The method of quasi-reversibility to solve the Cauchy problems for elliptic partial differential equations

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We use the method of quasi-reversibility, first introduced in [1], to solve the ill-posed Cauchy problems for an elliptic operator such as $P = -\Delta$. In particular, a non-conforming method is implemented, some a priori error estimates are derived, and a few numerical computations are presented.

1 Introduction

The objective is to solve an ill-posed Cauchy problem : for $A = \Delta$ or $A = \Delta + k^2$, and given data (g_0, g_1) on $\Gamma_0 \subset \partial\Omega$, find u such that $Au = 0$ in Ω, and both $u|_{\Gamma_0} = g_0$ and $(\partial u/\partial n)|_{\Gamma_0} = g_1$. By using an extension $\tilde{u}$ of data (g_0, g_1) and setting $f = -A\tilde{u}$, the problem is equivalent to find $\hat{u}$ such that $A\hat{u} = f$ in Ω with $\hat{u}|_{\Gamma_0} = 0$ and $(\partial\hat{u}/\partial n)|_{\Gamma_0} = 0$.
It is well-known that this problem is ill-posed : for given f, we have uniqueness but not necessarily existence of u (see Hadamard's example). Hence, this problem needs regularization, precisely : assume there exist u and f such that $Au = f$, with u and f unknown, and that we know some noisy data f^σ, which equals to f up to an error σ : a regularization method provides a quasi-solution which is close to u for small σ. So do the Tikhonov's regularization.

2 Tikhonov's regularization and method of quasi-reversibility

We consider two Hilbert spaces V and H, $A\ : V \to H$ a linear bounded operator which satisfies $\mathrm{Ker}(A) = \{0\}$, $\overline{\mathrm{R}(A)} = H$ but $R(A) \neq H$. If we consider the ill-posed abstract problem : find $u \in V$ such that $Au = f$, the regularized problem in the sense of Tikhonov (with exact data) consists in finding $u_\varepsilon \in V$ such that

$$(Au_\varepsilon, Av)_H + \varepsilon(u_\varepsilon, v)_V = (f, Av)_H,\ \ \forall v \in V.$$

With noisy data, that is $f \to f^\sigma$, we denote $u_\varepsilon \to u_\varepsilon^\sigma$. The Tikhonov's regularization defines a well-posed problem, we have convergence with exact data, that is $\lim_{\varepsilon\to 0} ||u_\varepsilon - u||_V = 0$, and stability with noisy data, that is for $f^\sigma \in H$ with $||f^\sigma - f||_H \leq \sigma$, there exists a regular choice $\varepsilon(\sigma)$, for example $\varepsilon(\sigma) = O(\sigma)$, with $\lim_{\sigma\to 0} ||u^\sigma_{\varepsilon(\sigma)} - u||_V = 0$.
The method of quasi-reversibility is a particular case of Tikhonov's regularization which is obtained by specifying $V = \{v \in H^2(\Omega),\ \ v|_{\Gamma_0} = 0,\ \partial_n v|_{\Gamma_0} = 0\}$, $H = L^2(\Omega)$, and $A = \Delta$ or $\Delta + k^2$.
We have two kinds of objective : the first one is to find an error estimate $err(\sigma, \varepsilon) = ||u - u^\sigma_\varepsilon||_{H^2}$, the second one is to make an *a priori* choice for ε as a function of σ. As far as the first objective is concerned, for a smooth domain, we have the following estimate [2] :

$$\forall\beta \in]0,1[,\ \exists C > 0, \forall\varepsilon > 0, \quad ||u_\varepsilon - u||_{H^1(\Omega)} \leq \frac{C}{(\log\frac{1}{\varepsilon})^\beta}.$$

If $f \to f^\sigma$ (noisy data) and $\varepsilon(\sigma) = O(\sigma)$ (regular choice), the above estimate is true with $u_\varepsilon \to u^\sigma_{\varepsilon(\sigma)}$ and $\varepsilon \to \sigma$.
As far as the second objective is concerned, by using spectral theory we know that for $||f^\sigma - f||_H \leq \sigma < ||f^\sigma||_H$, $\exists!\,\varepsilon(\sigma)$, $\quad ||Au^\sigma_{\varepsilon(\sigma)} - f^\sigma||_H = \sigma$, where u^σ_ε is the Tikhonov solution computed with data f^σ. This choice $\varepsilon(\sigma)$ is the so-called Morozov's choice. We can give an interpretation of Morozov's choice with duality, which is useful for computation. Indeed, we prove with theorem of Fenchel-Rockafellar that if $\varepsilon(\sigma)$ is the Morozov's choice, then

$$u^\sigma_{\varepsilon(\sigma)} = A^* p_\sigma \quad \text{and} \quad \varepsilon(\sigma) = \frac{\sigma}{||p_\sigma||_H}, \quad p_\sigma = \mathrm{argmin}_{p\in H}\Big(\frac{1}{2}||A^*p||^2_V + \sigma||p||_H - (f^\sigma, p)_H\Big).$$

3 Discretization of the quasi-reversibility method

The variational formulation of quasi-reversibility corresponds to a PDE of order 4, which would need conforming finite elements of class C^1. There are at least two other possibilities : use a mixed formulation as in [3], which enables classical

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C^0 finite elements, or use non-conforming finite elements, for example the Morley's finite element [4]. Thus, we now find a solution $u^\sigma_{h,\varepsilon} \in V_h$, where h is the size of the mesh and V_h is the finite dimensional space which discretizes V.
We again have two kinds of objective : the first one is to find an error estimate $err(\sigma, h, \varepsilon) = ||\pi_{V_h} u - u^\sigma_{h,\varepsilon}||_{V_h}$, the second one is to make an *a priori* choice for h and ε as functions of σ.
Based on a $2D$ polygonal domain and an associated regular triangulation $\mathcal{T}_h$, the Morley's finite element relies on the finite dimensional sets $H_h = \{f_h \in L^2(\Omega),\, \forall K \in \mathcal{T}_h,\, f_h|_K \in P_0(K)\}$ and $V_h = \{v_h \in L^2(\Omega),\, \forall K \in \mathcal{T}_h,\, v_h|_K \in P_2(K)$, where v_h is continuous at a_i (0 on $\overline{\Gamma_0}$) and $\partial v_h/\partial n$ continuous at b_i (0 on $\overline{\Gamma_0}$) $\}$. Here, the a_i are the vertices of the triangle, and b_i the middle of the edges. We also define the discretized operator $A_h\ : V_h \to H_h : \forall K \in \mathcal{T}_h, \forall v_h \in V_h,\, A_h v_h|_K = \Delta v_h$.
The discretized formulation of quasi-reversibility consists in finding $u^\sigma_{h,\varepsilon} \in V_h$ such that, if $(u_h, v_h)_{V_h} = \sum_{K \in \mathcal{T}_h} (u_h, v_h)_{H^2(K)}$ and $f^\sigma_h|_K = (1/|K|) \int_K f^\sigma \, dx$,

$$(A_h u^\sigma_{h,\varepsilon}, A_h v_h)_{L^2} + \varepsilon (u^\sigma_{h,\varepsilon}, v_h)_{V_h} = (f^\sigma_h, A_h v_h)_{L^2}, \quad \forall v_h \in V_h.$$

In order to cope with the first objective, we obtain two kinds of error estimates. With the assumptions $||f^\sigma - f||_{H^2} \leq \sigma$ and $(\sqrt{\varepsilon}||u^\sigma_\varepsilon||_{H^4}, ||\Delta u^\sigma_\varepsilon - f^\sigma||_{H^2}) \leq c(\sigma + \sqrt{\varepsilon})$, we obtain a first error estimate with $\lim_{\varepsilon \to 0} \eta_1(\varepsilon) = 0$ (η_1 defined below)

$$err(\sigma, h, \varepsilon) \leq \eta_1(\varepsilon) + c_1\Big(\frac{h}{\sqrt{\varepsilon}} + \frac{\sigma}{\sqrt{\varepsilon}} + \frac{h\sigma}{\varepsilon}\Big), \quad \eta_1(\varepsilon) := ||u - u_\varepsilon||_{H^2}.$$

In order to obtain a second error estimate, we denote by $s(h)$ the smallest non negative singular value of A_h, which satisfies $\lim_{h \to 0} s(h) = 0$. With the assumption $\lim_{h \to 0} \eta_2(h) = 0$ (η_2 defined below), we obtain

$$err(\sigma, h, \varepsilon) \leq \eta_2(h) + c_2\Big(\frac{\varepsilon}{s(h)^2} + \frac{\sigma}{s(h)} + \frac{\varepsilon\sigma}{s(h)^3}\Big), \quad \eta_2(h) := ||P_{\mathrm{Ker} A_h}(\pi_{V_h} u)||_{V_h} + c\frac{h}{s(h)}.$$

In order to cope with the second objective, the Morozov's choice is adapted to the discretized framework : once h is chosen such that $||f^\sigma - f^\sigma_h||_{L^2} \leq \sigma$, ε is uniquely defined by $||A_h u^\sigma_{h,\varepsilon} - f^\sigma_h||_{L^2} = \sigma$, and duality is used in order to compute $u^\sigma_{h,\varepsilon}$ for these Morozov's values $h(\sigma)$ and $\varepsilon(\sigma)$, as in the continuous problem.

4 Numerical example

We consider $A = \Delta$ on the square $\Omega =]0,1[\times]0,1[$, and we try to retrieve the function $u(x,y) = (\sin(\pi x)\sin(\pi y))^2$ with synthetic data $f^\sigma = f + b$, with $f = \Delta u$ and $b(x,y) = 2\sigma \sin(\pi x)\sin(\pi y)$. Γ_0 is $1/4$ of $\partial\Omega$. We compare the computed error surfaces $err^\sigma_{H^2}(h,\varepsilon) = ||\pi_{V_h} u - u^\sigma_{h,\varepsilon}||_{V_h}$ with our two theoretical estimates. In the case $\sigma = 0$, we know that $err^\sigma_{H^2}(h,\varepsilon) \leq \eta_1(\varepsilon) + c_1 h/\sqrt{\varepsilon}$ (first error estimate) with $\eta_1(\varepsilon) \simeq C/(\log 1/\varepsilon)^2$ (found numerically), and $err^\sigma_{H^2}(h,\varepsilon) \leq \eta_2(h) + c_2\varepsilon/s(h)$ (second error estimate) with $\eta_2(\varepsilon) \simeq Ch^{0.96}$ (found numerically). The numerical error surfaces fit reasonably well with the two theoretical estimates. The Morozov's choice turns out to be relevant.

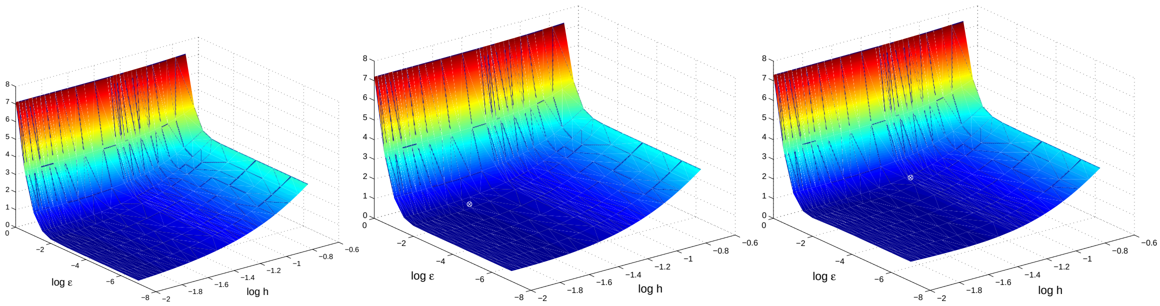

Fig. 1 Error surfaces $err^\sigma_{H^2}(h,\varepsilon)$ with $\sigma = 0, 2.5, 5\%$. The cross corresponds to the error for the Morovov's choice $h(\sigma)$ and $\varepsilon(\sigma)$.

References

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