

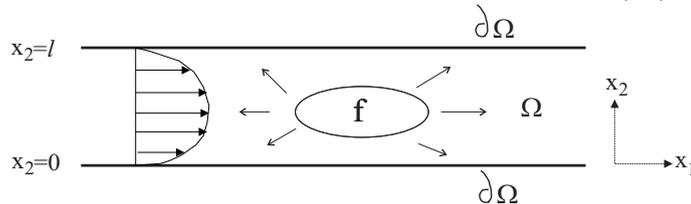
# Finite element simulation of time-harmonic aeroacoustics in a shear flow

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Our aim is to determine the acoustic field radiated by a source, in a fluid in flow, in time-harmonic regime ( $e^{-i\omega t}$ ). Most of the works mentioned in literature deal with Euler's linearized equations: they have been widely used to perform temporal simulations using finite difference methods [1, 2, 3, 4, 5, 6] or discontinuous Galerkin methods [7, 8]. It is more difficult to solve Euler's equations in time harmonic regime: the problem is to find a good treatment of unbounded domains (selection of the outgoing solution). This is satisfactorily done only in the case of a potential flow. Then acoustic perturbations are modeled by a scalar generalized Helmholtz equation. Then if the flow is uniform far from the source, the problem can be solved in a classical way, by coupling finite elements with integral [9] or modal [10] representation of the far-field. Our choice is different and it seems to us more adapted to solve the problem with finite elements in periodic regime. The equation we consider is due to Galbrun [11]. It is a second order system involving the displacement perturbation.

Our method is valid in 3D but in order to simplify the presentation we will present the equations in a two-dimensional problem. An infinite duct  $\Omega = \{-\infty < x < +\infty, 0 < y < h\}$  whose boundaries are rigid is considered. This duct is filled with a compressible fluid in a parallel and subsonic flow, characterized by the Mach number profile,  $-1 < M(x_2) < 1$ .



In time-harmonic regime, the displacement perturbation is supposed to read

$$\Re e \left( \mathbf{u}(x) e^{-i\omega t} \right), \quad \omega > 0$$

and we look for solving a problem of the following form (where  $\mathbf{f}$  is a source supposed to be compactly supported in the duct):

$$\begin{aligned} D^2 \mathbf{u} - \nabla(\operatorname{div} \mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where we have introduced  $D$  the convective derivative operator defined by  $D = -ik + M \partial x_1$  ( $k = \omega/c$  is the wave number,  $c$  the sound speed). The problem must be closed with radiation conditions at infinity in order to select the outgoing solution. This is achieved thanks to the use of the PML method (Perfectly Matched Layers) which allows to deal correctly with the artificial boundary conditions of the numerical simulation domain [12]. To sum up, our aim is to calculate the outgoing solution of (1) using finite elements and PMLs.

In the absence of flow ( $M = 0$ ), it is known that Galbrun's equation can not be discretized thanks to Lagrange finite elements. The solutions to this problem developed in the literature

consist either in using Raviart-Thomas elements,  $H_{div}$ -conforming, or in writing an equivalent regularized formulation of the problem.

With a flow, the situation is more delicate. The variational formulation of Galbrun's equation involves the following bilinear form, whose principal part does not have a fixed sign:

$$\int_{\Omega_b} \operatorname{div} \mathbf{u} \operatorname{div} \bar{\mathbf{v}} - M(x_2)^2 \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \bar{\mathbf{v}}}{\partial x_1} - 2ikM(x_2) \frac{\partial \mathbf{u}}{\partial x_1} \cdot \bar{\mathbf{v}} - k^2 \mathbf{u} \cdot \bar{\mathbf{v}}.$$

As a consequence there is no natural functional framework to formulate the problem, and moreover, we did not succeed in introducing adapted finite elements (similar to the Raviart-Thomas ones for the no-flow case). Such adapted finite elements have been proposed in a mixed approach introducing the pressure as a new unknown [13, 14]. They were proved to lead to stable mixed finite element in several applications. However a mixed formulation does not help to deal satisfactorily with the selection of the outgoing solution. Our approach, based on regularization, solves this last problem. In presence of a flow, we will be able to extend the regularization method, but in a way more delicate to perform [15, 16]. This time the idea is to consider temporarily  $\psi = \operatorname{curl} \mathbf{u}$  as a new unknown (which will be eliminated in the end):

$$\begin{aligned} \mathbf{D}^2 \mathbf{u} - \nabla(\operatorname{div} \mathbf{u}) + \mathbf{curl}(\operatorname{curl} \mathbf{u} - \psi) &= \mathbf{f} \quad (\Omega), \\ \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{u} - \psi &= 0 \quad (\partial\Omega). \end{aligned} \tag{2}$$

This equation can be written:

$$\mathbf{D}^2 \mathbf{u} - \Delta \mathbf{u} = \mathbf{curl} \psi + \mathbf{f}. \tag{3}$$

Then this equation can be discretized thanks to classical Lagrange finite elements. Taking the curl of Galbrun's equation, we get the following relation between  $\psi$  and  $\mathbf{u}$  (where  $M'$  stands for the derivative of the Mach number with respect to  $x_2$  and  $\operatorname{curl} \mathbf{f}$  is taken equal to 0):

$$\mathbf{D}^2 \psi - 2M' \mathbf{D} \left( \frac{\partial u_1}{\partial x_1} \right) = 0 \quad (\Omega). \tag{4}$$

It is then possible to calculate  $\psi$  versus  $\mathbf{u}$  thanks to a convolution formula  $\psi = \mathbf{A}u_1$  along the streamlines and to get a regularized problem versus  $\mathbf{u}$  associated to good mathematical properties [16]. In order to evaluate simply (by interpolation) the convolution formula in  $x_1$ , we use a structured mesh and quadrilateral-based Q2 elements. This formula raises two difficulties:

- the convolution operator  $\mathbf{A}$ , which links all the freedom degrees located on the same streamline ( $x_2 = \text{constant}$ ), leads to a matrix less sparse than a classical finite element matrix [16].
- this formula degenerates when the Mach number vanishes, which happens often in practice (close from the boundaries for instance). Note that this is not a problem for a theoretical point of view (then we can simply set  $\mathbf{A}u_1 = 0$  when  $M = 0$ ). For a practical point of view, the treatment of low Mach values is more delicate, since it requires the evaluation of highly oscillating integrals.

Concerning the first point, since it is not worth eliminating the unknown  $\psi$ , we use the following iterative algorithm initialized with  $\psi^0 = 0$ :

- $\mathbf{u}^{n+1}$  is calculated by inverting finite element matrices:

$$\begin{aligned} \mathbf{D}^2 \mathbf{u}^{n+1} - \nabla(\operatorname{div} \mathbf{u}^{n+1}) + \mathbf{curl}(\operatorname{curl} \mathbf{u}^{n+1} - \psi^n) &= \mathbf{f} \quad (\Omega_L), \\ \mathbf{u}^{n+1} \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{u}^{n+1} - \psi^n &= 0 \quad (\partial\Omega_L) \end{aligned}$$

- $\psi^{n+1}$  is calculated thanks to a finite element matrix product with the vector associated to  $u_1^{n+1}$ :

$$\psi^{n+1} = \mathbf{A}u_1^{n+1} \quad (\Omega_L).$$

We checked that this algorithm converges faster is the flow gradient is weaker, which is not surprising since  $\mathbf{A}$  vanishes for a uniform flow.

Concerning the second point, we have developed an approached model valid for flows whose Mach number is low enough (but not necessarily small in practice). This model is obtained by calculating an equivalent of the oscillating integral  $\mathbf{A}u_1$  when  $M$  tends toward 0. Then we show that we can replace the integral formula with the following differential formula

$$\psi = \frac{2iM'}{k} \frac{\partial u_1}{\partial x_1},$$

whose discretization is much simpler, particularly in the 3D case. This formula can also be deduced from (3) by replacing  $M$  with 0. However we keep Galbrun's equation (2) with  $M \neq 0$ : this means that our approximation consists in considering a weak interaction between acoustics and hydrodynamics for low Mach numbers while keeping a precise description of the acoustic propagation.

To compute the outgoing solution, a PML formulation is considered [12]. Let  $\Omega^L$  be the bounded domain composed of the physical domain  $\Omega_b$  and of surrounding layers  $\Omega_{\pm}^L$  of length  $L$  (see figure 1). The introduction of PML amounts to the transformation of the differential operator

$$\frac{\partial}{\partial x_1} \longrightarrow \alpha(x_1) \frac{\partial}{\partial x_1} \quad (5)$$

in the governing equations of the problem. The complex function  $\alpha$  is assumed to be unity in  $\Omega_b$  and, in order to simplify the subsequent study, constant and equal to the complex scalar  $\alpha^*$ , satisfying the following hypotheses

$$\text{Re}(\alpha^*) > 0, \quad \text{Im}(\alpha^*) < 0, \quad (6)$$

in  $\Omega \setminus \Omega_b$  (see [12] for a more thorough description and justification). More generally, the function  $\alpha$  depends on the variable  $x_1$  in the layers. Note that inverse upstream modes, not damped in

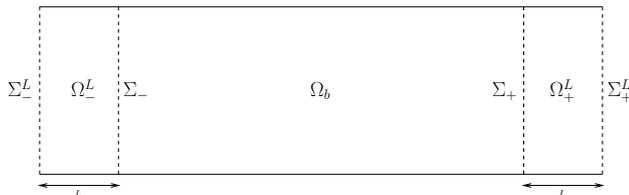


Figure 1: The bounded domain  $\Omega^L$ .

the upstream PML, cause instabilities in the case of temporal PML but not for time-harmonic PML.

Thanks to the introduction of PML, we have showed numerically that the law Mach number approximation (see figure 2 for a parabolic velocity profile  $0.2 \leq M \leq 0.3$  with a source at mid-height) remains valid even for non-small Mach numbers (see figure 3 for the solution with the exact model). In the exact model one can notice the presence of vortices downstream of the source which are not taken into account in the approximated model. However these vorticities do not produce acoustic radiation so that both models produce the same pressure field.

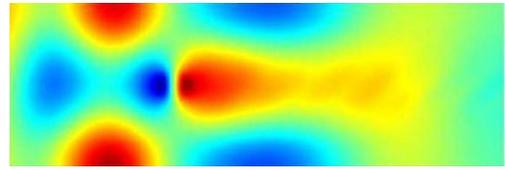
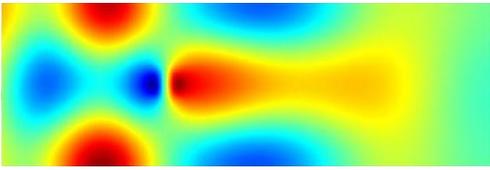


Figure 2: Real part of  $u_1$ , approximated model.

Figure 3: Real part of  $u_1$ , exact model.

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