(1-\theta_n)u_n \le u_{n+1}$, and using Proposition 2.1, we get

$$\begin{aligned}(1-\theta_n)\mathcal{M}u_n + \theta_n k &\le (1-\theta_n)\mathcal{M}u_n + \theta_n \mathcal{M}0 \\ &\le \mathcal{M}[(1-\theta_n)u_n] \le \mathcal{M}u_{n+1}.\end{aligned} \tag{20a}$$

We now prove that

$$(1-\theta_n+\mu\theta_n)u_{n+1} \le u_{n+2}, \tag{20b}$$

where u_{n+2} is the viscosity solution of (P_{n+2}). Since u_{n+1} is the viscosity solution of (P_{n+1}), and $f^{\alpha_i}(x) \ge 0$, for all x and for all α_i, we have that $(1-\theta_n+\mu\theta_n)u_{n+1}$ is a viscosity sub-solution of $\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}v(x)) = 0$. Moreover, by the construction of the sequence (17), and by (20a), we have

$$(1-\theta_n+\mu\theta_n)u_{n+1} \le (1-\theta_n)u_{n+1} + \mu\theta_n |u_0|_0, \tag{21a}$$

$$\mathcal{M}u_{n+1} \ge (1-\theta_n)\mathcal{M}u_n + \theta_n k. \tag{21b}$$

Taking the difference between (21a) and (21b), and knowing that u_{n+1} is the viscosity solution of (P_n), we have

$$\begin{aligned}&(1-\theta_n+\mu\theta_n)u_{n+1} - \mathcal{M}u_{n+1} \\ &\quad \le (1-\theta_n)u_{n+1} + \mu\theta_n|u_0|_0 - (1-\theta_n)\mathcal{M}u_n - \theta_n k \\ &\quad \le (1-\theta_n)u_{n+1} + \theta_n k - (1-\theta_n)\mathcal{M}u_n - \theta_n k \le 0.\end{aligned}$$

So we can say that $(1-\theta_n+\mu\theta_n)u_{n+1}$ is a viscosity sub-solution of (P_{n+2}). The comparison principle implies (20b), or equivalently

$$u_{n+1} - u_{n+2} \le \theta_n(1-\mu)u_{n+1}. \tag{22}$$

As in the proof of Theorem 4.2 of [15], by the inequalities $u_0 - u_1 \le u_0$ in $\mathbb{R}^N$, we obtain $u_1 - u_2 \le (1-\mu)u_1$ in $\mathbb{R}^N$. Then we can take $\theta_1 = 1-\mu$ and we obtain $u_2 - u_3 \le (1-\mu)^2 u_2$, and by induction we have

$$u_{n+1} - u_{n+2} \le (1-\mu)^{n+1}u_{n+1} \le (1-\mu)^{n+1}|u_0|_0. \quad \square \tag{23}$$

By (17) and (18), we can find a function $u \in C(\mathbb{R}^N)$, such that $|u_n - u|_0 \to 0$, when $n \to +\infty$. Proposition 2.1 and the stability of solutions imply that u is a viscosity solution of (P). Then we can say that u_n converges to u, the unique viscosity solution of (P), when $n \to +\infty$. We want to estimate an upper bound of $u_n - u$ for an arbitrary n. By (18) and since $(1-\mu) < 1$, we obtain that, for all $n \ge 0$,

$$u_n - u \le \sum_{i=n}^{+\infty}(1-\mu)^i|u_0|_0 = \frac{(1-\mu)^n}{1-(1-\mu)}|u_0|_0 = \frac{(1-\mu)^n}{\mu}|u_0|_0. \tag{24}$$

3.2. *Cascade for the Numerical Scheme*

As we have done for equation (P), we approach (S) by a sequence of equations (S_n). This kind of approach has already been used for the numerical study of the impulse control problem, see in particular [9].

Throughout the paper we suppose that every equation (S_n) has at least one solution $u_{hn} \in C_b\mathbb{R}^N$. We now give a useful lemma to obtain the uniqueness of u_{hn}, for all n. Consider the equation

$$\max\{S(h, x, \phi(x), \phi); \phi(x) - \psi(x)\} = 0, \qquad x \in \mathbb{R}^N, \tag{25}$$

where $\psi \in C_b(\mathbb{R}^N)$. We say that a function $\phi \in C_b(\mathbb{R}^N)$ is a sub-solution (resp. super-solution) of (25) if

$$\max\{S(h, x, \phi(x), \phi); \phi(x) - \psi(x)\} \leq 0 \quad (\text{resp. } \geq 0), \qquad \text{for all} \quad x \in \mathbb{R}^N.$$

Lemma 3.2. *Let S satisfy* (S1)–(S4), *and let u and v be respectively the sub- and super-solution of* (25). *Then*

$$u(x) \leq v(x), \qquad \textit{for all} \quad x \in \mathbb{R}^N.$$

The proof is a particular case of the proof of Proposition 3.5 where we take $g = 0$.

Let $u_{h0} \in C_b(\mathbb{R}^N)$ be a solution of

$$S(h, x, u_h(x), u_h) = 0, \qquad x \in \mathbb{R}^N. \tag{S_0}$$

By Lemma 3.2, u_{h0} is unique. Since $\mathcal{M}u_{h0}$ is bounded and continuous, by the same reason there exists a unique solution $u_{h1} \in C_b(\mathbb{R}^N)$ of

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_{h0}(x)\} = 0, \qquad x \in \mathbb{R}^N. \tag{26}$$

For $n = 2, 3, \ldots$, we denote by u_{hn} the unique continuous and bounded solution of

$$\max\{S(h, x, u_h(x), u_h); u_h(x) - \mathcal{M}u_{h(n-1)}(x)\} = 0, \qquad x \in \mathbb{R}^N. \tag{S_n}$$

The function u_{h1} is a sub-solution of (S_0), and then $u_{h1} \leq u_{h0}$ in $\mathbb{R}^N$. Using Remark 2.4 and assumption (S4), we verify that $u_h \equiv 0$ is a sub-solution of (26) in $\mathbb{R}^N$, and then we have $0 \leq u_{h1} \leq u_{h0}$ in $\mathbb{R}^N$. Proposition 2.1 implies that $0 \leq \mathcal{M}u_{h1} \leq \mathcal{M}u_{h0}$, then u_{h2} is a sub-solution of (26), and hence $u_{h2} \leq u_{h1}$ in $\mathbb{R}^N$. By induction on n, we obtain

$$0 \leq \cdots \leq u_{hn} \leq \cdots \leq u_{h2} \leq u_{h1} \leq u_{h0}. \tag{27}$$

As in Section 3.1, we suppose that $|u_0|_0 > k$. Then, since $u_{h0} \to u_0$ uniformly, we also have $|u_{h0}|_0 > k$ and we can choose $\mu \in (0, 1)$ such that $\mu|u_0|_0 < k$ and $\mu|u_{h0}|_0 < k$.

Theorem 3.3. *For all n and for h small enough*, *we have*

$$u_{hn} - u_{h(n+1)} \leq (1 - \mu)^n |u_{h0}|_0. \tag{28}$$

Proof. We use the same methods as in Theorem 3.1, taking some θ_n. The unique difference is that we have to show that $(1-\theta_n-\mu\theta_n)u_{h(n+1)}$ is a sub-solution of (S_{n+2}), which can be written

$$\max\{S(h,x,u_h(x),u_h);\, u_h(x)-\mathcal{M}u_{h(n+1)}(x)\}=0, \qquad x\in\mathbb{R}^N.$$

With the monotonicity of S, we obtain the result. □

Proposition 3.4. *Under assumptions* (S1)–(S4) *there exists a solution of the equation* (S).

Proof. By (27) and (18), we can find a function $u_h \in C_b(\mathbb{R}^N)$, such that $|u_{hn}-u_h|_0 \to +\infty$, when $n\to+\infty$. Proposition 2.1 and the stability of solutions implies that u_h is a solution of (S). □

We can prove a comparison principle for (S), with S written as in (9), and hence the uniqueness of its solution.

Proposition 3.5. *Let S satisfy* (S1)–(S4). *Let u and v be the solutions of*

$$\max\left\{\sup_{\alpha_i} S^{\alpha_i}(h,x,u(x),u);\, u(x)-\psi_1(x)\right\}=0, \qquad x\in\mathbb{R}^N, \tag{29}$$

and

$$\max\left\{\sup_{\alpha_i}(S^{\alpha_i}(h,x,v(x),v)+g(x));\, v(x)-\psi_2(x)\right\}=0, \qquad x\in\mathbb{R}^N, \tag{30}$$

where ψ_1, ψ_2 and g are elements of $C_b(\mathbb{R}^N)$. Then

$$|u-v|_0 \le \max\{|g|_0;\, |\psi_1-\psi_2|_0\}. \tag{31}$$

Proof. Since u and v are solutions of (29) and (30), respectively, we have that

$$\max\left\{\sup_{\alpha_i} S^{\alpha_i}(h,x,u(x),u);\, u(x)-\psi_1(x)\right\}\le 0,$$

$$\max\left\{\sup_{\alpha_i}(S^{\alpha_i}(h,x,v(x),v)+g(x));\, v(x)-\psi_2(x)\right\}\ge 0,$$

for all x in $\mathbb{R}^N$. Since $\max\{A;B\}-\max\{C;D\}\le\max\{A-C;B-D\}$, (29) and (30) imply

$$0\le\max\left\{\sup_{\alpha_i}(S^{\alpha_i}(h,x,v(x),v)+g(x))-\sup_{\alpha_i} S(h,x,u(x),u);\right.$$
$$\left. v(x)-\psi_2(x)-(u(x)-\psi_1(x))\right\}.$$

Hence we have the following two cases:

(a) $u(x) - v(x) \le \psi_1(x) - \psi_2(x)$, which implies $u(x) - v(x) \le |\psi_1 - \psi_2|_0$.
(b) $\sup_{\alpha_i} S^{\alpha_i}(h, x, u(x), u) \le 0$, and $\sup_{\alpha_i}(S^{\alpha_i}(h, x, v(x), v) + g(x)) \ge 0$. Then $\sup_{\alpha_i} S^{\alpha_i}(h, x, v(x), v) + |g|_0 \ge 0$, and, applying the monotonicity, $\sup_{\alpha_i} S(h, x, v(x) + |g|_0, v + |g|_0) \ge 0$. By Theorem 2.1 of [2], we obtain $u(x) - v(x) \le |g|_0$.

Combining the two cases we have

$$\sup_x (u(x) - v(x)) \le \max\left\{\sup_x g(x); \sup_x |\psi_1(x) - \psi_2(x)|\right\}.$$

On the other hand, we have

$$\max\left\{\sup_{\alpha_i} S^{\alpha_i}(h, x, u(x), u); u(x) - \psi_1(x)\right\} \ge 0,$$

$$\max\left\{\sup_{\alpha_i}(S^{\alpha_i}(h, x, v(x), v) + g(x)); v(x) - \psi_2(x)\right\} \le 0,$$

for all x in $\mathbb{R}^N$. We have the two following cases:

(a) $v(x) - u(x) \le \psi_2(x) - \psi_1(x)$, which implies $u(x) - v(x) \ge -|\psi_1 - \psi_2|_0$.
(b) $\sup_{\alpha_i} S^{\alpha_i}(h, x, u(x), u) \ge 0$ and $\sup_{\alpha_i}(S^{\alpha_i}(h, x, v(x), v) + g(x)) \le 0$. Since g is a positive function, we have $\sup_{\alpha_i} S(h, x, v(x), v) \le 0$, and, by Theorem 2.1 of [2], $u(x) - v(x) \ge 0$, which implies $u(x) - v(x) \ge -|g|_0$. □

We can now give the uniqueness result.

Proposition 3.6. *Under assumptions* (S1)–(S4), (S) *has a unique solution* $u_h \in C_b(\mathbb{R}^N)$.

Proof. Let u_h and v_h be solutions of (S). By (S5), νu_h is a solution of

$$\max\left\{\sup_{\alpha_i}(S^{\alpha_i}(h, x, \nu u_h(x), u_h) + (\nu - 1) f^{\alpha_i}(x)); \nu u_h(x) - \nu \mathcal{M} u_h(x)\right\} = 0,$$
$$x \in \mathbb{R}^N,$$

for $\nu \in (0, 1)$. Apply Proposition 3.5 to obtain

$$|\nu u_h - v_h|_0 \le \max\{|(\nu - 1) f^{\alpha_i}|_0; |\nu \mathcal{M} u_h - \mathcal{M} v_h|_0\}.$$

By Theorem 3.5 of [1], we know that $|\nu \mathcal{M} u_h - \mathcal{M} v_h|_0 < |\nu u_h - v_h|_0$, and hence $|\nu u_h - v_h|_0 \le |(\nu - 1) f|_0$. Letting ν go to 1, we have the result. □

We have proved that u_{hn} converges to the solution u_h of (S), for $n \to +\infty$, and by Proposition 3.6 this solution is unique. Moreover, we have

$$u_{hn} - u_h \leq \sum_{i=n}^{+\infty} (1-\mu)^i |u_{h0}|_0 = \frac{(1-\mu)^n}{\mu} |u_{h0}|_0. \tag{32}$$

4. The Upper Bound for the Cascade Problems

In this section we use the methods of [2] and [3] to obtain an upper bound of $u_n - u_{hn}$, for all n. We start with the case $n = 0$, and then we study the general case $n \geq 1$. Finally, we use these estimates to obtain the upper bound of $u - u_h$.

4.1. *Problem without Impulses*

Consider the problem (P_0) and its viscosity solution $u_0 \in C_{b,l}(\mathbb{R}^N)$. Let

$$L_{u_0} := \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_0|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1}.$$

We recall here the result of Lemma A.1 of [18].

Lemma 4.1. *L_{u_0} is an upper bound of the Lipschitz constant of u_0.*

Consider the scheme (S_0) and its solution $u_{h0} \in C_b(\mathbb{R}^N)$. We recall that L^{α_i} and S satisfy assumptions (A1), (A2) and (S1)–(S4). An upper bound of $u_0 - u_{h0}$ has been obtained in [2]. Here we need to rewrite some ideas of this paper, in order to detail constants which appear in various proofs. The auxiliary equation (see [19])

$$\sup_{\alpha_i \in \mathcal{A}, |e| \leq \varepsilon} L^{\alpha_i}(x + e, \mathcal{D}u_0^{\varepsilon}(x)) = 0, \qquad x \in \mathbb{R}^N, \tag{P_0P}$$

has a unique viscosity solution $u_0^{\varepsilon} \in C_{b,l}(\mathbb{R}^N)$. Let $u_{0\varepsilon}$ be the mollification of u_0^{ε}, defined as in (6). We now give a lemma useful in what follows. We recall that $\bar{\gamma}$ is defined in (14).

Lemma 4.2. *Let $g \in C_{b,l}(\mathbb{R}^N)$, and let its mollification be g_{ε}. Set $\varepsilon = |h|^{\bar{\gamma}}$. Then, J being defined in $(S3)$,*

$$Q(g_{\varepsilon}) \leq |J| K_c |g|_1 |h|^{\bar{\gamma}}. \tag{33}$$

Proof. Using (8) we get

$$Q(g_{\varepsilon}) = K_c |g|_1 \sum_{i \in J} \varepsilon^{1-i} |h|^{k_i} = K_c |g|_1 \sum_{i \in J} |h|^{\bar{\gamma}(1-i)+k_i}.$$

Since, by (14), $\bar{\gamma}(1-i) + k_i \geq \bar{\gamma}$, for all $i \in J$, we obtain the result. □

We recall here the result of Proposition 3.2 of [3], where we detail some constants.

Proposition 4.3. *Let $u_0 \in C_{b,l}(\mathbb{R}^N)$ be the viscosity solution of* (P_0), *and let $u_{h0} \in C_{b,l}(\mathbb{R}^N)$ be the solution of* (S_0). *Then we have*

$$u_0(x) - u_{h0}(x) \leq \bar{C}_0 |h|^{\bar{\gamma}}, \qquad \forall x \in \mathbb{R}^N, \tag{$\bar{E}_0$}$$

$$\bar{C}_0 := |J| K_c |u_0^\varepsilon|_1 + R, \tag{34}$$

where R depends only on the constant K of assumption (A1).

Proof. In [2] the authors verify that $u_{0\varepsilon}$ is a classical sub-solution of (P_0). By the consistency hypothesis (S3), (8) and Lemma 4.2, for $\varepsilon = |h|^{\bar{\gamma}}$,

$$S(h, x, u_{0\varepsilon}(x), u_{0\varepsilon}) \leq Q(u_{0\varepsilon}) \leq |J| K_c |u_0^\varepsilon|_1 |h|^{\bar{\gamma}}, \qquad x \in \mathbb{R}^N.$$

Monotonicity implies that $u_{0\varepsilon} - |J| K_c |u_0^\varepsilon|_1 |h|^{\bar{\gamma}} \leq u_{h0}$. By Lemma A.1 of [2], we have that $|u_0 - u_{0\varepsilon}| \leq R\varepsilon$, where R depends only on K defined in (A1). So we have the result. □

4.2. *Problem with n Impulses*, $n \geq 1$

Consider now the problem with n impulses (P_n), for $n \geq 1$, and its viscosity solution $u_n \in C_{b,l}(\mathbb{R}^N)$. We generalize here the method of [2], by introducing the perturbed equation

$$\max \left\{ \sup_{\alpha_i, |e| \leq \varepsilon} L^{\alpha_i}(x + e, \mathcal{D}u_n^\varepsilon(x)); u_n^\varepsilon(x) - \mathcal{M}u_{n-1}(x) \right\} = 0, \tag{$P_n P$}$$

whose unique viscosity solution in $C_{b,l}(\mathbb{R}^N)$ is denoted u_n^ε. We recall that, for the problem without impulses, u_0^ε is the solution of $(P_0 P)$. The next result, proved in Appendix A, gives upper bounds of the Lipschitz constants of u_n and u_n^ε.

Lemma 4.4. *Let u_n and u_n^ε denote the viscosity solutions of* (P_n) *and* $(P_n P)$, *respectively, for $n \geq 1$. Then the upper bounds of the Lipschitz constants of u_n and u_n^ε are*

$$L_{u_n} = L_{u_0}, \tag{35}$$

$$L_{u_n^\varepsilon} = \max \left(L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_n^\varepsilon|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \right). \tag{36}$$

Using the same methods as for sequence (17), we can show that

$$0 \leq \cdots \leq u_n^\varepsilon \leq \cdots \leq u_2^\varepsilon \leq u_1^\varepsilon. \tag{37}$$

Combining with (36), we get

$$0 \leq \cdots \leq L_{u_n^\varepsilon} \leq \cdots \leq L_{u_2^\varepsilon} \leq L_{u_1^\varepsilon}. \tag{38}$$

The following result is proved in Appendix A.

Proposition 4.5. *Let u_n and u_n^ε be the viscosity solutions of (P_n) and (P_nP), respectively, and let A_{u_n,u_n^ε} be as defined in* (61). *Then*

$$|u_n - u_n^\varepsilon|_0 \le A_{u_n,u_n^\varepsilon}\varepsilon.$$

Relations (38), (61), (17) and (36) imply the following result.

Lemma 4.6. $0 \le \cdots \le A_{u_n,u_n^\varepsilon} \le \cdots \le A_{u_2,u_2^\varepsilon} \le A_{u_1,u_1^\varepsilon}$.

Proof. This follows from the expression of coefficients A_{u_i,u_i^ε}, $i = 1, \dots, n$, given in (61), combined with Lemma 4.4 and relation (38). □

We can now give the error estimate of the upper bound. We recall that $\bar{C}_0$ was defined in (34).

Proposition 4.7. *Let $u_n \in C_{b,l}(\mathbb{R}^N)$ be the unique viscosity solution of (P_n), and let $u_{hn} \in C_b(\mathbb{R}^N)$ be the unique solution of (S_n), $n \ge 1$. Then we have*

$$u_n(x) - u_{hn}(x) \le \bar{C}_n|h|^{\bar{\gamma}}, \tag{$\bar{E}_n$}$$

$$\bar{C}_n = \bar{C}_{n-1} + A_{u_n,u_n^\varepsilon} + L_{u_n^\varepsilon} + L_{u_0}. \tag{39}$$

Proof. For all $n \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $u_{n\varepsilon}$ the mollification of u_n^ε. We prove the proposition by induction over n. Take $n = 1$. We show that $u_{1\varepsilon} - \bar{C}_0|h|^{\bar{\gamma}} - L_{u_0}\varepsilon$ is a sub-solution of (26). Applying the classical methods (see [2], [3] and [5]), we have that $u_{1\varepsilon} - L_{u_0}\varepsilon$ is a classical sub-solution of (P_1). Using the consistency hypothesis (S3), Proposition 4.3, the equality $Q(u_{1\varepsilon} - L_{u_0}\varepsilon) = Q(u_{1\varepsilon})$ and the monotonicity of S, we obtain

$$\begin{cases} S(h, x, u_{1\varepsilon}(x) - L_{u_0}\varepsilon - Q(u_{1\varepsilon}), u_{1\varepsilon} - L_{u_0}\varepsilon - Q(u_{1\varepsilon})) \le 0, \\ u_{1\varepsilon}(x) - L_{u_0}\varepsilon - \bar{C}_0|h|^{\bar{\gamma}} \le \mathcal{M}u_{h0}(x). \end{cases}$$

We deduce that $u_{1\varepsilon}(x) - L_{u_0}\varepsilon - \max\{\bar{C}_0|h|^{\bar{\gamma}}, Q(u_{1\varepsilon})\}$ is a sub-solution of (S_1). We now set $\varepsilon = |h|^{\bar{\gamma}}$. By Lemma 4.2, and by (37), (38) and (34), we obtain

$$Q(u_1^\varepsilon) \le |J|K_c|u_1^\varepsilon|_1|h|^{\bar{\gamma}} \le |J|K_c|u_0^\varepsilon|_1|h|^{\bar{\gamma}} \le \bar{C}_0|h|^{\bar{\gamma}}.$$

Then $\max\{\bar{C}_0|h|^{\bar{\gamma}}, Q(u_{1\varepsilon})\} = \bar{C}_0|h|^{\bar{\gamma}}$, which implies $u_{1\varepsilon}(x) - \bar{C}_0|h|^{\bar{\gamma}} - L_{u_0}\varepsilon \le u_{h1}(x)$, for all x. Hence, with (7) and Proposition 4.5,

$$\begin{aligned} u_1(x) - u_{h1}(x) &= u_1(x) - u_1^\varepsilon(x) + u_1^\varepsilon(x) - u_{1\varepsilon}(x) + u_{1\varepsilon}(x) - u_{h1}(x) \\ &\le A_{u_1,u_1^\varepsilon}\varepsilon + L_{u_1^\varepsilon}\varepsilon + L_{u_0}\varepsilon + \bar{C}_0|h|^{\bar{\gamma}} \\ &= (A_{u_1,u_1^\varepsilon} + L_{u_1^\varepsilon} + L_{u_0} + \bar{C}_0)|h|^{\bar{\gamma}}. \end{aligned}$$

We obtain that (39) holds for $n = 1$.

Now we suppose the proposition is true for $n-1$. The same methods as before, the assumption of induction and Lemma 4.4 give us the result. □

So we have obtained that, for all $n \geq 1$, $u_n - u_{hn} \leq \bar{C}_n |h|^{\bar{\gamma}}$. We set

$$\bar{D}_{n-1} := \bar{C}_n - \bar{C}_{n-1} = A_{u_n, u_n^\varepsilon} + L_{u_n^\varepsilon} + L_{u_0}.$$

Lemma 4.6 and relation (38) imply that $\bar{D}_n \leq \bar{D}_0$, and hence, by (39),

$$\bar{C}_n \leq \bar{C}_0 + n\bar{D}_0. \tag{40}$$

5. The Lower Bound for the Cascade Problems

In this section we use the methods of [2] and [3] to obtain a lower bound of $u_n - u_{hn}$, for all n. We start with the case $n = 0$, and then we study the general case $n \geq 1$. Finally, we use these estimates to obtain the lower bound of $u - u_h$.

5.1. *Problem without Impulses*

Consider problem (P_0) of the solution $u_0 \in C_{b,l}(\mathbb{R}^N)$, and scheme (S_0) of the solution $u_{h0} \in C_b(\mathbb{R}^N)$. We recall that L^{α_i} and S satisfy assumptions (A1), (A2) and (S1)–(S4). A lower bound of $u_0 - u_{h0}$ has been obtained in [2]. Here we need to rewrite some parts of this paper, in order to give explicit bounds of constants appearing in various proofs. Consider the following switching system, which approaches (P_0):

$$\max\left\{L^{\alpha_i}(x, \mathcal{D}v_i^0(x));\, v_i^0(x) - \min_{j\neq i}\{v_j^0(x) + \ell\}\right\} = 0, \tag{SS_0}$$

for $x \in \mathbb{R}^N$, $i \in \mathcal{I} = \{1, \ldots, M\}$ and $\ell > 0$. Let $v^0 = (v_1^0, \ldots, v_M^0)$ be the unique viscosity solution of (SS_0), $v^0 \in C_{b,l}(\mathbb{R}^N)^M$. By Remark 2.4, we have that $(0, \ldots, 0)$ is a viscosity sub-solution of (SS_0), hence $0 \leq v_i^0(x)$, for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$.

For every i, v_i^0 converges to u_0, when $\ell \to 0$. We rewrite here the result of Theorem 2.3 of [3], which gives this rate of convergence.

Lemma 5.1. *Let u_0 and v^0 be the viscosity solutions of* (P_0) *and* (SS_0), *respectively. Then, for all i, we have*

$$0 \leq v_i^0 - u_0 \leq C\ell^{1/3}, \tag{H_0}$$

where C depends only on K, defined in (A1).

5.1.1. *Error Estimate*.

Consider the following perturbation of the switching system (SS_0):

$$\max\left\{\inf_{|e|\leq\varepsilon} L^{\alpha_i}(x+e, \mathcal{D}w_i^{\varepsilon 0}(x));\, w_i^{\varepsilon 0}(x) - \min_{j\neq i}\{w_j^{\varepsilon 0}(x) + \ell\}\right\} = 0. \tag{SS_0P}$$

We denote by $w^{0\varepsilon} = (w_1^{0\varepsilon}, \ldots, w_M^{0\varepsilon})$ the unique viscosity solution of (SS_0P) in $C_{b,l}(\mathbb{R}^N)^M$. We have $0 \leq w_i^{0\varepsilon}(x)$, for all i and for all x.

The following result is proved in the appendix.

Lemma 5.2. *Let v^0 and $w^{0\varepsilon}$ be the viscosity solutions of* (SS_0) *and* (SS_0P), *respectively. Then* $\max_i |v_i^0 - w_i^{0\varepsilon}|_0 \leq \varepsilon A_{v^0,w^{0\varepsilon}}$, *where* $A_{v^0,w^{0\varepsilon}}$ *is defined in* (61).

Consider $\underline{\gamma}$ defined in (14). We have the following result.

Lemma 5.3. *Given* $g \in C_{b,l}(\mathbb{R}^N)$, *its mollification* g_ε *and* $\varepsilon = |h|^{3\underline{\gamma}}$, *we have that, for* J *defined in* $(S3)$,

$$Q(g_\varepsilon) \leq |J| K_c |g|_1 |h|^{\underline{\gamma}}. \tag{41}$$

Proof. By (8), we know that

$$Q(g_\varepsilon) = K_c |g|_1 \sum_{i \in J} \varepsilon^{1-i} |h|^{k_i} = K_c |g|_1 \sum_{i \in J} |h|^{3(1-i)\underline{\gamma}+k_i}.$$

Since $3(1-i)\underline{\gamma} + k_i \geq \underline{\gamma}$, for all $i \in J$, we obtain the result. □

We recall here Lemma 3.4 of [3], which gives some auxiliary results to obtain the error estimate.

Lemma 5.4. *Assume* (A1) *and* (A2), *let* $w_{\varepsilon i}^0 = \rho_\varepsilon * w_i^{0\varepsilon}$, *for* $i \in \mathcal{I}$. *Moreover, assume that* $\varepsilon \leq (4 \sup_i [w_i^{0\varepsilon}]_1)^{-1} \ell$. *Then, for every* $x \in \mathbb{R}^N$, *if* $j = \operatorname{argmin}_{i \in \mathcal{I}} w_{\varepsilon i}^0(x)$, *then*

$$L^{\alpha_j}(x, w_{\varepsilon j}^0(x), Dw_{\varepsilon j}^0(x), D^2 w_{\varepsilon j}^0(x)) \geq 0.$$

We now recall the result of Theorem 3.5 of [3], where we detail some constants.

Proposition 5.5. *Let* $u_0 \in C_{b,l}(\mathbb{R}^N)$ *be the viscosity solution of* (P_0) *and let* $u_{h0} \in C_{b,l}(\mathbb{R}^N)$ *be the solution of* (S_0). *Then we have*

$$u_{h0}(x) - u_0(x) \leq \underline{C}_0 |h|^{\underline{\gamma}}, \qquad \forall x \in \mathbb{R}^N, \tag{$\underline{E}_0$}$$

$$\underline{C}_0 = |J| K_c |w^{0\varepsilon}|_1 + R, \tag{42}$$

where R *depends only on* K *defined in* $(A1)$, *and* J *is defined in* $(S3)$.

Proof. We recall the ideas of Theorem 3.5 of [3]. We set

$$m := \sup_{y \in \mathbb{R}^N} \{u_{h0}(y) - g_0(y)\},$$

where $g_0 = \min_{i \in \mathcal{I}} w_{\varepsilon i}^0$. We now set $\varepsilon = |h|^{3\underline{\gamma}}$. Computations of Theorem 3.5 of [3], combined with Lemmas 5.3 and 5.4, give

$$m \leq |J| K_c |w_i^{\varepsilon 0}|_1 |h|^{\underline{\gamma}}, \tag{43}$$

where J is defined in (S3). Applying Lemmas 5.1 and 5.2 and (43), we have that, for all $i \in \mathcal{I}$,

$$\begin{aligned}
&\sup_x (u_{h0}(x) - u_0(x)) \\
&\qquad \le m + \sup_x (w^0_{\varepsilon i}(x) - w^{0\varepsilon}_i(x)) + \sup_x (w^{0\varepsilon}_i(x) - v^0_i(x)) + \sup_x (v^0_i(x) - u_0(x)) \\
&\qquad \le |J| K_c |w^{\varepsilon 0}_i|_1 \sum_{i \in J} \varepsilon^{1-i} |h|^{k_i} + C\varepsilon + A_{v^0, w^{0\varepsilon}} \varepsilon + C\ell^{1/3},
\end{aligned}$$

where C depends only on K defined in (A1). In agreement with Lemma 5.4, $\ell = 4\varepsilon L_{w^{0\varepsilon}} = 4|h|^{3\underline{\gamma}} L_{w^{0\varepsilon}}$, where $L_{w^{0\varepsilon}}$ is an upper bound of the Lipschitz constant of $w^{0\varepsilon}$. By Lemma 5.3, we have

$$\sup_x (u_{h0}(x) - u_0(x)) \le R_0[2|h|^{3\underline{\gamma}} + |h|^{\underline{\gamma}}] + |J| K_c |w^{0\varepsilon}_0|_1 |h|^{\underline{\gamma}},$$

where R_0 depends only on K defined in (A1). Setting $R = 3R_0$, we obtain the result. □

5.2. *Problem with n Impulses*, $n \ge 1$

We generalize here the methods of [2]. Consider problem (P_n) and its solution $u_n \in C_{b,l}(\mathbb{R}^N)$, defined in Section 3.1. We know that L_{u^0} is an upper bound of the Lipschitz constant of u_n, for all n.

Then consider the scheme (S_n) of solution $u_{hn} \in C_b(\mathbb{R}^N)$, defined in Section 3.2. We recall that L^{α_i} and S satisfy assumptions (A1), (A2) and (S1)–(S4). Consider the following switching system which approaches (P_n):

$$\max\left\{ L^{\alpha_i}(x, \mathcal{D}v^n_i(x));\ v^n_i(x) - \min_{j \ne i}\{v^n_j(x) + \ell\};\ v^n_i(x) - \mathcal{M}u_{n-1}(x) \right\} = 0, \quad (SS_n)$$

for $x \in \mathbb{R}^N$ and $i \in \mathcal{I} = \{1, \ldots, M\}$. Under assumptions (A1) and (A2), (SS_n) has a unique viscosity solution $v^n = (v^n_1, \ldots, v^n_M) \in C_{b,l}(\mathbb{R}^N)^M$. By Remark 2.4 it is easy to see that $(0, \ldots, 0)$ is a viscosity sub-solution of (SS_n), and that v^n is a viscosity sub-solution of $(SS_{(n-1)})$, for all n. We can then build the following sequence:

$$0 \le \cdots \le v^n_i(x) \le \cdots \le v^1_i(x) \le v^0_i(x),$$

for all i and for all x.

5.2.1. *Convergence of the Switching System.*

Using the same methods as in Theorem 2.3 of [3], we introduce an auxiliary switching system

$$\begin{aligned}
\max\Bigg\{ &\sup_{|e| \le \varepsilon} L^{\alpha_i}(x + e, \mathcal{D}v^{n\varepsilon}_i(x));\ v^{n\varepsilon}_i(x) \\
&- \min_{j \ne i}\{v^{n\varepsilon}_j(x) + \ell\};\ v^{n\varepsilon}_i(x) - \mathcal{M}u_{n-1}(x) \Bigg\},
\end{aligned} \tag{44}$$

and denote by $v^{n\varepsilon} = (v_1^{n\varepsilon}, \dots, v_M^{n\varepsilon})$ its viscosity solution in $C_{b,l}(\mathbb{R}^N)^M$. As before, we have that $n \mapsto v_i^{n\varepsilon}(x)$ is nonincreasing, for all i and for all x. Let $v_{\varepsilon i}^n = \rho_\varepsilon * v_{\varepsilon i}^n$, for all $i \in \mathcal{I}$.

We can now give the following result about the convergence.

Proposition 5.6. *Let u_n and v^n be the solutions of (P_n) and (SS_n), respectively. Then, for all i, we have*

$$0 \le v_i^n - u_n \le H_{v^n, v^{n\varepsilon}} \ell^{1/3}, \qquad (H_n)$$

where $H_{v^n, v^{n\varepsilon}}$ is defined in (61).

Proof. We start by giving the proof for $n = 1$. Consider $w = (u_1, \dots, u_1)$ (a vector with M components equal to u_1). Then, for every i, we have

$$\begin{cases} L^{\alpha_i}(x, \mathcal{D}u_1(x)) \le 0, \\ u_1(x) \le u_1(x) + \ell \\ u_1(x) \le \mathcal{M}u_0(x), \end{cases} \Rightarrow \quad u_1 \text{ is a sub-solution of (SS1)} \quad \Rightarrow \quad u_1(x) \le v_i^1(x),$$

for all $x \in \mathbb{R}^N$, $i \in \mathcal{I}$. We show that, for all i, $v_{i\varepsilon}^1 - C\ell\varepsilon^{-2} - L_{u_0}\varepsilon$ is a sub-solution of (P_1), where

$$C = C_\rho \ell \sup_{\alpha_i}(|\sigma^{\alpha_i}|_0 + |b^{\alpha_i}|_0 + |c^{\alpha_i}|_0). \qquad (45)$$

With classical methods (see [2], [3] and [5]), we have that $v_{i\varepsilon}^1$ is, for all i, a sub-solution, in the classical sense, of

$$L^{\alpha_i}(x, \mathcal{D}v(x)) = 0, \qquad \forall x \in \mathbb{R}^N. \qquad (46a)$$

The definition of a switching system implies that $|v_i^{1\varepsilon} - v_j^{1\varepsilon}| \le \ell$, for all i, j. Combining with (8), we obtain

$$|L^{\alpha_i}(x, \mathcal{D}v_{\varepsilon j}^1(x)) - L^{\alpha_i}(x, \mathcal{D}v_{\varepsilon i}^1(x))| \le \frac{C\ell}{\varepsilon^2}, \qquad \forall i, j \in \mathcal{I} \quad \text{and} \quad \forall x \in \mathbb{R}^N.$$

Since $v_{\varepsilon i}^1$ is a sub-solution of (46a), this implies

$$L^{\alpha_i}(x, \mathcal{D}v_{\varepsilon j}^1(x)) \le \frac{C\ell}{\varepsilon^2}, \qquad \forall i, j \quad \text{and} \quad \forall x \in \mathbb{R}^N. \qquad (46b)$$

Consequently $v_{\varepsilon i}^1 - C\ell\varepsilon^{-2}$ is a classical sub-solution of $\sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}w(x)) = 0$. Moreover, by the definition of the auxiliary system, we have that $v_i^{1\varepsilon}(x) - \mathcal{M}u_0(x) \le 0$, for all $i \in \mathcal{I}$, and for all $x \in \mathbb{R}^N$. Let $u_{\varepsilon 0}$ be the mollification of u_0, defined as in (6). Then, we have $v_{\varepsilon i}^1(x) - \mathcal{M}u_{0\varepsilon}(x) \le 0$, which implies $v_{\varepsilon i}^1(x) - \mathcal{M}u_0(x) \le L_{u_0}\varepsilon$, and also

$$v_{\varepsilon i}^1(x) - L_{u_0}\varepsilon - C\ell\varepsilon^{-2} - \mathcal{M}u_0(x) \le 0, \qquad \forall x \in \mathbb{R}^N.$$

Hence, for all $x \in \mathbb{R}^N$, we have

$$\begin{cases} \sup_{\alpha_i} L^{\alpha_i}(x, \mathcal{D}(v^1_{\varepsilon i} - C\ell\varepsilon^{-2})(x)) \le L_{u_0}\varepsilon, \\ v^1_{\varepsilon i}(x) - \mathcal{M}u_0(x) \le L_{u_0}\varepsilon + C\ell\varepsilon^{-2}. \end{cases}$$

So $v^1_{\varepsilon i} - L_{u_0}\varepsilon - C\ell\varepsilon^{-2}$ is a viscosity sub-solution of (P1), and we have $v^1_{\varepsilon i}(x) - L_{u_0}\varepsilon - C\ell\varepsilon^{-2} \le u_1(x)$, for all $x \in \mathbb{R}^N$. Finally we obtain

$$v^1_i(x) - u_1(x) \le \frac{C_\rho \ell}{\varepsilon^2} \sup_{\alpha_i}(|\sigma^{\alpha_i}|_0 + |b^{\alpha_i}|_0 + |c^{\alpha_i}|_0) + (L_{u_0} + L_{v^{1\varepsilon}} + A_{v^1,v^{1\varepsilon}})\varepsilon,$$

for all x in $\mathbb{R}^N$. Minimizing with respect to ε, we obtain

$$v^1_i(x) - u_1(x) \le H_{v^1,v^{1\varepsilon}}\ell^{1/3}.$$

The result for $n > 1$ can be proved similarly, using $L_{u_{n-1}} = L_{u_0}$ as an upper bound of the Lipschitz constant of u_{n-1}. □

5.2.2. *Error Estimates*. Consider the following perturbed switching system which approaches (P_n):

$$\max\left\{ \inf_{|e|\le\varepsilon} L^{\alpha_i}(x+e, \mathcal{D}w^{n\varepsilon}_i(x));\ w^{n\varepsilon}_i(x) - \min_{j\ne i}\{w^{n\varepsilon}_j(x) + \ell\};\ w^{n\varepsilon}_i(x) - \mathcal{M}u_{n-1}(x) \right\} = 0, \qquad (SS_nP)$$

and its unique viscosity solution $w^{n\varepsilon} \in C_{b,l}(\mathbb{R}^N)^M$. As before, we can prove that $0 \le \cdots \le w^{n\varepsilon}_i(x) \le \cdots \le w^{1\varepsilon}_i(x) \le w^{0\varepsilon}_i(x)$, for all i and x. Let $g^n := v^n, v^{n\varepsilon}, w^{n\varepsilon}$. Then we set

$$L_{g^n} := \max\left(\sup_i \frac{[c^{\alpha_i}]_1 |g^n_i|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1};\ L_{u_0} \right). \qquad (47)$$

We have the following results, which are showed in the Appendices.

Lemma 5.7. *Let* $g^n := v^n, v^{n\varepsilon}, w^{n\varepsilon}$. *Then* $\max_i [g^n_i]_1 \le L_{g^n}$.

Lemma 5.8. *Let* v^n, $v^{n\varepsilon}$ *and* $w^{n\varepsilon}$ *be the viscosity solutions of* (SS_n), (44) *and* (SS_nP), *respectively. Then we have*

$$\max_i |v^n_i - v^{n\varepsilon}_i|_0 \le A_{v^n,v^{n\varepsilon}}\varepsilon, \qquad \max_i |v^n_i - w^{n\varepsilon}_i|_0 \le A_{v^n,w^{n\varepsilon}}\varepsilon,$$

where $A_{v^n,v^{n\varepsilon}}$ *and* $A_{v^n,w^{n\varepsilon}}$ *are defined in* (61).

The following result is proved in Theorems A.1 and B.3.

Lemma 5.9. *Let* $g^i := v^i$, $v^{i\varepsilon}$, $w^{i\varepsilon}$, *and let* L_{g^i} *be defined as in* (47). *Then*

$$L_{g^n} \le \cdots \le L_{g^2} \le L_{g^1},$$

$$A_{v^n, w^{n\varepsilon}} \le \cdots \le A_{v^1, w^{1\varepsilon}} \le A_{v^0, w^{0\varepsilon}},$$

$$A_{v^n, v^{n\varepsilon}} \le \cdots \le A_{v^1, v^{1\varepsilon}} \le A_{v^0, v^{0\varepsilon}}.$$

We can now give the lower bound.

Proposition 5.10. *Let* $u_n \in C_{b,l}(\mathbb{R}^N)$ *be the viscosity solution of* (P_n) *and let* $u_{hn} \in C_{b,l}(\mathbb{R}^N)$ *be the solution of* (S_n), $n \ge 1$. *Then we have*

$$u_{hn}(x) - u_n(x) \le \underline{C}_n |h|^{\underline{\gamma}}, \qquad \forall x \in \mathbb{R}^N, \tag{$\underline{E}_n$}$$

$$\underline{C}_n = \underline{C}_{n-1} + 12 L_{w^{n\varepsilon}} + 4 L_{u_0} + A_{v^n, w^{n\varepsilon}} + H_{v^n, v^{n\varepsilon}} (6 L_{w^{n\varepsilon}})^{1/3}. \tag{48}$$

Proof. The proof is by induction over n. Let $n = 1$, and let

$$m := \sup_{y \in \mathbb{R}^N} \{u_{h1}(y) - g(y)\}, \tag{49}$$

where $g = \min_{i \in \mathcal{I}} w^1_{\varepsilon i}$. For $\eta \ge 0$, let

$$m_\eta := \sup_{y \in \mathbb{R}^N} \{u_{h1}(y) - g(y) - \eta \phi(y)\},$$

where $\eta > 0$ is a small constant, and $\phi(x) = (1 + |x|^2)^{1/2}$. Let x_0 be such that $m_\eta = u_{h1}(x_0) - g(x_0) - \eta\phi(x_0)$. Then we also have $m_\eta = u_{h1}(x_0) - w^1_{\varepsilon i_0}(x_0) - \eta\phi(x_0)$, where $w^1_{\varepsilon i_0}(x_0) = \min_{j \in \mathcal{I}} w^1_{\varepsilon j}(x_0)$. After some computations (see Theorem 3.4 of [3]), we can say that if $\varepsilon \le (6 L_{w^{1\varepsilon}})^{-1} \ell$, then

$$w^{1\varepsilon}_{i_0}(y) - \min_{j \ne i_0} \{w^{1\varepsilon}_j(y) + \ell\} < 0, \qquad \forall y \in B(x_0, 2\varepsilon). \tag{50}$$

Then equation i_0 in the system (SS_1P) becomes

$$\max\left\{ \inf_{|e| \le \varepsilon} L^{\alpha_{i_0}}(y+e, \mathcal{D}w^{1\varepsilon}_{i_0}(y)); \, w^{1\varepsilon}_{i_0}(y) - \mathcal{M}u_0(y) \right\} = 0, \qquad y \in B(x_0, 2\varepsilon). \tag{51}$$

We have to study two cases.

Case 1. There exists $\bar{x} \in B(x_0, 2\varepsilon)$ such that

$$w^{1\varepsilon}_{i_0}(\bar{x}) = \mathcal{M}u_0(\bar{x}), \quad \text{i.e.} \quad w^{1\varepsilon}_{i_0}(\bar{x}) = k + \inf_{\xi} \{u_0(\bar{x} + \xi) + c(\xi)\}.$$

Then, for all $y \in B(x_0, 2\varepsilon)$,

$$w^{\varepsilon 1}_{i_0}(y) + 4(L_{w^{1\varepsilon}} + L_{u_0})\varepsilon \ge k + \inf_{\xi} \{u_0(y + \xi) + c(\xi)\}.$$

Consider now $\mathcal{M}u_{h0}(y) - \mathcal{M}u_0(y)$. By Proposition 5.5, we have that $\mathcal{M}u_0(y) \geq \mathcal{M}u_{h0}(y) - \underline{C}_0|h|^{\underline{\gamma}}$. Then we obtain

$$\begin{aligned} w_{i_0}^{\varepsilon 1}(y) + 4(L_{w^{1\varepsilon}} + L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}} \\ \geq\ k + \inf_{\xi}\{u_{h0}(y+\xi) + c(\xi)\}, \qquad \forall y \in B(x_0, 2\varepsilon). \end{aligned}$$

Since $u_{h1}(y) \leq k + \inf_\xi\{u_{h0}(y+\xi) + c(\xi)\}$, for all $y \in B(x_0, 2\varepsilon)$, hence

$$\begin{aligned} u_{h1}(x_0) - w_{\varepsilon i_0}^1(x_0) &\leq 4(L_{w^{1\varepsilon}} + L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}} + L_{w^{1\varepsilon}}\varepsilon \\ &= (5L_{w^{1\varepsilon}} + 4L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}}, \end{aligned}$$

which implies

$$m_\eta \leq (5L_{w^{1\varepsilon}} + 4L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}} - \eta\phi(x). \tag{52}$$

Case 2. For all $y \in B(x_0, 2\varepsilon)$, we have

$$w_{i_0}^{1\varepsilon}(y) < \mathcal{M}u_0(y).$$

The classical methods (see [3] and [5]) imply that

$$\sup_{\alpha_i} L^{\alpha_i}(x_0, \mathcal{D}w_{\varepsilon i_0}^1(x_0)) \geq 0.$$

We can apply the consistency hypothesis, to obtain

$$\begin{aligned} &-C\eta \leq S(h, x_0, (w_{\varepsilon i_0}^1 + \eta\phi)(x_0), w_{\varepsilon i_0}^1 + \eta\phi) + Q(w_{\varepsilon i_0}^1 + \eta\phi) \\ &\quad\Rightarrow\quad S(h, x_0, (w_{\varepsilon i_0}^1 + \eta\phi)(x_0), w_{\varepsilon i_0}^1 + \eta\phi) \geq -Q(w_{\varepsilon i_0}^1) + O(l\eta). \end{aligned}$$

Monotonicity implies that

$$\begin{aligned} S(h, x_0, (w_{\varepsilon i_0}^1 + \eta\phi)(x_0), w_{\varepsilon i_0}^1 + \eta\phi) &\leq S(h, x_0, u_{h1}(x_0) - m_\eta, u_{h1} - m_\eta) \\ &\leq -m_\eta + S(h, x_0, u_{h1}(x_0), u_{h1}) \\ &\leq -m_\eta. \end{aligned}$$

The last inequality follows from the definition of (S1). Then we have

$$m_\eta \leq Q(w_{\varepsilon i_0}^1) + O(\eta). \tag{53}$$

Conclusion. By (52) and (53), we obtain that

$$m_\eta \leq \max\{(5L_{w^{1\varepsilon}} + 4L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}} - \eta\phi(x);\ Q(w_{\varepsilon i_0}^1) + O(\eta)\}.$$

We set $\varepsilon = |h|^{3\underline{\gamma}}$. Then, if η goes to zero, we can conclude that

$$m \leq \max\left\{(5L_{w^{1\varepsilon}} + 4L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}};\ K_c|w^{1\varepsilon}|_1 \sum_{i\in J} \varepsilon^{1-i}|h|^{k_i}\right\}.$$

Hence

$$\begin{aligned} u_{h1} - u_1 &= u_{h1} - w^1_{\varepsilon i} + w^1_{\varepsilon i} - u_1 \\ &\le m + w^1_{\varepsilon i} - w^{1\varepsilon}_i + w^{1\varepsilon}_i - v^1_i + v^1_i - u_1 \\ &\le \max\left\{(5L_{w^{1\varepsilon}} + 4L_{u_0})\varepsilon + \underline{C}_0|h|^{\underline{\gamma}};\ K_c|w^{1\varepsilon}|_1 \sum_{i\in J} \varepsilon^{1-i}|h|^{k_i}\right\} \\ &\quad + \ell + L_{w^{1\varepsilon}}\varepsilon + A_{v^1,w^{1\varepsilon}}\varepsilon + H_{v^1,v^{1\varepsilon}}\ell^{1/3}. \end{aligned}$$

Setting $\ell = (6Lw^{1\varepsilon})$, we obtain

$$\begin{aligned} u_{h1} - u_1 \le \max\{&(12L_{w^{1\varepsilon}} + 4L_{u_0})|h|^{3\underline{\gamma}} + (\underline{C}_0 + H_{v^1,v^{1\varepsilon}}(6L_{w^{1\varepsilon}})^{1/3})|h|^{\underline{\gamma}}; \\ &(7L_{w^{1\varepsilon}})|h|^{3\underline{\gamma}} + (|J|K_c|w^{1\varepsilon}|_1 + H_{v^1,v^{1\varepsilon}}(6L_{w^{1\varepsilon}})^{1/3})|h|^{\underline{\gamma}}\}. \end{aligned}$$

Since $|J|K_c|w^{1\varepsilon}|_1 \le |J|K_c|w^{0\varepsilon}|_1 \le \underline{C}_0$, the maximum is attained by the first term. Then we have the result.

Suppose now that $(\underline{E}_n)$ and (48) hold for $n-1$. The same methods as before, the induction and the fact that $L_{u_{n-1}} = L_{u_0}$ give the result. □

We set

$$\underline{D}_{n-1} := \underline{C}_n - \underline{C}_{n-1} = 12L_{w^{n\varepsilon}} + 4L_{u_0} + A_{w^n} + H_n(6L_{w^{n\varepsilon}})^{1/3}. \tag{$\underline{D}_n$}$$

Lemma 5.9 implies that

$$\underline{C}_n \le \underline{C}_0 + n\underline{D}_0. \tag{54}$$

6. Proof of Theorem 2.7

Before giving the proof of Theorem 2.7, consider the following result. Let ϕ: $\mathbb{R} \to \mathbb{R}$, $\phi(x) = \nu a^x + bx + c$, where $0 < a < 1$, $b \in \mathbb{R}^+$, $\nu > 0$ and $c \ge 0$. Let $m := \min_{n\in\mathbb{N}} \phi(n)$. Then we have the following elementary lemma that we state without proof.

Lemma 6.1.

(i) *ϕ attains its minimum over $\mathbb{R}$ at $r := \log_a(-b/\nu \ln a)$, where $-b/\nu \ln a > 0$, since $a < 1$.*

(ii) *If $-b/\nu \ln a \ge 1$, then $r \le 0$, and hence $m = \phi(0) = \nu + c$.*

(iii) *If $-b/\nu \ln a < 1$, then*

$$\begin{aligned} m &\le \phi(\lceil r\rceil) = a^{\lceil r\rceil} + b\lceil r\rceil \le a^{r+1} + b(r+1) \\ &= -\frac{ab}{\ln a} + b\left(\log_a\left(\frac{b}{\nu\ln a}\right) + 1\right) + c. \end{aligned}$$

Proof of Theorem 2.7. We already proved in Propositions 3.4 and 3.6 that (S) has a unique solution.

We start by proving the upper bound of (15). Consider the following decomposition:

$$\sup_x(u(x) - u_h(x)) \le \sup_x(u(x) - u_n(x)) + \sup_x(u_n(x) - u_{hn}(x)) + \sup_x(u_{hn}(x) - u_h(x)), \tag{55a}$$

for all $n < +\infty$. Using $u - u_n \le 0$, $u_n - u_{hn} \le \bar{C}_n|h|^{\bar{\gamma}}$, $u_{hn} - u_{h\infty} \le ((1-\mu)^n/\mu)|u_{h0}|_0$ and (40), we obtain

$$\sup_x(u(x) - u_h(x)) \le (\bar{C}_0 + n\bar{D}_0)|h|^{\bar{\gamma}} + ((1-\mu)^n/\mu)|u_{h0}|_0. \tag{55b}$$

Let $\phi(n) = (\bar{C}_0 + n\bar{D}_0)|h|^{\bar{\gamma}} + ((1-\mu)^n/\mu)|u_{h0}|_0$, and let $m := \min_{n\in\mathbb{N}} \phi(n)$. Applying Lemma 6.1 and the fact that $r \le \lceil r \rceil \le r+1$, we obtain that

- $u - u_h \le \left(\bar{C}_0 + \dfrac{\bar{D}_0}{(-\ln(1-\mu))}\right)|h|^{\bar{\gamma}}, \quad$ if $\quad -\dfrac{\bar{D}_0\mu|h|^{\bar{\gamma}}}{|u_{h0}|_0 \ln(1-\mu)} \ge 1;$
- $u - u_h \le \left[-\dfrac{(1-\mu)\bar{D}_0}{\ln(1-\mu)} + \bar{C}_0 + \bar{D}_0\left(\log_{(1-\mu)}\left(-\dfrac{\mu\bar{D}_0|h|^{\bar{\gamma}}}{|u_{h0}|_0\ln(1-\mu)}\right) + 1\right)\right]|h|^{\bar{\gamma}},$ otherwise.

Hence we have the result. We now prove the lower bound. Consider the following decomposition:

$$\sup_x(u_h(x) - u(x)) \le \sup_x(u_h(x) - u_{hn}(x)) + \sup_x(u_{hn}(x) - u_n(x)) + \sup_x(u_n(x) - u(x)), \tag{56a}$$

for all $n < +\infty$. Since $u_h - u_{hn} \le 0$, $u_{hn} - u_n \le \underline{C}_n|h|^{\underline{\gamma}}$, $u_n - u \le ((1-\mu)^n/\mu)|u_0|_0$ and (54), we obtain

$$u_h - u \le \frac{(1-\mu)^n}{\mu}|u_0|_0 + \underline{C}_0|h|^{\underline{\gamma}} + n\underline{D}_0|h|^{\underline{\gamma}}. \tag{56b}$$

Applying Lemma 6.1, we obtain that

- $u_h - u \le \left(\underline{C}_0 + \dfrac{\underline{D}_0}{(-\ln(1-\mu))}\right)|h|^{\underline{\gamma}}, \quad$ if $\quad -\dfrac{\underline{D}_0\mu|h|^{\underline{\gamma}}}{|u_0|_0\ln(1-\mu)} \ge 1;$
- $u_h - u \le \left[-\dfrac{(1-\mu)\underline{D}_0}{\ln(1-\mu)} + \underline{C}_0 + \underline{D}_0\left(\log_{(1-\mu)}\left(-\dfrac{\mu\underline{D}_0|h|^{\underline{\gamma}}}{|u_0|_0\ln(1-\mu)}\right) + 1\right)\right]|h|^{\underline{\gamma}},$ otherwise.

Hence we have the result. □

Appendix A. The Upper Bounds of Lipschitz Constants

Proof of Lemma 4.4. We prove this lemma by induction. Let $n = 1$, and set

$$m_{\varepsilon_1} := \sup_{x,y} \phi(x, y) := \sup_{x,y\in\mathbb{R}^N} \left\{ u_1(x) - u_1(y) - \frac{\delta}{2}|x - y|^2 - \frac{\varepsilon_1}{2}(|x|^2 + |y|^2) \right\}.$$

Let $m_{\varepsilon_1} = \phi(x_0, y_0)$. By Ishii's lemma (see [10]), there exist $X, Y \in \mathcal{S}^N$ such that

$$0 \le \max\left\{ \sup_{\alpha_i} L^{\alpha_i}(y_0, u_1(y_0), p_y, Y); u_1(y_0) - \mathcal{M}u_0(y_0) \right\} - \max\left\{ \sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X); u_1(x_0) - \mathcal{M}u_0(x_0) \right\}, \tag{57}$$

where

$$p_x = \delta(x_0 - y_0) + \varepsilon_1 x_0, \qquad p_y = \delta(x_0 - y_0) - \varepsilon_1 y_0, \tag{58}$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \delta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon_1 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{59}$$

Then (57) implies

$$0 \le \max\left\{ \sup_{\alpha_i}[L^{\alpha_i}(y_0, u_1(y_0), p_y, Y) - L^{\alpha_i}(x_0, u_1(x_0), p_x, X)]; u_1(y_0) - \mathcal{M}u_0(y_0) - u_1(x_0) - \mathcal{M}u_0(x_0) \right\}.$$

We can reduce this to study two different cases.

Case 1: $u_1(y_0) - \mathcal{M}u_0(y_0) - (u_1(x_0) - \mathcal{M}u_0(x_0)) \ge 0$. This last inequality implies that $u_1(x_0) - u_1(y_0) \le L_{u_0}|x_0 - y_0|$. Then we deduce that

$$m_{\varepsilon_1} \le L_{u_0}|x_0 - y_0| - \delta|x_0 - y_0|^2. \tag{60}$$

Setting $r := |x_0 - y_0|$, and noting that $\max_r(L_{u_0}r - \delta r^2) = L_{u_0}^2/4\delta$, we obtain

$$m_{\varepsilon_1} \le \frac{L_{u_0}^2}{4\delta}.$$

Applying Lemma 2.3 of [18], for fixed δ, we have that

$$\lim_{\varepsilon_1\to 0} m_{\varepsilon_1} = \sup_{x,y\in\mathbb{R}^N} \{u_1(x) - u_1(y) - \delta|x - y|^2\} := m,$$

and hence $m \le L_{u_0}^2/4\delta$. Then we have, by definition of m,

$$u_1(x) - u_1(y) \le \frac{L_{u_0}^2}{4\delta} + \delta|x - y|^2, \qquad \forall x, y \in \mathbb{R}^N.$$

Use $\min_\delta(L_{u_0}^2/4\delta + \delta|x-y|^2) = L_{u_0}|x-y|$ to obtain

$$u_1(x) - u_1(y) \le L_{u_0}|x-y|, \qquad \forall x, y \in \mathbb{R}^N.$$

Case 2: $\sup_{\alpha_i} L^{\alpha_i}(y_0, u_1(y_0), p_y, Y) - \sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X) \ge 0$. This is the standard case (see Lemma A.1 of [18]), and we have that

$$u_1(x) - u_1(y) \le \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_1|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} |x-y|, \qquad \forall x, y, \in \mathbb{R}^N.$$

In conclusion, we obtain

$$L_{u_1} = \max\left\{ L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_1|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \right\}.$$

Since by (17) $|u_1|_0 \le |u_0|_0$, using the definition of L_{u_0}, we have $L_{u_1} = L_{u_0}$.

We now compute $L_{u_1^\varepsilon}$. With the same methods as before, we obtain

$$L_{u_1^\varepsilon} = \max\left(L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_1^\varepsilon|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \right).$$

In this case we do not have the estimate between $|u_0|_0$ and $|u_1^\varepsilon|_0$, hence we must give the result in this form.

Suppose now that the lemma is true for $n-1$, i.e.

$$L_{u_{n-1}} = L_{u_0}, \quad L_{u_{n-1}^\varepsilon} = \max\left(L_{u_0}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_{n-1}^\varepsilon|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \right).$$

Applying the same method as before, we can show that

$$L_{u_n} = \max\left(L_{u_{n-1}}; \sup_{\alpha_i} \frac{[c^{\alpha_i}]_1 |u_{n-1}|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1} \right).$$

Induction and definition of (17) give the result. The same for $L_{u_n^\varepsilon}$. □

Proof of Lemma 5.7. We start by computing L_{v^1}. We set

$$\begin{aligned} m_{\varepsilon_1} &:= \sup_{i,x,y} \phi_i(x,y) \\ &:= \sup_{x,y\in\mathbb{R}^N, i\in\mathcal{I}} \left\{ v_i^1(x) - v_i^1(y) - \frac{\delta}{2}|x-y|^2 + \frac{\varepsilon_1}{2}(|x|^2 + |y|^2) \right\}. \end{aligned}$$

Let $m = \phi_j(x_0, y_0)$, i.e. (j, x_0, y_0) attains the supremum. Let $A := \{i \in \mathcal{I}, (i, x_0, y_0)$ attains the supremum$\}$. Then, by Lemma A.2 of [3], there exists $i_0 \in A$, such that $v_{i_0}^1(y_0) < \min_{j \ne i_0}\{v_j^1(y_0) + l\}$. Hence we have $m = \phi_{i_0}(x_0, y_0)$. The definition of viscosity solution and Ishii's lemma imply the existence of $X, Y \in \mathcal{S}^N$ such that

$$\max\{L^{\alpha_{i_0}}(x_0, v^1_{i_0}(x_0), p_x, X); v^1_{i_0}(x_0) - \min_{j\neq i}\{v^1_j(x_0) + l\};$$
$$v^1_{i_0}(x_0) - \mathcal{M}u_0(x_0)\} \le 0,$$
$$\max\{L^{\alpha_{i_0}}(y_0, v^1_{i_0}(y_0), p_y, X); v^1_{i_0}(y_0) - \mathcal{M}u_0(y_0)\} \ge 0,$$

where p_x, p_y, X and Y satisfy (58) and(59). Then we can reduce this to study two cases.

Case 1: $v^1_{i_0}(y_0) - \mathcal{M}u_0(y_0) - (v^1_{i_0}(x_0) - \mathcal{M}u_0(x_0)) \ge 0$. This last inequality implies that $v^1_{i_0}(x_0) - v^1_{i_0}(y_0) \le L_{u_0}|x_0 - y_0|$. From now on, we continue as in case 1 of the preceding proof, and we have

$$v^1_i(x) - v^1_i(y) \le L_{u_0}|x - y|, \qquad \forall x, y \in \mathbb{R}^N, \quad \forall i \in \mathcal{I}.$$

Case 2: $L^{\alpha_{i_0}}(y_0, v^1_{i_0}(y_0), p_y, Y) - L^{\alpha_{i_0}}(x_0, v^1_{i_0}(x_0), p_x, X) \ge 0$. This is the standard case (see Lemma A.2 of [3]), and we have

$$v^1_i(x) - v^1_i(y) \le \sup_{\alpha_i, i} \frac{[c^{\alpha_i}]_1 |v^1_i|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1}|x - y|, \qquad \forall x, y, \in \mathbb{R}^N, \quad \forall i \in \mathcal{I}.$$

Then we obtain

$$L_{v^1} = \max\left(L_{u_0}; \sup_{\alpha_i, i} \frac{[c^{\alpha_i}]_1 |v^1_i|_0 + [f^{\alpha_i}]_1}{1 - [\sigma^{\alpha_i}]_1^2 - [b^{\alpha_i}]_1}\right).$$

The same computations lead us to obtain $L_{v^{1\varepsilon}}$ and $L_{w^{1\varepsilon}}$. For $n > 1$, we apply exactly the same method. We only need to recall that $L_{u_{n-1}} = L_{u_0}$. □

Theorem A.1. *The sequences $(L_{v^n})_n$, $(L_{v^{n\varepsilon}})_n$ and $(L_{w^{n\varepsilon}})_n$ are nonincreasing.*

Proof. We prove this theorem for $(L_{v^n})_n$, the other cases are similar. Using Lemma 5.7, and since $(v^n_i)_n$ is a decreasing sequence, we obtain that $(L_{v^n})_n$ is decreasing, and then we have the result. □

Appendix B. Constants A_i

We begin this section by introducing the following notation. Let $\psi, \varphi \in C_{b,l}(\mathbb{R}^N)^M$, $M \ge 1$. We define constants $A_{\psi,\varphi}$ and $H_{\psi,\varphi}$ as follows:

$$A_{\psi,\varphi} := \sqrt{2k_1 k_2^{\psi,\varphi}} + k_3^{\psi,\varphi}, \qquad H_{\psi,\varphi} := \frac{3}{2^{2/3}} h_1 h_2^{\psi,\varphi}, \tag{61}$$

where

$$k_1 = \sup_{\alpha_i}\{[\sigma^{\alpha_i}]_1^2 + [b^{\alpha_i}]_1\},$$

$$k_2^{\psi,\varphi} = \sup_{\alpha_i}\{\tfrac{1}{4}(L_\psi + L_\varphi)^2(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1) + \tfrac{1}{2}(L_\psi + L_\varphi)(|\psi|_0 \wedge |\varphi|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\},$$

$$k_3^{\psi,\varphi} = \sup_{\alpha_i}\{|\psi|_0 \wedge |\varphi|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1\},$$

$$h_1 := (C_\rho \sup_{\alpha_i}(|\sigma^{\alpha_i}|_0 + |b^{\alpha_i}|_0 + |c^{\alpha_i}|_0))^{1/3},\ C_\rho \text{ depends only on } \rho,$$

$$h_2^{\psi,\varphi} := (L_\varphi + A_{\psi,\varphi} + L_{u_0})^{2/3}.$$

We give here an extension of the comparison principle of Lemma A.1 of [2].

Proposition B.1. *Let u_n and v_n be the viscosity solutions of two equations like (P_n), for $n \geq 1$, with coefficients σ, b, c, f and $\bar\sigma, \bar b, \bar c, \bar f$, respectively. Then we have*

$$\sup_x\{u_n(x) - v_n(x)\} \leq (2k_1 k_2^{u_n,v_n})^{1/2} + k_3^{u_n,v_n},$$

where

- $k_1 = \sup_{\alpha_i}\{|\bar\sigma^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar b^{\alpha_i} - b^{\alpha_i}|_0^2\}$,
- $k_2^{u_n,v_n} = \sup_{\alpha_i}\{((L_{u_n} + L_{v_n})^2/4)(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1) + ((L_{u_n} + L_{v_n})/2)(|u_n|_0 \wedge |v_n|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,
- $k_3^{u_n,v_n} = \sup_{\alpha_i}\{|u_n|_0 \wedge |v_n|_0|\bar c^{\alpha_i} - c^{\alpha_i}|_0 + |\bar f^{\alpha_i} - f^{\alpha_i}|_0\}$.

Proof. We prove the proposition for $n = 1$. We apply the same methods as in Theorem A.1 of [2]; we set

$$m := \sup_{x,y}\phi(x, y) := \sup_{x,y}\{u_1(x) - v_1(y) - \delta|x - y|^2 - \varepsilon_1(|x|^2 + |y|^2)\}.$$

Let $m = \phi(x_0, y_0)$. Applying the notion of viscosity solution and Ishii's lemma, there exist $X, Y \in \mathcal{S}^N$ such that

$$0 \leq \max\left\{\sup_{\alpha_i}\bar L^{\alpha_i}(y_0, v_1(y_0), p_y, Y); v_1(y_0) - \mathcal{M}u_0(y_0)\right\} - \max\left\{\sup_{\alpha_i}L^{\alpha_i}(x_0, u_1(x_0), p_x, X); u_1(x_0) - \mathcal{M}u_0(x_0)\right\}, \tag{62}$$

where (p_x, p_y, X, Y) satisfy (58) and (59). Using $2\phi(x_0, y_0) \geq \phi(x_0, x_0) + \phi(y_0, y_0)$, we obtain

$$|x_0 - y_0| \leq \frac{L_{u_1} + L_{v_1}}{2}\delta^{-1}. \tag{63}$$

Now we have to study two different cases.

Case 1: $v_1(y_0) - \mathcal{M}u_0(y_0) - (u_1(x_0) - \mathcal{M}u_0(x_0)) \geq 0$. This last inequality implies that $u_1(x_0) - v_1(y_0) \leq L_{u_0}|x_0 - y_0|$, and, using (63), we have $u_1(x_0) - v_1(y_0) \leq L_{u_0}(L_{u_1} + L_{v_1})(2\delta)^{-1}$, which implies

$$m \leq \tfrac{1}{2}(L_{u_1} + L_{v_1})L_{u_0}\delta^{-1}. \tag{64}$$

Case 2: $\sup_{\alpha_i} L^{\alpha_i}(y_0, v_1(y_0), p_y, Y) - \sup_{\alpha_i} L^{\alpha_i}(x_0, u_1(x_0), p_x, X) \geq 0$. This is the standard case, and we use the same computations as in Theorem A.1 of [2], detailing all constants. For the bounds of $-\mathrm{tr}[\bar{a}^{\alpha_i}(y_0)Y - a^{\alpha_i}(x_0)X]$, $(b^{\alpha_i}(x_0)p_x - \bar{b}^{\alpha_i}(y_0)p_y)$, $(\bar{c}^{\alpha_i}(y_0)v_1(y_0) - c^{\alpha_i}(x_0)u_1(x_0))$, $(f^{\alpha_i}(x_0) - \bar{f}^{\alpha_i}(y_0))$, we use the estimates given in Theorem A.1 of [2]. Finally we obtain

$$\begin{aligned} m &\leq 2\delta \sup_{\alpha_i}\{|\bar{\sigma}^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar{b}^{\alpha_i} - b^{\alpha_i}|_0^2\} \\ &\quad + \frac{1}{\delta}\sup_{\alpha_i}\left\{(2[\sigma^{\alpha_i}]_1^2 + 4 + 2[b^{\alpha_i}]_1)\left(\frac{L_{v_1} + L_{u_1}}{2}\right)^2 \right. \\ &\qquad\qquad \left. + \ (|u_1|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1)\left(\frac{L_{v_1} + L_{u_1}}{2}\right)\right\} \\ &\quad + \sup_{\alpha_i}\{|v_1|_0|\bar{c}^{\alpha_i} - c^{\alpha_i}|_0 + |\bar{f}^{\alpha_i} - f^{\alpha_i}|_0\} + \varepsilon_1(1 + |x_0|^2 + |y_0|^2). \end{aligned}$$

If we add the two cases, we have

$$m \leq 2k_1\delta + \frac{k_2}{\delta} + k_3 + \varepsilon_1 k_4,$$

where

- $k_1 = \sup_{\alpha_i}\{|\bar{\sigma}^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar{b}^{\alpha_i} - b^{\alpha_i}|_0^2\}$,
- $k_2 = \sup_{\alpha_i}\{((L_{u_1}+L_{v_1})^2/4)(2[\sigma^{\alpha_i}]_1^2+4+2[b^{\alpha_i}]_1)+((L_{u_1}+L_{v_1})/2)(|u_1|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,
- $k_3 = \sup_{\alpha_i}\{|v_1|_0|\bar{c}^{\alpha_i} - c|_0 + |\bar{f}^{\alpha_i} - f^{\alpha_i}|_0\}$,
- $k_4 = (1 + |x_0|^2 + |y_0|^2)$.

Since $\min_\delta\{2k_1\delta + k_2/\delta\} = \sqrt{2k_1k_2}$, letting ε_1 go to zero, we obtain

$$m \leq \sqrt{2k_1k_2} + k_3.$$

Reversing $|u_1|_0$ and $|v_1|_0$, we also have the symmetric inequality, hence we have the result, with $k_i^{u_1,v_1}$ defined as before. For the general case, we have only to recall that $L_{u_{n-1}} = L_{u_0}$, for all n. □

Proof of Proposition 4.5. We apply the precedent proposition, using that $|\bar{g} - g| \leq [g]_1\varepsilon$, for $g = \sigma, b, c, f$. Then we have the result. □

Consider now the switching systems. We give here an extension of Lemma A.1 of [2].

Proposition B.2. *Let v^n and w^n be solutions of two equations (SS_n), for $n \geq 1$, with coefficients σ, b, c, f and $\bar\sigma, \bar b, \bar c, \bar f$, respectively. Then we have*

$$\sup_{x,i}\{v_i^n(x) - w_i^n(x)\} \leq (2k_1 k_2^{v^n,w^n})^{1/2} + k_3^{v^n,w^n},$$

where

- $k_1 = \sup_{\alpha_i}\{|\bar\sigma^{\alpha_i} - \sigma^{\alpha_i}|_0^2 + |\bar b^{\alpha_i} - b^{\alpha_i}|_0^2\}$,
- $k_2^{v^n,v^n} = \sup_{\alpha_i}\{((L_{v^n} + L_{w^n})^2/4)(2[\sigma^{\alpha_i}]_1^2{+}4{+}2[b^{\alpha_i}]_1){+}((L_{u_n} + L_{v_n})/2)(|v^n|_0\wedge |w^n|_0[c^{\alpha_i}]_1 + [f^{\alpha_i}]_1 + L_{u_0})\}$,
- $k_3^{v^n,w^n} = \sup_{\alpha_i}\{|v^n|_0 \wedge |w^n|_0|\bar c^{\alpha_i} - c^{\alpha_i}|_0 + |\bar f_i^{\alpha} - f^{\alpha_i}|_0\}$.

Proof. We prove the proposition for $n = 1$. We apply the same methods as in Theorem A.1 of [2]; we set

$$m := \sup_{x,y,i} \phi_i(x, y) := \sup_{x,y,i}\{v_i^1(x) - w_i^1(y) - \delta|x-y|^2 - \varepsilon_1(|x|^2 + |y|^2)\}.$$

Let $m = \phi_j(x_0, y_0)$, i.e. (j, x_0, y_0) attains the supremum. Let $A := \{i \in \mathcal{I}, (i, x_0, y_0)$ attains the supremum$\}$. Then, by Lemma A.2 of [3], there exists $i_0 \in A$, such that $w_{i_0}^1(y_0) < \min_{j\neq i_0}\{w_j^1(y_0) + l\}$. Applying the notion of a viscosity solution, and Ishii's lemma, there exist $X, Y \in \mathcal{S}^N$ such that

$$\begin{aligned} 0 \leq{}& \max\{\bar L^{\alpha_{i_0}}(y_0, w_{i_0}^1(y_0), p_y, Y); w_{i_0}^1(y_0) - \mathcal{M}u_0(y_0)\} \\ &- \max\left\{L^{\alpha_{i_0}}(x_0, v_{i_0}^1(x_0), p_x, X); v_{i_0}^1(x_0)\right. \\ &\qquad\left. - \min_{j\neq i_0}\{v_j^1(x_0) + \ell\}; v_{i_0}^1(x_0) - \mathcal{M}u_0(x_0)\right\}, \end{aligned} \tag{65}$$

where p_x, p_y, X and Y satisfy (58) and (59). Continuing as in Proposition B.1, we obtain the result. □

Proof of Lemma 5.8. We apply the precedent theorem, using that $|\bar g - g| \leq [g]_1\varepsilon$, for $g = \sigma, b, c, f$. □

Theorem B.3. *We have that*

$$A_{v^n} \leq \cdots \leq A_{v^2} \leq A_{v^1},$$

$$A_{w^n} \leq \cdots \leq A_{w^2} \leq A_{w^1}.$$

Proof. The form of A_g and L_g, $g = v^i, w^i, i \geq 0$, defined in (61) and (47) respectively, imply the result. □

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