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Abstract

We present a steering algorithm for general nonholonomic systems which are not required to possess special properties such as flatness or exact nilpotentizability. The method makes use of local steering laws, with suitable contraction properties, designed on the basis of a continuous approximation of the system.

1 Introduction

Nonholonomic systems attract the attention of the scientific community for the theoretical challenges arising from the research on the control of these systems and for their relevance in applications. In particular, the problem of generating feasible trajectories joining two system configurations (referred to as nonholonomic path planning) has been solved for specific classes of driftless systems by effective techniques. These include a Lie-theoretical method for steering nilpotentizable systems [10], open-loop control (e.g., sinusoidal inputs [12]) for chained-form transformable systems and trajectory generation for flat systems [5].

However, there exist nonholonomic robots — also called *general* in this paper — whose kinematic model does not fall into any of the aforementioned classes. For example, mobile robots with more than one trailer cannot be transformed in chained form unless each trailer is hinged to the midpoint of the previous wheel axle — a particular arrangement, very unusual in real trailer vehicles, known as 'on-hooking'. Another such example are robotic systems that perform object manipulation by rolling contacts [13]: even the simplest mechanism in this category, the so-called plate-ball system, does not admit a chained-form transformation. More in general, for 2-input systems, as soon as the dimension of the state space reaches 5, exact nilpotentizability becomes the exception rather than the rule (whereas all systems up to dimension 4 possess this property [11]).

Techniques for steering general nonholonomic systems include the iterative method of [10], the generic loop method of [14] and the continuation method of [17] and [4]. However, the practical applicability of these methods is limited. In fact, the first two essentially require an a priori estimate of some "critical distance" which is generally unknown[1], while the third imposes strong assumptions on the system.

In a preceding work [9], we also proposed a global algorithm for steering *regular* nonholonomic systems, that is, systems without any singular points. The algorithm is an iterative scheme based on uniform nilpotent approximations, and the regularity assumption is essential to obtain the uniformity property. In this paper, we extend this algorithm to general nonholonomic systems with singularities. For such systems, the main difficulty is the construction of a uniform nilpotent approximation, and is dealt with a technique introduced in [18].

The paper is organized as follows. In Section 2 we fix the notation and recall the basic definitions, and in Section 3 we describe the different steps of an approximate steering algorithm. Section 4 is devoted to the construction of a local approximate steering methods based on the use of approximations. The uniform character

[1] In [14], this is "masked" by the fact that an optimization problem is solved at each iteration.

of both the nilpotent approximation and the error estimates is dealt with in Section 5. The above mentioned local methods are used in Section 6 for devising a globally convergent steering algorithm. Finally, Appendix A contains the proof of some technical results of Section 5.

2 Nonholonomic control systems

We recall some basic tools used in sub-Riemannian geometry following [1].

Let Ω be an open connected subset of $\mathbb{R}^n$, and $VF(\Omega)$ the set of C^∞ vector fields on Ω. Let T be a positive real number. Consider a nonholonomic control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i, \qquad x \in \Omega, \tag{1}$$

where $g_1, \dots, g_m$ belong to $VF(\Omega)$ and the input $u(t) = (u_1(t), \dots, u_m(t))$ is an integrable vector function on $[0,T]$ which takes values in $\mathbb{R}^m$. This system is characterized by the m-tuple $g = (g_1, \dots, g_m) \in VF(\Omega)^m$.

Definition 1. The *length* of an input u is defined as

$$\ell(u) = \int_0^T \sqrt{u_1^2(t) + \ldots + u_m^2(t)}\, dt.$$

Given $x_a \in \Omega$, let $x(t, x_a, u)$, $t \in [0,T]$ be a trajectory of (1) originating from x_a under an input function u. We define its *length* as

$$\ell(x(\cdot, x_a, u)) = \ell(u).$$

A point $x = x(t, x_a, u)$, for $t \in [0,T]$, is *accessible* from x_a.

Definition 2. System (1) induces a *sub-Riemannian distance* d on Ω, defined as

$$d(x_1, x_2) = \inf_u \ell(x(\cdot, x_1, u)), \tag{2}$$

where the infimum is taken over all inputs u such that the trajectory $x(\cdot, x_1, u)$ is defined on $[0,T]$ and $x(T, x_1, u) = x_2$.

Note that $d(x_1, x_2) < \infty$ if and only if x_1 and x_2 are accessible from each other. Chow's Theorem states that any two points in Ω are accessible from each other if the elements of the Lie Algebra $\mathcal{L}_g$ generated by the g_i's form an n-dimensional vector space at each point. As system (1) is driftless, Chow's condition implies controllability in any usual sense [15]. Throughout this paper, we assume that system (1) is controllable.

Take $x_a \in \Omega$ and let $L^s(x_a)$ be the vector space generated by the values at x_a of the brackets of the elements of g of length $\leq s$, $s = 1, 2, \dots$ (input vector fields are brackets of length 1). Controllability guarantees that there exists a smallest integer $r = r(x_a)$ such that $\dim L^r(x_a) = n$. This integer is called the *degree of nonholonomy* at x_a.

Definition 3. Let $n_s(x) = \dim L^s(x)$, $s = 1, \dots, r$, the sequence $(n_1(x), \dots, n_r(x))$ is the *growth vector* of g at x.

Point x_a is said to be *regular* if the growth vector remains constant in a neighborhood of x_a; otherwise x_a is *singular*. Points at which the degree of nonholonomy changes are singular. Regular points form an open and dense set in Ω.

Consider a smooth real-valued function f. Call *first-order nonholonomic derivatives* of f the Lie derivatives $g_i f$ of f along g_i, $i = 1, \dots, m$. Call $g_i(g_j f)$, $i, j = 1, \dots, m$, the *second-order nonholonomic derivatives* of f, and so on.

Definition 4. A function f is *of order* $\geq s$ at x_a if its nonholonomic derivatives of order $\leq s-1$ vanish at x_a. If f is of order $\geq s$ and not of order $\geq s+1$ at x_a, it is *of order* s at x_a.

Equivalently, if f is of order $\geq s$ at x_a, then $f(x) = O(d^s(x_a, x))$.

Definition 5. A vector field h is *of order* $\geq q$ at x_a if, for every s and every f of order s at x_a, hf has order $\geq q+s$ at x_a. If h is of order $\geq q$ but not $\geq q+1$, it is *of order* q at x_a.

It is easy to show that every element of g has order ≥ -1, bracket $[g_i, g_j]$, $i, j = 1, \ldots, m$, has order ≥ -2, and so on.

Definition 6. Let the integer w_j, $j = 1, \ldots, n$, be defined by setting $w_j = s$ if $n_{s-1} < j \leq n_s$, with $n_s = n_s(x_a)$ and $n_0 = 0$. Local coordinates $z_1, \ldots, z_n$ centered at x_a form a system of *privileged coordinates* if the order of z_j at x_a equals w_j (called the *weight* of coordinate z_j), for $j = 1, \ldots, n$.

The order of functions and vector fields expressed in privileged coordinates can be computed in an algebraic way:

- The order of the monomial $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ is equal to its weighted degree $w(\alpha) = w_1\alpha_1 + \cdots + w_n\alpha_n$.
- The order of a function $f(z)$ at $z = 0$ (the image of x_a) is the least weighted degree of the monomials actually appearing in the Taylor expansion of f at 0.
- The order of a vector field $h(z) = \sum_{j=1}^n h_j(z)\partial_{z_j}$ at $z = 0$ is the least weighted degree of the monomials actually appearing in the Taylor expansion of h at 0:

$$h(z) \sim \sum_{\alpha,j} a_{\alpha,j} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j},$$

  considering the term $a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$ as a monomial and assigning to ∂_{z_j} the weight $-w_j$.

Definition 7. Given the system $z_1, \ldots, z_n$ of privileged coordinates at x_a, the function

$$\|z\|_{x_a} = |z_1|^{1/w_1} + \cdots + |z_n|^{1/w_n},$$

where $w_1, \ldots, w_n$ are the coordinate weights at x_a, is called *pseudonorm* at x_a.

Denote by $B(x, R)$ the open sub-Riemannian ball of radius R centered at x.

Definition 8. A *continuously varying system of privileged coordinates on* Ω is a mapping Φ, with values in $\mathbb{R}^n$, defined and continuous on a neighborhood of the diagonal in $\Omega \times \Omega$, and such that the partial mapping $z = \Phi(x_a, \cdot)$ is a system of privileged coordinates at x_a. In this case, there exists a continuous function $\rho : \Omega \to (0, +\infty)$ such that the coordinates $\Phi(x_a, \cdot)$ are defined on $B(x_a, \rho(x_a))$; we call ρ an *injectivity radius* of Φ.

Privileged coordinates provide an estimate of the sub-Riemannian distance d, according to the following result.

Theorem 1 (Ball-Box Theorem)**.** *Consider $g \in VF(\Omega)^m$, a point $x_a \in \Omega$ and a system of privileged coordinates z at x_a. There exist positive constants $C_d(x_a)$ and $\varepsilon_d(x_a)$ such that, for all x with $d(x_a, x) < \varepsilon_d(x_a)$,*

$$\frac{1}{C_d(x_a)} \|z(x)\|_{x_a} \leq d(x_a, x) \leq C_d(x_a) \|z(x)\|_{x_a}. \tag{3}$$

If Ω contains only regular points and if Φ is a continuously varying system of privileged coordinates on Ω, then there exist continuous positive functions $C_d(\cdot)$ and $\varepsilon_d(\cdot)$ on Ω such that inequality (3) holds with $z = \Phi(x_a, \cdot)$ at all (x, x_a) satisfying $d(x, x_a) < \varepsilon_d(x_a)$.

3 Steering by approximations

Consider a nonholonomic control system $\dot{x} = \sum_{i=1}^m g_i(x)\, u_i$ on Ω. The path planning problem is: given two configurations $x_0, x_1 \in \Omega$, find an input u steering the system from x_0 to x_1.

The principle of an approximate steering algorithm is the following. Solve first the path planning problem for a control problem "approximating" the original one (in a meaning to be specified); apply then the resulting input to the original system from x_0; iterate this procedure from the current point.

We will now define precisely the different steps of such an algorithm, and give properties ensuring its convergence.

3.1 Local approximate steering methods

Let $g = (g_1, \ldots, g_m) \in VF(\Omega)^m$.

Definition 9. A m-tuple $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_m)$ defined on a neighborhood of x_a is a *first-order approximation of g at x_a* if the vector fields $g_i - \hat{g}_i$, $i = 1, \ldots, m$, are of order ≥ 0 at x_a. A *first-order approximation of g on Ω* is a mapping A that associates to each $x_a \in \Omega$ a first-order approximation $\hat{g} = A(x_a)$ of g at x_a defined on a ball $B(x_a, \rho(x_a))$. The function $\rho : \Omega \to (0, +\infty)$ is called the *approximation radius* of A.

Since first-order approximations are always used in this paper, they are referred to simply as 'approximations'. Useful properties of approximations are continuity and nilpotency.

Definition 10. Let $A : x_a \mapsto \hat{g}$ be an approximation on Ω.

- We say that A is *continuous* if the mapping
$$(x_a, x) \mapsto \hat{g}(x) \in \mathbb{R}^n,$$
is defined and continuous on a neighborhood of the diagonal in $\Omega \times \Omega$, and the approximation radius ρ of A is continuous.
- We say that A is *nilpotent of step $s \in \mathbb{N}$* if, for all $x_a \in \Omega$, the Lie algebra generated by $\hat{g}$ is nilpotent of step s.

Privileged coordinates allow to measure the error obtained when we replace g by a first-order approximation (see [1, Prop. 7.29]).

Lemma 2. *Consider a point $x_a \in \Omega$, a system of privileged coordinates z at x_a, and a first-order approximation $\hat{g}$ of g at x_a. Then, there exist positive constants $C_e(x_a)$ and $\varepsilon_e(x_a)$ such that, for all $x \in \Omega$ with $d(x_a, x) < \varepsilon_e(x_a)$ and all integrable control functions $u(\cdot)$ with $\ell(u) < \varepsilon_e(x_a)$, we have*

$$\|z(x(T, x, u)) - z(\hat{x}(T, x, u))\|_{x_a} \leq C_e(x_a) \max\left(\|z(x)\|_{x_a}, \ell(u)\right) \ \ell(u)^{1/r}, \tag{4}$$

where r is the degree of nonholonomy at x_a and $x(\cdot, x, u)$ and $\hat{x}(\cdot, x, u)$ are the trajectories of $\dot{x} = \sum_{i=1}^m g_i(x)\, u_i$ and $\dot{x} = \sum_{i=1}^m \hat{g}_i(x)\, u_i$ respectively.

If Ω contains only regular points, Φ is a continuously varying system of privileged coordinates on Ω and A a continuous approximation on Ω, then there exist continuous positive functions $C_e(\cdot)$ and $\varepsilon_e(\cdot)$ such that inequality (4) holds, with $z = \Phi(x_a, \cdot)$ and $\hat{g} = A(\overline{x})$, for all (x, x_a) with $d(x, x_a) < \varepsilon_e(x_a)$ and all integrable control functions $u(\cdot)$ with $\ell(u) < \varepsilon_e(x_a)$.

We also need to define precisely the notion of steering law for an approximation.

Definition 11. Let $A : x_a \mapsto \hat{g}$ be an approximation on Ω and ρ its approximation radius. A *steering law* of A is a mapping which, to every pair $x, x_a \in \Omega$ satisfying $d(x_a, x) < \rho(x_a)$, associates an integrable control function $\hat{u}(t)$, $t \in [0, T]$ (henceforth called a *steering control*) such that the trajectory $\hat{x}(\cdot, x, \hat{u})$ is defined on $[0, T]$ and satisfies $\hat{x}(T, x, \hat{u}) = x_a$. In other terms, $\hat{u}(\cdot)$ steers $A(x_a)$ from x to x_a.

For example, a systematic design of the steering law is possible when nilpotent approximations are used [10].

Given g, an approximation A of g, and a steering law for A, we define a local approximate steering method for g as follow.

Definition 12. Fix $x_a \in \Omega$. For a point $x \in B(x_a, \rho(x_a))$, let $\hat{u}(\cdot)$ be the steering control of $A(x_a)$ between x and x_a. The *local approximate steering* (LAS) method associated to A and its steering law is the function defined by:

$$\mathrm{AppSteer}(x, x_a) = x(T, x, \hat{u}).$$

Definition 13. A LAS method is *contractive* if, for any $x_a \in \Omega$, there exist a positive constant $\mu(x_a)$ such that, for any x sufficiently close to x_a, $d(x_a, x) < \mu(x_a)$ implies

$$d(x_a, \mathrm{AppSteer}(x, x_a)) \leq d(x_a, x)^{1+\beta},$$

where β is a positive constant independent of x_a. A LAS method is *uniformly contractive* on a set $K \subset \Omega$ if it is contractive and if $\mu(\cdot)$ has a constant positive value μ_K on K.

3.2 Toward convergent local and global algorithms

Assume first that we have a contractive LAS method AppSteer. We can easily build a locally convergent approximate steering algorithm, as follows. Let e be a given tolerance.

Local_Approximate_Steering(x_0, x_1)

1. $k := 0$;
2. $x^k := x_0$;
3. while $d(x^k, x_1) > e$
4. $\quad x^{k+1} = \text{AppSteer}(x^k, x_1)$;
5. $\quad k := k + 1$;

This algorithm converges if $d(x_0, x_1) < \mu(x_1)$.

Assume now that we have a uniformly contractive LAS method AppSteer on a set $K \subset \Omega$. Based on the local algorithm, the construction of a global approximate steering algorithm on K is inspired to the following idea[2]. Consider a parameterized path[3] $\gamma \subset K$ connecting x_0 to x_1, and choose a finite sequence of intermediate goals $\{x_0^d = x_0, x_1^d, \ldots, x_n^d = x_1\}$ on γ, such that $d(x_{i-1}^d, x_i^d) < \mu_K/2$, $i = 0, \ldots, n$. It is possible to prove that the iterated application of the uniformly contractive LAS method AppSteer(x^{i-1}, x_i^d) from the current state to the next subgoal (having set $x_i^d = x_1$, $\forall i \geq n$) yields a sequence x^i which converges to x_1.

To turn the above idea into a real algorithm, three issues are still remaining:

- we have to explain how to construct a uniformly contractive LAS method;
- the sub-Riemannian distance d is used in both local and global algorithm. This distance is in general not computable, so we need estimates of d, like the one of Theorem 1. However the latter estimate is not usable here because it does not hold uniformly: the radius of validity $\varepsilon_d(x_a)$ of the estimate tends to zero when x_a tends to singular points [7]. So we need computable uniform estimates of the sub-Riemannian distance d;
- the a priori knowledge of μ_K is required, but in practice is not available. The algorithm should achieve global convergence without knowing μ_K.

These three issues will be addressed in Sect. 4, 5, and 6 respectively. We will restrict ourselves to a particular – but large – class of systems, that we describe now.

3.3 A class of generic systems

Consider two integers m and n, $2 \leq m < n$, and an open connected subset Ω of $\mathbb{R}^n$. For $j \geq 1$, denote by $\widetilde{n}_j$ the dimension of the linear space generated by all words of length not greater than j in the free Lie algebra with m generators. Recall that $\widetilde{n}_1 = m$ and, for $j > 1$,

$$\widetilde{n}_j - \widetilde{n}_{j-1} = \frac{1}{j} \sum_{d|j} \mu(d) m^{j/d},$$

where μ is the Moebius function (see [3]). Denote also $r = r(m,n)$ the integer such that $\widetilde{n}_{r-1} < n \leq \widetilde{n}_r$.

Definition 14. The class $\Lambda_{gen}(\Omega)$ is the set of m-tuple $g \in VF(\Omega)^m$ which growth vector is equal to:

(i) $(\widetilde{n}_1, \ldots, \widetilde{n}_{r-1}, n-1, n)$ on a subset of codimension one or greater than one,

(ii) $(\widetilde{n}_1, \ldots, \widetilde{n}_{r-1}, n)$ elsewhere.

[2] A similar idea is proposed in [10].

[3] If no such path exists, K is not arc-connected and the steering problem has no solution in K.

The class $\Lambda_{reg}(\Omega)$ is the set of m-tuple $g \in VF(\Omega)^m$ which growth vector is constant, that is the class of *regular m-tuples.*

Notice that, for a m-tuple $g \in \Lambda_{gen}(\Omega)$, the case (i) corresponds to singular points and the case (ii) to regular ones. On the other hand, a m-tuple $g \in \Lambda_{reg}(\Omega)$ has no singular points.

The class of control systems (1) associated to m-tuples in $\Lambda_{gen}(\Omega) \cup \Lambda_{reg}(\Omega)$ is wide enough for most of the applications in robotics. It first contains the regular systems, i.e. systems associated to regular m-tuples, which frequently occur (for instance flat systems, left-invariant systems on Lie groups, etc). Second, a generic control system (1) can be considered as associated to a $g \in \Lambda_{gen}(\Omega)$. Indeed, the growth vector of a generic element of $VF(\Omega)^m$ is characterized by the following result of [19].

Lemma 3. *For every g in an open and dense subset of $VF(\Omega)^m$ endowed with the Whitney C^∞ topology, there exist a set $\Omega_1 \subset \Omega$ of codimension greater than one such that g belongs to $\Lambda_{gen}(\Omega \backslash \Omega_1)$.*

Notice that $\Omega\backslash\Omega_1$ is open and connected in $\mathbb{R}^n$. For motion planning purposes, we can ignore Ω_1 and work on $\Omega\backslash\Omega_1$. But we must take account of the set Ω_2 of codimension one where the growth vector is equal to $(\widetilde{n}_1, \ldots, \widetilde{n}_{r-1}, n-1, n)$, since $\Omega\backslash\Omega_2$ may be non connected.

The case of a regular m-tuple g has already been settled in a previous article [9] (it can also be treated as a simple adaptation of the present work). We will then focus on the class $\Lambda_{gen}(\Omega)$.

4 A local approximate steering method

From now on, we fix an open connected subset Ω of $\mathbb{R}^n$ and a m-tuple $g = (g_1, \ldots, g_m)$ in $\Lambda_{gen}(\Omega)$.

4.1 Regular and singular domains

Denote by $\mathcal{L}(1, \ldots, m)$ the free Lie algebra generated by the alphabet $1, \ldots, m$, and by $\mathcal{L}^s(1, \ldots, m)$, $s \geq 1$, the subset of words of length not greater than s. Denote also by $|I|$ the length of a word $I \in \mathcal{L}(1, \ldots, m)$. Choose a P. Hall basis $\mathcal{H}$ of $\mathcal{L}(1, \ldots, m)$; in particular, for any $s \geq 1$, $\mathcal{H}$ contains exactly $\widetilde{n}_s$ words of length not greater than s, which form a basis of $\mathcal{L}^s(1, \ldots, m)$ (by definition, $\widetilde{n}_s = \dim \mathcal{L}^s(1, \ldots, m)$).

Let $I_1, \ldots, I_{\widetilde{n}_{r-1}}$ be all the words of length smaller than r in $\mathcal{H}$. Among all the n-tuples $(I_1, \ldots, I_{\widetilde{n}_{r-1}}, I_{\widetilde{n}_{r-1}+1}, \ldots, I_n)$ of words in $\mathcal{H}$, call $\mathcal{J}_1^R, \ldots, \mathcal{J}_N^R$ the ones with $|I_{\widetilde{n}_{r-1}+1}| = \cdots = |I_n| = r$, and $\mathcal{J}_1^S, \ldots, \mathcal{J}_{N'}^S$ the ones with $|I_{\widetilde{n}_{r-1}+1}| = \cdots = |I_{n-1}| = r$, $|I_n| = r+1$.

Let E_g be the *evaluation map* which assigns to each $I \in \mathcal{L}(1, \ldots, m)$ the vector field $E_g(I) = g_I$ obtained by plugging in the g_i, $i = 1, \ldots, m$, for the corresponding letter i. For a n-tuple $\mathcal{J}$ of words in $\mathcal{L}(1, \ldots, m)$, we define $\mathcal{V}_\mathcal{J}$ as the set of points $x \in \Omega$ such that $\operatorname{rank}\{g_I(x) \ : \ I \in \mathcal{J}\} = n$. Each $\mathcal{V}_\mathcal{J}$ is an open set, eventually empty. By definition of $\Lambda_{gen}(\Omega)$, we have:

$$\text{Regular set of } g = \bigcup_{i=1}^{N} \mathcal{V}_{\mathcal{J}_i^R}, \qquad \text{Singular set of } g \subset \bigcup_{i=1}^{N'} \mathcal{V}_{\mathcal{J}_i^S},$$

$$\Omega = \bigcup_{i=1}^{N} \mathcal{V}_{\mathcal{J}_i^R} \cup \bigcup_{i=1}^{N'} \mathcal{V}_{\mathcal{J}_i^S}.$$

We call $\mathcal{V}_{\mathcal{J}_1^R}, \ldots, \mathcal{V}_{\mathcal{J}_N^R}$ the *regular domains*, and $\mathcal{V}_{\mathcal{J}_1^S}, \ldots, \mathcal{V}_{\mathcal{J}_{N'}^S}$ the *singular domains*.

4.2 Continuous approximation on a domain $\mathcal{V}_\mathcal{J}$

Consider a n-tuples of words $\mathcal{J} = (I_1, \ldots, I_n)$ equal to one of the $\mathcal{J}_i^R$'s or to one of the $\mathcal{J}_j^S$'s.

Let x_a be a point in $\mathcal{V}_\mathcal{J}$ and $n_s = \dim L^s(x_a)$, $s = 1, \ldots, r+1$. By construction, $n_s = \widetilde{n}_s$ for $s = 1, \ldots, r-1$, $n_{r+1} = n$, and n_r is equal to either n if x_a is regular, or $n-1$ if x_a is singular. The vector fields $g_{I_1}, \ldots, g_{I_n}$ have then the following property:

$$g_{I_1}(x), \ldots, g_{I_{n_s}}(x) \text{ is a basis of } L^s(x), \quad s = 1, \ldots, r+1,$$

for any x in a neighborhood of x_a (vector fields having this property are called an *adapted frame at* x_a, see [1] or [8, p. 14]).

Define local coordinates z at x_a as follows:

- compute local coordinates $y = (y_1, \dots, y_n)$ as

$$y = \Gamma^{-1}(x - x_a),$$

where Γ is the $n \times n$ matrix whose elements Γ_{ij} are defined by $g_{I_j}(x_a) = \sum_{i=1}^n \Gamma_{ij} \partial_{x_i|x_a}$.

- build $z = (z_1, \dots, z_n)$ via the recursive formula ($j = 1, \dots, n$)

$$z_j = y_j + \sum_{k=2}^{|I_j|-1} h_k(y_1, \dots, y_{j-1}), \tag{5}$$

where, for $k = 2, \dots, |I_j| - 1$,

$$h_k(y_1, \dots, y_{j-1}) = - \sum_{\substack{|\alpha|=k \\ w(\alpha)<|I_j|}} g_{I_1}^{\alpha_1} \dots g_{I_{j-1}}^{\alpha_{j-1}} (y_j + \sum_{q=2}^{k-1} h_q)(x_a) \prod_{i=1}^{j-1} \frac{y_i^{\alpha_i}}{\alpha_i!},$$

with $|\alpha| = \sum_{i=1}^n \alpha_i$ and $w(\alpha) = \alpha_1 |I_1| + \dots + \alpha_n |I_n|$.

It results from [1] (see also [18]) that the mapping $\Phi_{\mathcal{J}} : (x_a, x) \mapsto z$ is a continuously varying system of privileged coordinates on $\mathcal{V}_{\mathcal{J}}$.

Notice that the polynomial change of coordinates (5) from y to z has a triangular structure. This implies that the inverse change of coordinates from z to y has exactly the same form. As a consequence, the mapping $z = \Phi_{\mathcal{J}}(x_a, \cdot)$ is defined on the whole $\mathbb{R}^n$, i.e. $\Phi_{\mathcal{J}}$ has an infinite injectivity radius.

We define a *weighted degree* relative to $\mathcal{J}$ as follows: a monomial function $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is of weighted degree $w(\alpha) = \alpha_1 |I_1| + \dots + \alpha_n |I_n|$, and a monomial vector field $z_1^{\alpha_1} \cdots z_n^{\alpha_n} \partial_{z_i}$ on $\mathbb{R}^n$ is of weighted degree $w(\alpha) - |I_i|$.

Fix now $x_a \in \mathcal{V}_{\mathcal{J}}$. We use z to denote the system of privileged coordinates $\Phi_{\mathcal{J}}(x_a, \cdot)$ at x_a, and, with a little abuse of notations, $g_i(z)$ to denote $z_* g_i$. For $i = 1, \dots, m$, we expand $g_i(z)$ in Taylor series at 0 as:

$$g_i(z) = g_i^{(-)} + g_i^{(0)},$$

where $g_i^{(-)}$ contains all the monomial vector fields of negative weighted degree. We denote then by $\hat{g}_i$ the vector field $z^* g_i^{(-)}$ on Ω.

Proposition 4. *The mapping $A_{\mathcal{J}} : x_a \mapsto \hat{g} = (\hat{g}_1, \dots, \hat{g}_m)$ is a continuous approximation of g on $\mathcal{V}_{\mathcal{J}}$. Moreover, it is nilpotent of step $r + 1$.*

Definition 15. We call $A_{\mathcal{J}}$ the *non-homogeneous nilpotent approximation* on $\mathcal{V}_{\mathcal{J}}$.

Proof of Proposition 4. The continuity is clear, it results from the one of $\Phi_{\mathcal{J}}$. Moreover, if x_a is a regular point and $\mathcal{J}$ is one of the $\mathcal{J}_i^R$'s or if x_a is a singular point and $\mathcal{J}$ is one of the $\mathcal{J}_i^S$'s, then the weights $w_1 = |I_1|, \dots, w_n = |I_n|$ are equal to the nonholonomic orders at x_a of the privileged coordinates z. In these cases, $\hat{g}$ is the usual homogeneous nilpotent approximation at x_a in the coordinates z (see [1]) and the conclusion follows.

Assume now that x_a is a regular point and $\mathcal{J}$ is one of the $\mathcal{J}_i^S$'s. The weights $w_1 = |I_1|, \dots, w_{n-1} = |I_{n-1}|$ are equal to the nonholonomic orders at x_a of the privileged coordinates $z_1, \dots, z_{n-1}$, but $w_n = r + 1$ is greater than the nonholonomic order of z_n, which is equal to r.

To prove that $\hat{g}$ is an approximation of g at x_a, it suffices to show that, in the Taylor expansion of $g_i(z)$, all monomial vector fields of negative nonholonomic order belongs to $g_i^{(-)}$. Equivalently, it suffices to show that a monomial vector field with negative nonholonomic order is of negative weighted degree. Now the nonholonomic order of a monomial vector field $z_1^{\alpha_1} \cdots z_n^{\alpha_n} \partial_{z_i}$ is equal to

- $\alpha_1 |I_1| + \dots + \alpha_{n-1} |I_{n-1}| + \alpha_n r - |I_i|$, i.e. its weighted degree minus α_n, if $i \neq n$;

- $\alpha_1|I_1| + \cdots + \alpha_{n-1}|I_{n-1}| + (\alpha_n - 1)r$, i.e. its weighted degree minus $(\alpha_n - 1)$, if $i = n$.

Therefore, when the nonholonomic order is negative, we have $\alpha_n = 0$, and so the weighted degree is negative too.

Finally, the nilpotency is a consequence of the definition. Indeed, in z coordinates, consider two monomial vector fields g and g' of negative weighted degree, respectively $-s$ and $-s'$. Then the Lie bracket $[g, g']$ is a sum of terms of weighted degree $-(s + s')$. Thus $g_I^{(-)}$ is a sum of terms of weighted degree not greater than $-|I|$. Since a weighted degree can not be smaller than $-(r + 1)$, $g_I^{(-)} = 0$ if $|I| > r + 1$. □

4.3 Construction of AppSteer

Recall that a local approximate steering method is defined by the given of an approximation A of g, and a steering law for A. To obtain contractive LAS method we also need some conditions on the steering law of the approximation.

Definition 16. Let A be an approximation on Ω and $\hat{d}_{x_a}$ the sub-Riemannian distance associated to $A(x_a)$. We say that a steering law of A is *quasi-optimal* if there exists a constant $C_\ell > 0$ and a continuous positive function $\varepsilon_\ell(\cdot)$ such that, for any $x_a, x \in \Omega$ with $d(x_a, x) < \varepsilon_\ell(x_a)$, the control $\hat{u}(\cdot)$ steering $\hat{g} = A(x_a)$ from x to x_a satisfies:

$$\ell(\hat{u}) \leq C_\ell\, \hat{d}_{x_a}(x, x_a) = C_\ell\, \hat{d}_{x_a}(\hat{x}(0, x, \hat{u}), \hat{x}(T, x, \hat{u})).$$

Note that, due to the definition of the sub-Riemannian distance $\hat{d}_{x_a}$, quasi-optimal steering laws always exist. Furthermore, if A is a nilpotent approximation, one can verify that the steering laws proposed in [10] are quasi-optimal (use the characterizations (8) and (18)) .

Consider one of the n-tuple $\mathcal{J} = (I_1, \ldots, I_n)$ in $\{\mathcal{J}_1^R, \ldots, \mathcal{J}_N^R, \mathcal{J}_1^S, \ldots, \mathcal{J}_{N'}^S\}$, and choose a steering law of $A_\mathcal{J}$. Since $A_\mathcal{J}$ is nilpotent, we can choose for instance the steering law proposed in [10]. We define $\text{AppSteer}_\mathcal{J}$ as the LAS method on $\mathcal{V}_\mathcal{J}$ associated to $A_\mathcal{J}$ and its steering law.

For $x \in \Omega$, set $\det_\mathcal{J}(x) = \det(g_{I_1}(x), \ldots, g_{I_n}(x))$. By definition, x belongs to $\mathcal{V}_\mathcal{J}$ if and only if $\det_\mathcal{J}(x) \neq 0$.

Definition 17. The LAS method AppSteer on Ω is defined as

$$\text{AppSteer}(x, x_a) = \text{AppSteer}_\mathcal{J}(x, x_a),$$

where $\mathcal{J}$ is the first n-tuple (for the lexicographic order) in $\{\mathcal{J}_1^R, \ldots, \mathcal{J}_N^R, \mathcal{J}_1^S, \ldots, \mathcal{J}_{N'}^S\}$ satisfying

$$\det_\mathcal{J}(x_a) = \max(\det_{\mathcal{J}_1^R}(x_a), \ldots, \det_{\mathcal{J}_N^R}(x_a), \det_{\mathcal{J}_1^S}(x_a), \ldots, \det_{\mathcal{J}_{N'}^S}(x_a)).$$

5 Uniformity properties

In this section we will provide a uniform estimate for the sub-Riemannian distance and prove that AppSteer is uniformly contractive. We first distinguish regular domains from singular ones and show that the corresponding LAS methods $\text{AppSteer}_\mathcal{J}$ are uniformly contractive.

5.1 Regular domains

Let $\mathcal{J} = (I_1, \ldots, I_n)$ be one of the $\mathcal{J}_i^R$'s. Using Theorem 1 and Lemma 2, there exist continuous positive functions $C_\mathcal{J}(\cdot)$ and $\varepsilon_\mathcal{J}(\cdot)$ on the regular domain $\mathcal{V}_\mathcal{J}$ such that, for all x_a, x in $\mathcal{V}_\mathcal{J}$ satisfying $d(x_a, x) < \varepsilon_\mathcal{J}(x_a)$,

$$\frac{1}{C_\mathcal{J}(x_a)}\, \|z(x)\|_{x_a} \leq d(x_a, x) \leq C_\mathcal{J}(x_a)\, \|z(x)\|_{x_a} \tag{6}$$

and, for all integrable control functions $u(\cdot)$ with $\ell(u) < \varepsilon_\mathcal{J}(x_a)$,

$$\|z(x(T, x, u)) - z(\hat{x}(T, x, u))\|_{x_a} \leq C_\mathcal{J}(x_a) \max\left(\|z(x)\|_{x_a}, \ell(u)\right)\, \ell(u)^{1/r}, \tag{7}$$

where $z = \Phi_\mathcal{J}(x_a, \cdot)$, $\hat{g} = A_\mathcal{J}(x_a)$, and $x(\cdot, x, u)$, $\hat{x}(\cdot, x, u)$ are the trajectories of $\dot{x} = \sum_{i=1}^m g_i(x)\, u_i$ and $\dot{x} = \sum_{i=1}^m \hat{g}_i(x)\, u_i$ respectively.

Remark. Note that, for any $x_a \in \mathcal{V}_{\mathcal{J}}$, the coordinates $\Phi_{\mathcal{J}}(x_a, \cdot)$ are also privileged coordinates at x_a of $\hat{g} = A_{\mathcal{J}}(x_a)$. Since both $\Phi_{\mathcal{J}}(x_a, \cdot)$ and $\hat{g} = A_{\mathcal{J}}(x_a)$ vary continuously with x_a, we obtain the following uniform estimate of $\hat{d}_{x_a}$ (the sub-Riemannian distance associated to $A_{\mathcal{J}}(x_a)$), up to reducing $\varepsilon_{\mathcal{J}}(\cdot)$ and increasing $C_{\mathcal{J}}(\cdot)$: for all x_a, x in $\mathcal{V}_{\mathcal{J}}$ satisfying $d(x_a, x) < \varepsilon_{\mathcal{J}}(x_a)$,

$$\frac{1}{C_{\mathcal{J}}(x_a)} \|z(x)\|_{x_a} \leq \hat{d}_{x_a}(x_a, x) \leq C_{\mathcal{J}}(x_a) \|z(x)\|_{x_a}, \quad \text{with } z = \Phi_{\mathcal{J}}(x_a, \cdot).$$

As a consequence, a steering law of $A_{\mathcal{J}}$ is quasi-optimal if and only if there exist continuous positive functions $C_\ell(\cdot)$ and $\varepsilon_\ell(\cdot)$ on $\mathcal{V}_{\mathcal{J}}$ such that, for any $x_a, x \in \mathcal{V}_{\mathcal{J}}$ with $d(x_a, x) < \varepsilon_\ell(x_a)$, the control $\hat{u}(\cdot)$ steering $\hat{g} = A_{\mathcal{J}}(x_a)$ from x to x_a satisfies:

$$\ell(\hat{u}) \leq C_\ell(x_a) \|z(x)\|_{x_a}, \quad \text{with } z = \Phi_{\mathcal{J}}(x_a, \cdot). \tag{8}$$

This characterization is useful in particular to check that a given steering law is quasi-optimal.

The property of uniform contraction follows for $\mathrm{AppSteer}_{\mathcal{J}}$.

Proposition 5. *Let $\mathcal{V}_{\mathcal{J}}$ be a regular domain. Then the LAS method* $\mathrm{AppSteer}_{\mathcal{J}}$ *is uniformly contractive on every compact subset of $\mathcal{V}_{\mathcal{J}}$.*

Proof. Given $x_a \in \Omega$, we set $z = \Phi_{\mathcal{J}}(x_a, \cdot)$ and $\hat{g} = A_{\mathcal{J}}(x_a)$. For $x \in B(x_a, \rho)$, let $\hat{u}(\cdot)$ be the steering control of $\hat{g}$ between x and x_a, i.e. $x(T, x, \hat{u}) = \mathrm{AppSteer}_{\mathcal{J}}(x, x_a)$. Let us apply (7) to the input $\hat{u}$. By construction $\hat{x}(T, x, \hat{u}) = x_a$, and $z(x_a) = 0$; therefore, if $d(x_a, x)$ and $\ell(\hat{u})$ are smaller than $\varepsilon_{\mathcal{J}}(x_a)$, then

$$\|z(x(T, x, \hat{u}))\|_{x_a} \leq C_{\mathcal{J}}(x_a) \max\left(\|z(x)\|_{x_a}, \ell(\hat{u})\right) \ell(\hat{u})^{1/r}.$$

Now, the steering law of $A_{\mathcal{J}}$ is quasi-optimal and thus satisfies (8). Using then (6) two times, we obtain

$$\frac{1}{C_{\mathcal{J}}(x_a)} d(x_a, x(T, x, \hat{u})) \leq C_{\mathcal{J}}(x_a) C_\ell(x_a)^{1+1/r} \|z(x)\|_{x_a} \leq C_{\mathcal{J}}(x_a)^{2+1/r} C_\ell(x_a)^{1+1/r} d(x_a, x)^{1+1/r},$$

provided that $d(x_a, x)$ is smaller than $\varepsilon_{\mathcal{J}}(x_a)$ (which is assumed smaller than $\varepsilon_\ell(x_a)$).

Let K be a compact subset of $\mathcal{V}_{\mathcal{J}}$. Since the functions $\varepsilon_{\mathcal{J}}$, $C_{\mathcal{J}}$, and C_ℓ are all continuous, there exists a positive constant μ_K such that, for any $x_a \in K$ and $x \in \mathcal{V}_{\mathcal{J}}$, $d(x_a, x) < \mu_K$ implies

$$d(x_a, x(T, x, \hat{u})) = d(x_a, \mathrm{AppSteer}_{\mathcal{J}}(x, x_a)) \leq d(x_a, x)^{1+\frac{1}{2r}}.$$

Thus the LAS method $\mathrm{AppSteer}_{\mathcal{J}}$ is uniformly contractive on K. $\square$

5.2 Singular domains

Let now $\mathcal{J} = (I_1, \dots, I_n)$ be one of the $\mathcal{J}_i^S$'s. On the singular domain $\mathcal{V}_{\mathcal{J}}$ the weights $w_1, \dots, w_n$ defining the pseudo-norm $\|\cdot\|_{x_a}$ are not the same at singular points as at regular ones; thus the function

$$(x_a, x) \mapsto \|\Phi_{\mathcal{J}}(x_a, x)\|_{x_a}$$

can not be continuous at (x_a, x) if x_a is a singular point. As a consequence, the statement of Theorem 1, and so the one of Lemma 2, can not hold uniformly.

We need then to define a notion generalizing the pseudo-norm in a continuous way around singular points.

Let $J_n, \dots, J_{\tilde{n}_r}$ be the words in the P. Hall basis $\mathcal{H}$ completing $I_1, \dots, I_{n-1}$ in a basis of $\mathcal{L}^r(1, \dots, m)$. On $\mathcal{V}_{\mathcal{J}}$, we write the vector fields $g_{J_n}, \dots, g_{J_{\tilde{n}_r}}$ as

$$g_{J_k} = \sum_{i=1}^{n} \alpha_k^i g_{I_i}, \quad k = n, \dots, \widetilde{n}_r, \tag{9}$$

the components α_k^i being smooth functions on $\mathcal{V}_{\mathcal{J}}$. We set $\alpha = \max_{n \leq j \leq \tilde{n}_r} |\alpha_j^n|$.

For each $x_a \in \mathcal{V}_{\mathcal{J}}$, we define the functions $D_{x_a}^{\mathcal{J}}(\cdot)$ on Ω and $D_{x_a}^{\mathcal{J}}(\cdot, \cdot)$ on $\Omega \times \Omega$ as

$$
\begin{aligned}
D^{\mathcal{J}}_{x_a}(x) &= |z_1(x)|^{1/|I_1|} + \cdots + |z_{n-1}(x)|^{1/|I_{n-1}|} + \min\big(\big|\frac{z_n(x)}{\alpha(x_a)}\big|^{1/r}, |z_n(x)|^{1/(r+1)}\big)\\
D^{\mathcal{J}}_{x_a}(x,y) &= |z_1(x)-z_1(y)|^{1/|I_1|} + \cdots + |z_{n-1}(x)-z_{n-1}(y)|^{1/|I_{n-1}|} +\\
&\qquad + \min\big(\big|\frac{z_n(x)-z_n(y)}{\alpha(x_a)}\big|^{1/r}, |z_n(x)-z_n(y)|^{1/(r+1)}\big)
\end{aligned}
$$

where $z = \Phi_{\mathcal{J}}(x_a, \cdot)$. The function $D^{\mathcal{J}}_{x_a}(\cdot) = D^{\mathcal{J}}_{x_a}(x_a, \cdot)$ is a kind of uniform generalization of the pseudo-norm.

Proposition 6. *Let $\mathcal{V}_{\mathcal{J}}$ be a singular domain. There exist continuous positive functions $C_{\mathcal{J}}(\cdot)$ and $\varepsilon_{\mathcal{J}}(\cdot)$ on $\mathcal{V}_{\mathcal{J}}$ such that, for any x_a, x in $\mathcal{V}_{\mathcal{J}}$ satisfying $d(x_a, x) < \varepsilon_{\mathcal{J}}(x_a)$, we have*

$$
\frac{1}{C_{\mathcal{J}}(x_a)} D^{\mathcal{J}}_{x_a}(x) \leq d(x_a, x) \leq C_{\mathcal{J}}(x_a) D^{\mathcal{J}}_{x_a}(x).
$$

Proposition 7. *Let $\mathcal{V}_{\mathcal{J}}$ be a singular domain. Then the LAS method* AppSteer$_{\mathcal{J}}$ *is uniformly contractive on every compact subset of $\mathcal{V}_{\mathcal{J}}$.*

The proofs of both above results relies on the desingularization of the m-tuple g by a lifting method introduced in [7]. For the sake of clarity, they are postponed to Appendix A.

5.3 Global result

Similarly to the construction of AppSteer, we define a "uniform pseudo-norm" on Ω by choosing at every point x_a one of the functions $D^{\mathcal{J}}_{x_a}$.

Definition 18. Let $x_a \in \Omega$, and $\mathcal{J}$ be the first n-tuple (for the lexicographic order) in the set $\{\mathcal{J}^R_1, \dots, \mathcal{J}^R_N, \mathcal{J}^S_1, \dots, \mathcal{J}^S_{N'}\}$ satisfying

$$
\det{}_{\mathcal{J}}(x_a) = \max(\det{}_{\mathcal{J}^R_1}(x_a), \dots, \det{}_{\mathcal{J}^R_N}(x_a), \det{}_{\mathcal{J}^S_1}(x_a), \dots, \det{}_{\mathcal{J}^S_{N'}}(x_a)).
$$

We define the functions $D_{x_a}(\cdot, \cdot)$ on $\Omega \times \Omega$ as

$$
D_{x_a}(x,y) = \begin{cases} \|\Phi_{\mathcal{J}}(x_a, x) - \Phi_{\mathcal{J}}(x_a, y)\|_{x_a}, & \text{if } \mathcal{J} \text{ is one the } \mathcal{J}^R_i\text{'s},\\ D^{\mathcal{J}}_{x_a}(x,y), & \text{if } \mathcal{J} \text{ is one the } \mathcal{J}^S_i\text{'s}, \end{cases},
$$

and $D_{x_a}(\cdot) = D_{x_a}(x_a, \cdot)$.

Theorem 8. *Let K be a compact subset of Ω. Then the LAS method* AppSteer *is uniformly contractive on K.*

Moreover there exists positive constants C_K and ε_K, such that, for any $x_a \in K$ and $x \in \Omega$ satisfying $d(x_a, x) < \varepsilon_K$, we have

$$
\frac{1}{C_K} D_{x_a}(x) \leq d(x_a, x) \leq C_K D_{x_a}(x).
$$

Proof. For every $\mathcal{J}$ in $\{\mathcal{J}^R_1, \dots, \mathcal{J}^R_N, \mathcal{J}^S_1, \dots, \mathcal{J}^S_{N'}\}$, the set of points $x_a \in \Omega$ such that

$$
\det{}_{\mathcal{J}}(x_a) = \max(\det{}_{\mathcal{J}^R_1}(x_a), \dots, \det{}_{\mathcal{J}^R_N}(x_a), \det{}_{\mathcal{J}^S_1}(x_a), \dots, \det{}_{\mathcal{J}^S_{N'}}(x_a))
$$

is a compact subset $K_{\mathcal{J}}$ of $\mathcal{V}_{\mathcal{J}}$. Therefore the points $x_a \in \Omega$ such that AppSteer$(x_a, \cdot) =$ AppSteer$_{\mathcal{J}}(x_a, \cdot)$ all belongs to $K_{\mathcal{J}}$. The first part of the theorem follows then directly from Propositions 5 and 7.

The second part results from (6) and Proposition 6 by setting

$$
\varepsilon_K = \min_{\mathcal{J}} \inf_{K_{\mathcal{J}}} \varepsilon_{\mathcal{J}} \quad \text{and} \quad C_K = \max_{\mathcal{J}} \sup_{K_{\mathcal{J}}} C_{\mathcal{J}}.
$$

□

Remark. As a corollary of Theorem 8, the property of uniform contraction of AppSteer can also be written in the following way. Up to reducing ε_K, for any $x_a \in K$ and $x \in \Omega$ satisfying $d(x_a, x) < \varepsilon_K$, we have

$$
\begin{aligned}
d(\mathrm{AppSteer}(x, x_a), x_a) &\leq \frac{1}{2} d(x, x_a) &\quad (10)\\
D_{x_a}(\mathrm{AppSteer}(x, x_a)) &\leq \frac{1}{2} D_{x_a}(x). &\quad (11)
\end{aligned}
$$

6 The global approximate steering algorithm

In this section, we devise an algorithm to steer system (1) from any $x_0 \in \Omega$ to the origin (assumed w.l.o.g. to be the goal) using the uniformly contractive LAS method AppSteer designed in Sect. 4. The algorithm is described in Fig. 1.

The parameterized path $t \mapsto \delta_{0,t}(x)$ is defined as follows. Let $\mathcal{J}$ be the n-tuple of words in $\mathcal{H}$ such that $D_0 = D_0^{\mathcal{J}}$ (and so $\mathrm{AppSteer}(\cdot, 0) = \mathrm{AppSteer}_{\mathcal{J}}(\cdot, 0)$). Then

$$\delta_{0,t}(x) = (t^{|I_1|} z_1(x), \ldots, t^{|I_n|} z_n(x)),$$

where $z = \Phi_{\mathcal{J}}(0, \cdot)$. The parameter t give an estimate of the distance D_0 on the path since, for $t, s \geq 0$,

$$D_0(\delta_{0,t}(x), \delta_{0,s}(x)) \leq |t - s| D_0(x).$$

The function Subgoal is the following.

Subgoal($\overline{x}, \eta_i, j$)

1. $t_j := \max(0, 1 - \frac{j\eta_i}{D_0(\overline{x})})$;
2. $x^d := \delta_{0,t_j}(\overline{x})$

The formula for generating t_j guarantees that $D_0(x^d, x^d_{j-1}) \leq \eta_i$ and that $x^d = 0$ for j large enough.

Global($x_0, 0$)	
1. $i := 0$; $j := 1$;	
2. $x_i := x_0$; $\overline{x} := x_0$;	
3. $\eta_i := D_0(x_0)$;	*initial choice of the maximum step size;*
4. while $D_0(x_i) > e$	*while the pseudonorm at 0 of the state is above a given tolerance e...;*
5. $x^d := \mathrm{Subgoal}(\overline{x}, \eta_i, j)$;	*choose the subgoal x^d at a distance η_i from x^d_{j-1};*
6. $x := \mathrm{AppSteer}(x_{i-1}, x^d)$;	*steer the system from x_{i-1} using an approximate steering control with destination x^d;*
7. if $D_{x^d}(x) > \frac{1}{2} D_{x^d}(x_{i-1})$	*if the system is not approaching the subgoal...;*
8. then $\eta_i := \frac{\eta_i}{2}$;	*reduce the maximum step size;*
$\overline{x} := x_{i-1}$; $j := 1$;	*and change the path $\delta_{0,t}(\overline{x})$;*
9. else $x_i := x$; $x^d_i := x^d$;	
$i := i + 1$; $j := j + 1$;	

Figure 1: The approximate steering algorithm

The global convergence of the approximate steering algorithm is established in the following result. For the sake of simplicity we assume to work on a compact set $K \subset \Omega$. Alternatively, this can be guaranteed by adding a step to the algorithm, as proposed at the end of the section.

Theorem 9. *Assume that the sequences $(x_i)_{i \geq 0}$ and $(x^d_i)_{i \geq 0}$ resulting from the use of the algorithm Global$(x_0, 0)$ both belong to a compact set $K \subset \Omega$. Then the algorithm terminates in a finite number of steps for any choice of the tolerance e.*

Proof. Note first that, if the conditional statement of Step 7 is not true for every i greater than some i_0, then $x_i^d = 0$ after a finite number of iterations. In this case, the error $D_0(x_i)$ is reduced at each iteration and the algorithm stops when it becomes smaller than the given tolerance e. This happens in particular if $d(x_{i-1}, x^d) < \varepsilon_K$ for all i greater than i_0, since condition (11) is then verified.

Another preliminary remark is that, due to the continuity of the control distance and of the function D_0, there exists $\overline{\eta} > 0$ such that, for any $x, y \in K$,

$$D_0(x, y) < \overline{\eta} \Rightarrow d(x, y) < \frac{\varepsilon_K}{2}. \tag{12}$$

In the following, we will prove by induction that if, for some i_0, $\eta_{i_0} < \overline{\eta}$, then, for all $i > i_0$,

$$d(x_{i-1}, x_i^d) < (1/2 + \cdots + (1/2)^{i-i_0})\varepsilon_K < \varepsilon_K.$$

We assume w.l.o.g. $i_0 = 0$ and $\overline{x} = x_0$. For $i = 1$, by construction $x^d = \text{Subgoal}(x_0, \eta_0, 1)$ and

$$D_0(x_0, x^d) \leq \eta_0 < \overline{\eta}.$$

In view of (12) we have then $d(x_0, x^d) < \varepsilon_K/2$, and so $x_1^d = x^d$ by (11). Therefore $d(x_0, x_1^d) < \varepsilon_K/2$.

Assume now that for $i > 1$ we have:

$$d(x_{i-2}, x_{i-1}^d) < (1/2 + \cdots + (1/2)^{i-1})\varepsilon_K. \tag{13}$$

Let $x^d = \text{Subgoal}(\overline{x}, \eta_i, j)$. We can write

$$d(x_{i-1}, x^d) \leq d(x_{i-1}, x_{i-1}^d) + d(x_{i-1}^d, x^d).$$

By construction, it is

$$D_0(x_{i-1}^d, x^d) \leq \eta_i < \overline{\eta},$$

which implies $d(x_{i-1}^d, x^d) < \varepsilon_K/2$. The induction hypothesis (13) implies that

$$d(x_{i-1}, x_{i-1}^d) \leq \frac{1}{2} d(x_{i-2}, x_{i-1}^d).$$

Finally, we have

$$\begin{aligned} d(x_{i-1}, x^d) &\leq \frac{1}{2} d(x_{i-2}, x_{i-1}^d) + d(x_{i-1}^d, x^d) \\ &\leq (1/2 + \cdots + (1/2)^i)\varepsilon_K. \end{aligned}$$

In view of (11), the conditional statement of Step 7 is not true, and so $x_i^d = x^d$.

If, for some i, $\eta_i \geq \overline{\eta}$ the conditional statement of Step 7 could be false. In this case, η_i is decreased as in Step 8. The updating law of η_i guarantees that after a finite number of iterations of Step 8, there holds $\eta_i < \overline{\eta}$. This ends the proof. □

When the working space Ω is equal to the whole $\mathbb{R}^n$, the assumption that the algorithm stays in a compact set can be removed. This requires a simple modification of the last step of the algorithm.

We choose a real number R close to one, precisely $(\frac{1}{2})^{1/(r+1)^2} < R < 1$. For every nonnegative integer k, we set $R_k = 1 + R + \cdots + R^k$. The algorithm is modified as follows. Introduce first a new variable k, and add the initialization $k := 0$ to Step 1. Replace then Step 9 by Step 9' below (Figure 2).

This step guarantees that the sequences $(x_i)_{i \geq 0}$ and $(x_i^d)_{i \geq 0}$ of the algorithm both belong to the compact set

$$K = \{x \in \mathbb{R}^n \ : \ D_0(x) \leq \frac{1}{1-R} D_0(x_0)\}.$$

Moreover, at each iteration of the algorithm, the new variable k is such that

$$D_0(x_i) \geq R_k D_0(x_0) \quad \Rightarrow \quad \eta_i \leq \frac{D_0(x_0)}{2^k}.$$

9'. else

9'.1. if $D_0(x) \geq R_{k+1} D_0(x_0) \quad \eta_i := \frac{\eta_i}{2}$;

9'.2. if $R_k D_0(x_0) \leq D_0(x) < R_{k+1} D_0(x_0)$

$x_i := x$; $x_i^d := x^d$; $i := i+1$; $j := j+1$;

$\eta_i := \frac{\eta_{i-1}}{2}$; $k := k+1$;

9'.3. if $D_0(x) \leq R_k D_0(x_0)$

$x_i := x$; $x_i^d := x^d$; $i := i+1$; $j := j+1$;

Figure 2: Step 9'

Proposition 10. *The modified algorithm Global (with Step 9' instead of Step 9) terminates in a finite number of iterations for any choice of x_0 and of the tolerance e.*

Proof. Notice that Step 9'.3 is identical to Step 9. It is therefore enough to show that, after a finite number of iterations, only Step 9'.3 occurs in Step 9'.

Another preliminary remark is that the distance D_0 give a rough estimate of the sub-Riemannian distance. Indeed it follows from Theorem 8 that, for any x, y in K close enough one from each other,

$$\frac{1}{C_0} D_0(x,y)^{r+1} \leq d(x,y) \leq C_0 D_0(x,y)^{1/(r+1)}, \tag{14}$$

where C_0 is a positive constant. As a consequence, Relation (12) holds if $\overline{\eta} \leq (\varepsilon_K/(2C_0))^{r+1}$.

Let us choose a positive $\overline{\eta}$ as above. We will prove that if, for some i_0, $\eta_{i_0} < \overline{\eta}$, then Steps 9'.1 and 9'.2 appear only in a finite number of iterations.

Recall first that, from the proof of Theorem 9, we get, for every $i > i_0$,

$$D_0(x_i^d) \leq D_0(x_{i_0}) \quad \text{and} \quad d(x_{i-1}, x_i^d) \leq \varepsilon_K.$$

In view of (14), an obvious adaptation of the latter proof yields, for every $i > i_0$, $d(x_{i-1}, x_i^d) \leq 2C_0 \eta_{i_0}^{1/(r+1)}$, and so $D_0(x_{i-1}, x_i^d) \leq (2C_0^2)^{1/(r+1)} \eta_{i_0}^{1/(r+1)^2}$. Finally we get

$$D_0(x_i) \leq D_0(x_{i+1}^d) + D_0(x_i, x_{i+1}^d) \leq D_0(x_{i_0}) + (2C_0^2)^{1/(r+1)} \eta_{i_0}^{1/(r+1)^2}. \tag{15}$$

On the other hand, there exists an integer k_0 such that $\eta_{i_0} \geq \frac{D_0(x_0)}{2^{k_0}}$, which implies $D_0(x_{i_0}) \leq R_{k_0} D_0(x_0)$. Up to reducing $\overline{\eta}$, and so increasing k_0, we can assume

$$(2C_0^2)^{1/(r+1)} (\frac{D_0(x_0)}{2^{k_0}})^{1/(r+1)^2} \leq R^{k_0+1} D_0(x_0),$$

since we have chosen $R > (\frac{1}{2})^{1/(r+1)^2}$. Using (15), it holds, for every $i \geq i_0$,

$$D_0(x_i) \leq R_{k_0} D_0(x_0) + R^{k_0+1} D_0(x_0) = R_{k_0+1} D_0(x_0).$$

Therefore Steps 9'.1 and 9'.2 can appear in at most $k_0 + 1$ iterations.

Applying again the arguments of the proof of Theorem 9, the conclusion follows. $\square$

A Proofs of Propositions 6 and 7

Both proofs rely on the desingularization of the m-tuple g by a lifting method described in the following result from [7].

Proposition 11. *Set $\widetilde{n} = \widetilde{n}_r + 1$, $\widetilde{\Omega} = \Omega \times \mathbb{R}^{\widetilde{n}-n}$, and let $a \in \Omega$. There exist a neighborhood $\widetilde{U} \subset \widetilde{\Omega}$ of $(a, 0)$; a neighborhood $U \subset \Omega$ of a with $U \times \{0\} \subset \widetilde{U}$; local coordinates y on U and (y, v) on $\widetilde{U}$; and smooth functions b_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, \widetilde{n} - n$ on $\mathbb{R}^{\widetilde{n}}$, such that the m-tuple of vector fields $\xi = (\xi_1, \ldots, \xi_m)$ on $\widetilde{U}$ defined in (y, v) coordinates by*

$$\xi_i(y, v) = g_i(y) + \sum_{j=1}^{\widetilde{n}-n} b_{ij}(y, v)\partial_{v_j}$$

has a growth vector equal to $(\widetilde{n}_1, \ldots, \widetilde{n}_r, \widetilde{n})$ on $\widetilde{U}$. In particular ξ has no singular points.

Moreover, denoting $\pi : \widetilde{\Omega} \to \Omega$ the canonical projection and $\widetilde{d}$ the sub-Riemannian distance defined by ξ on $\widetilde{U}$, we have, for all x^1, x^2 in U,

$$d(x^1, x^2) = \inf_{\widetilde{x}^2 \in \pi^{-1}(x^2)} \widetilde{d}((x^1, 0), \widetilde{x}^2).$$

Remark. In this proposition, and in the following, we use $h(y)$ instead of $y_* h$ to denote a vector field h in local coordinates y.

The construction of AppSteer$_{\mathcal{J}}$ is based on the particular system of privileged coordinates $\Phi_{\mathcal{J}}$. To simplify the paper, we will replace in the proofs $\Phi_{\mathcal{J}}$ by another system of privileged coordinates Φ, easier to work with. With $\Phi_{\mathcal{J}}$, the arguments would be very similar but more involved. For instance, the case $n = 5$, $m = 2$ has already been treated in [18] using $\Phi_{\mathcal{J}}$.

The new system of privileged coordinates Φ is defined as follow. For every $x_a \in \mathcal{V}_{\mathcal{J}}$, consider the local diffeomorphism

$$\psi_{x_a} : z = (z_1, \ldots, z_n) \mapsto e^{z_n g_{I_n}} \circ \cdots \circ e^{z_1 g_{I_1}}(x_a)$$

defined on a neighborhood of 0 in $\mathbb{R}^n$, and set $\Phi(x_a, x) = \psi_{x_a}^{-1}(x)$ for x near x_a. Then Φ is a continuously varying system of privileged coordinates on $\mathcal{V}_{\mathcal{J}}$ [6].

Proof of Proposition 6. Fix a point $a \in \mathcal{V}_{\mathcal{J}}$, and consider the neighborhoods U, $\widetilde{U}$, the coordinates (y, v), and the m-tuple ξ given in Proposition 11. We assume $\widetilde{U} \subset \mathcal{V}_{\mathcal{J}} \times \mathbb{R}^{\widetilde{n}-n}$.

Let E_ξ be the evaluation map which assigns to each $I \in \mathcal{L}(1, \ldots, m)$ the vector field $E_\xi(I) = \xi_I$ obtained by plugging in the ξ_i, $i = 1, \ldots, m$, for the corresponding letter i. In coordinates (y, v), the vector field ξ_I is written as

$$\xi_I(y, v) = g_I(y) + \sum_{j=1}^{\widetilde{n}-n} b_{Ij}(y, v)\partial_{v_j},$$

where the b_{Ij}'s are smooth functions.

Recall that $J_n, \ldots, J_{\widetilde{n}_r}$ are the words in the P. Hall basis $\mathcal{H}$ completing $I_1, \ldots, I_{n-1}$ in a basis of $\mathcal{L}^r(1, \ldots, m)$. Therefore the vector fields $\xi_{I_1}, \ldots, \xi_{I_{n-1}}, \xi_{J_n}, \ldots, \xi_{J_{\widetilde{n}_r}}, \xi_{I_n}$ form an adapted frame at every point of $\widetilde{U}$.

Let $x_a \in U$, and $\widetilde{x}_a = (x_a, 0) \in \widetilde{U}$. Define vector fields $\zeta_n, \ldots, \zeta_{\widetilde{n}_r}$ on $\widetilde{U}$ by

$$\zeta_k(y, v) = \alpha_k^n(x_a)\xi_{I_n} + \sum_{j=1}^{\widetilde{n}-n} b_{J_k j}(y, v)\partial_{v_j}, \quad k = n, \ldots, \widetilde{n}_r,$$

where α_k^n are the functions introduced in (9). Since $\zeta_k(\widetilde{x}_a) = \xi_{J_k}(\widetilde{x}_a) - \sum_{i=1}^{n-1} \alpha_k^i(x_a)\xi_{I_i}(\widetilde{x}_a)$, the vector fields $\xi_{I_1}, \ldots, \xi_{I_{n-1}}, \zeta_n, \ldots, \zeta_{\widetilde{n}_r}, \xi_{I_n}$ form an adapted frame at $\widetilde{x}_a$. As a consequence (see [6]), the mapping

$$\psi_{\widetilde{x}_a} : \widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}) \mapsto e^{\widetilde{z}_{\widetilde{n}}\xi_{I_n}} \circ e^{\widetilde{z}_{\widetilde{n}_r}\zeta_{\widetilde{n}_r}} \circ \cdots \circ e^{\widetilde{z}_n\zeta_n} \circ e^{\widetilde{z}_{n-1}\xi_{I_{n-1}}} \circ \cdots \circ e^{\widetilde{z}_1\xi_{I_1}}(\widetilde{x}_a)$$

is a local diffeomorphism at $0 \in \mathbb{R}^{\widetilde{n}}$, and $\widetilde{\Phi} : (\widetilde{x}_a, \widetilde{x}) \mapsto \psi_{\widetilde{x}_a}^{-1}(\widetilde{x})$ is a continuously varying system of privileged coordinates on $\widetilde{U}$.

Since $\widetilde{U}$ contains only regular points (for ξ), Theorem 1 applies: there exist continuous positive functions $C_{\widetilde{d}}(\cdot)$ and $\varepsilon_{\widetilde{d}}(\cdot)$ on $\widetilde{U}$ such that, for any $\widetilde{x}_a, \widetilde{x}$ in $\widetilde{U}$ satisfying $\widetilde{d}(\widetilde{x}_a, \widetilde{x}) < \varepsilon_{\widetilde{d}}(\widetilde{x}_a)$,

$$\frac{1}{C_{\widetilde{d}}(\widetilde{x}_a)} \|\widetilde{z}(\widetilde{x})\|_{\widetilde{x}_a} \le \widetilde{d}(\widetilde{x}_a, \widetilde{x}) \le C_{\widetilde{d}}(\widetilde{x}_a) \|\widetilde{z}(\widetilde{x})\|_{\widetilde{x}_a},$$

where $\widetilde{z} = \widetilde{\Phi}(\widetilde{x}_a, \cdot)$ and $\|\widetilde{z}\|_{\widetilde{x}_a} = |\widetilde{z}_1|^{1/|I_1|} + \cdots + |\widetilde{z}_{n-1}|^{1/|I_{n-1}|} + |\widetilde{z}_n|^{1/r} + \cdots + |\widetilde{z}_{\widetilde{n}-1}|^{1/r}| + |\widetilde{z}_{\widetilde{n}}|^{1/(r+1)}$.

Therefore, using the last property of Proposition 11, there exist continuous positive functions $\varepsilon_{\mathcal{J}}(\cdot)$ and $C_{\mathcal{J}}(\cdot)$ on U such that, for any x_a, x in U satisfying $d(x_a, x) < \varepsilon_{\mathcal{J}}(x_a)$, we have

$$\frac{1}{C_{\mathcal{J}}(x_a)} \inf_{\widetilde{x} \in \Pi_x} \|\widetilde{\Phi}(\widetilde{x}_a, \widetilde{x})\|_{\widetilde{x}_a} \le d(x_a, x) \le C_{\mathcal{J}}(x_a) \inf_{\widetilde{x} \in \Pi_x} \|\widetilde{\Phi}(\widetilde{x}_a, \widetilde{x})\|_{\widetilde{x}_a}$$

where $\widetilde{x}_a = (x_a, 0)$ and $\Pi_x = \{\widetilde{x} \in \pi^{-1}(x) \ : \ \widetilde{d}(\widetilde{x}_a, \widetilde{x}) \le \varepsilon_{\mathcal{J}}(x_a)\}$ (choose for instance $\varepsilon_{\mathcal{J}}(x_a) = \varepsilon_{\widetilde{d}}(\widetilde{x}_a)/2C_{\widetilde{d}}(\widetilde{x}_a)$ and $C_{\mathcal{J}}(x_a) = C_{\widetilde{d}}(\widetilde{x}_a)$).

Let us compute now the coordinates $z = \Phi(x_a, x)$ in function of the coordinates $\widetilde{z} = \widetilde{\Phi}(\widetilde{x}_a, \widetilde{x})$ when $x = \pi(\widetilde{x})$. Due to the particular form of the vector fields ξ_{I_i} and ζ_j, we have

$$\begin{aligned} \pi(\psi_{\widetilde{x}_a}(\widetilde{z})) &= e^{\widetilde{z}_{\widetilde{n}} g_{I_n}} \circ e^{\widetilde{z}_{\widetilde{n}_r} \alpha^n_{\widetilde{n}_r}(x_a) g_{I_n}} \circ \cdots \circ e^{\widetilde{z}_n \alpha^n_n(x_a) g_{I_n}} \circ e^{\widetilde{z}_{n-1} g_{I_{n-1}}} \circ \cdots \circ e^{\widetilde{z}_1 g_{I_1}}(x_a) \\ &= e^{(\widetilde{z}_{\widetilde{n}} + \sum_{j=n}^{\widetilde{n}_r} \widetilde{z}_j \alpha^n_j(x_a)) g_{I_n}} \circ e^{\widetilde{z}_{n-1} g_{I_{n-1}}} \circ \cdots \circ e^{\widetilde{z}_1 g_{I_1}}(x_a) \\ &= \psi_{x_a}(\widetilde{z}_1, \ldots, \widetilde{z}_{n-1}, \widetilde{z}_{\widetilde{n}} + \sum_{j=n}^{\widetilde{n}_r} \widetilde{z}_j \alpha^n_j(x_a)). \end{aligned} \tag{16}$$

Equivalently, $z_1 = \widetilde{z}_1, \ldots, z_{n-1} = \widetilde{z}_{n-1}$, $z_n = \widetilde{z}_{\widetilde{n}} + \sum_{j=n}^{\widetilde{n}_r} \widetilde{z}_j \alpha^n_j(x_a)$. As a consequence,

$$\begin{aligned} \inf_{\widetilde{x} \in \Pi_x} \|\widetilde{\Phi}(\widetilde{x}_a, \widetilde{x})\|_{\widetilde{x}_a} &= \inf_{(\widetilde{z}_n, \ldots, \widetilde{z}_{\widetilde{n}_r}) \in \mathbb{R}^{\widetilde{n}-n}} \Big(|z_1|^{1/|I_1|} + \cdots + |z_{n-1}|^{1/|I_{n-1}|} + \\ &\qquad + |\widetilde{z}_n|^{1/r} + \cdots + |\widetilde{z}_{\widetilde{n}_r}|^{1/r}| + |z_n - \sum_{j=n}^{\widetilde{n}_r} \widetilde{z}_j \alpha^n_j(x_a)|^{1/(r+1)} \Big) \\ &= D^{\mathcal{J}}_{x_a}(x). \end{aligned} \tag{17}$$

Thus, there exist continuous positive functions $C_{\mathcal{J}}(\cdot)$ and $\varepsilon_{\mathcal{J}}(\cdot)$ on a neighborhood U of a such that, for any x_a, x in U satisfying $d(x_a, x) < \varepsilon_{\mathcal{J}}(x_a)$, we have

$$\frac{1}{C_{\mathcal{J}}(x_a)} D^{\mathcal{J}}_{x_a}(x) \le d(x_a, x) \le C_{\mathcal{J}}(x_a) D^{\mathcal{J}}_{x_a}(x).$$

This holds for every point $a \in \mathcal{V}_{\mathcal{J}}$. Using then a continuous partition of unity, we get Proposition 6. □

Remark. Note that, for any $x_a \in \mathcal{V}_{\mathcal{J}}$, the coordinates $\Phi_{\mathcal{J}}(x_a, \cdot)$ are also privileged coordinates at x_a of $\hat{g} = A_{\mathcal{J}}(x_a)$. Since both $\Phi_{\mathcal{J}}(x_a, \cdot)$ and $\hat{g} = A_{\mathcal{J}}(x_a)$ vary continuously with x_a, we obtain the following uniform estimate of $\hat{d}_{x_a}$ (the sub-Riemannian distance associated to $A_{\mathcal{J}}(x_a)$), up to reducing $\varepsilon_{\mathcal{J}}(\cdot)$ and increasing $C_{\mathcal{J}}(\cdot)$: for all x_a, x in $\mathcal{V}_{\mathcal{J}}$ satisfying $d(x_a, x) < \varepsilon_{\mathcal{J}}(x_a)$,

$$\frac{1}{C_{\mathcal{J}}(x_a)} D^{\mathcal{J}}_{x_a}(x) \le \hat{d}_{x_a}(x_a, x) \le C_{\mathcal{J}}(x_a)\, D^{\mathcal{J}}_{x_a}(x), \quad \text{with } z = \Phi_{\mathcal{J}}(x_a, \cdot).$$

As a consequence, a steering law of $A_{\mathcal{J}}$ is quasi-optimal if and only if there exist continuous positive functions $C_\ell(\cdot)$ and $\varepsilon_\ell(\cdot)$ on $\mathcal{V}_{\mathcal{J}}$ such that, for any $x_a, x \in \mathcal{V}_{\mathcal{J}}$ with $d(x_a, x) < \varepsilon_\ell(x_a)$, the control $\hat{u}(\cdot)$ steering $\hat{g} = A_{\mathcal{J}}(x_a)$ from x to x_a satisfies:

$$\ell(\hat{u}) \le C_\ell(x_a)\, D^{\mathcal{J}}_{x_a}(x). \tag{18}$$

Before proving Proposition 7, we will show in the next lemma that the non-homogeneous nilpotent approximation $A_{\mathcal{J}}$ admits a lifting which is an approximation of the lifting of g. As in the proof of Proposition 6 above, the point $a \in \mathcal{V}_{\mathcal{J}}$ is fixed, and the neighborhoods U, $\widetilde{U}$, the coordinates (y, v), and the m-tuple ξ are the one given in Proposition 11.

Lemma 12. *There exists a continuous approximation $\widetilde{A} : \widetilde{x}_a \mapsto \widehat{\xi}$ of ξ on $\widetilde{U}$ such that $\widehat{\xi}_i$, $i = 1, \dots, m$, in coordinates (y, v) are written as*

$$\widehat{\xi}_i(y, v) = \hat{g}_i(y) + \sum_{j=1}^{\widetilde{n}-n} \widehat{b}_{ij}(y, v)\partial_{v_j},$$

where $\hat{g} = A_{\mathcal{J}}(x_a)$, and the $\widehat{b}_{ij}$'s are smooth functions on $\mathbb{R}^{\widetilde{n}}$.

Proof. Let $V \subset T\widetilde{U}$ be the distribution on $\widetilde{U}$ spanned by the vector fields $\partial_{v_1}, \dots, \partial_{v_{\widetilde{n}-n}}$. For every $\widetilde{x}_a \in \widetilde{U}$ and $i = 1, \dots, m$, we are looking for vector fields $\widehat{\xi}_i$, $i = 1, \dots, m$, written in coordinates (z, v) as

$$\widehat{\xi}_i(z, v) = \hat{g}_i(z) + \sum_{j=1}^{\widetilde{n}-n} b_{ij}(z, v)\partial_{v_j} + V_i(z, v),$$

where $V_i \in V$, and z are the privileged coordinates $\Phi(x_a, \cdot)$, $x_a = \pi(\widetilde{x}_a)$. The lemma will be proved if we show that we can choose the vector fields V_i depending continuously on $\widetilde{x}_a$ and so that $\xi_i - \widehat{\xi}_i$ is of non-negative nonholonomic order at $\widetilde{x}_a$ (for the nonholonomic order defined by ξ on $\widetilde{U}$).

Notice first that, by definition of $\hat{g}$, the vector field $g_i(z) - \hat{g}_i(z)$ is a sum of monomial vector fields of non-negative weighted degree, that is

$$g_i(z) - \hat{g}_i(z) = \sum_{j=1}^{n} R_{ij}(z)\partial_{z_j}, \quad \text{where } R_{ij}(z) = \sum_{w(\alpha) \geq |I_j|} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Therefore $(\xi_i - \widehat{\xi}_i)(z, v) = \sum_{j=1}^{n} R_{ij}(z)\partial_{z_j} - V_i(z, v)$.

Recall also that $z_1 = \widetilde{z}_1, \dots, z_{n-1} = \widetilde{z}_{n-1}$, $z_n = \widetilde{z}_{\widetilde{n}} + \sum_{j=n}^{\widetilde{n}_r} \widetilde{z}_j \alpha_j^n(x_a)$, where $\widetilde{z} = \widetilde{\Phi}(\widetilde{x}_a, \cdot)$ are the privileged coordinates at $\widetilde{x}_a$ defined in the proof of Proposition 6 above. For $j = 1, \dots, n-1$, set $Z_j = \partial_{z_j} - \partial_{\widetilde{z}_j}$. For $k = 1, \dots, n$, there holds $dz_k(Z_j) = 0$, and so $Z_j \in V$. Moreover, due to the definition of coordinates z and to Equation (16), we have $\partial_{z_n} = g_{I_n}$, and $\partial_{\widetilde{z}_j} - \alpha_j^n(x_a)g_{I_n} \in V$, for $j = n, \dots, \widetilde{n}-1$. Equivalently, the vector fields $Z'_j = \alpha_j^n(x_a)\partial_{z_n} - \partial_{\widetilde{z}_j}$, $j = n, \dots, \widetilde{n}-1$, belong to V. To summarize,

$$(\xi_i - \widehat{\xi}_i)(z, v) = \sum_{j=1}^{n-1} R_{ij}(z)\partial_{\widetilde{z}_j} + \sum_{j=1}^{n-1} R_{ij}(z)Z_j + R_{in}(z)\partial_{z_n} - V_i(z, v). \tag{19}$$

Denote by $\widetilde{\text{ord}}$ the ξ-nonholonomic order at $\widetilde{x}_a$. We have $\widetilde{\text{ord}}(z_1) = \widetilde{\text{ord}}(\widetilde{z}_1) = |I_1|, \dots, \widetilde{\text{ord}}(z_{n-1}) = \widetilde{\text{ord}}(\widetilde{z}_{n-1}) = |I_{n-1}|$, and $\widetilde{\text{ord}}(z_n) \geq r$. Moreover, $\widetilde{\text{ord}}(\partial_{z_n}) \geq -(r+1)$ since ∂_{z_n} is non zero. Since the weighted degree of z_j is $|I_j|$, we have $\widetilde{\text{ord}}(R_{ij}(z)) \geq |I_j|$ for $j = 1, \dots, n-1$, and $R_{in}(z) = c_i z_n + R'_{in}(z)$, where c_i is a constant and $\widetilde{\text{ord}}(R'_{in}(z)) \geq r+1$. Now, due to the definition of Z'_j, we have

$$c_i z_n \partial_{z_n} = c_i \widetilde{z}_n \partial_{z_n} + c_i \sum_{j=n}^{\widetilde{n}-1} \widetilde{z}_j (\partial_{\widetilde{z}_j} + Z'_j).$$

Hence Equation (19) rewrites as

$$(\xi_i - \widehat{\xi}_i)(z, v) = \left[\sum_{j=1}^{n-1} R_{ij}(z)\partial_{\widetilde{z}_j} + R'_{in}(z)\partial_{z_n} + c_i \widetilde{z}_n \partial_{z_n} + c_i \sum_{j=n}^{\widetilde{n}-1} \widetilde{z}_j \partial_{\widetilde{z}_j}\right] + \\ + \left[\sum_{j=1}^{n-1} R_{ij}(z)Z_j + c_i \sum_{j=n}^{\widetilde{n}-1} \widetilde{z}_j Z'_j - V_i(z, v)\right].$$

In the second-hand member of this equality, the first term into bracket is of non-negative order and the second one belongs to V. We then choose the vector field V_i as

$$V_i(z,v) = \sum_{j=1}^{n-1} R_{ij}(z) Z_j + c_i \sum_{j=n}^{\widetilde{n}-1} \widetilde{z}_j Z'_j \ \in V,$$

which depends continuously on $\widetilde{x}_a$, and we obtain $\widetilde{\operatorname{ord}}(\xi_i - \widehat{\xi}_i) \geq 0$. This proves the lemma. □

Proof of Proposition 7. Once again, we fix a point $a \in \mathcal{V}_{\mathcal{J}}$, and so the neighborhoods U, $\widetilde{U}$, the coordinates (y, v), and the m-tuple ξ.

Given x_a, x in U close enough, let $u(\cdot)$ be the steering control of $\hat{g} = A_{\mathcal{J}}(x_a)$ between x and x_a, and $x_b = \mathrm{AppSteer}_{\mathcal{J}}(x_a, x)$. The steering law of $A_{\mathcal{J}}$ being quasi-optimal, we have $\ell(u) \leq C_{qo}\, \hat{d}_{x_a}(x_a, x)$.

Let $\widetilde{x}_a = (x_a, 0)$. Consider the approximation $\widetilde{A}$ given in Lemma 12, and set $\widehat{\xi} = \widetilde{A}(\widetilde{x}_a)$. Due to the form of $\widehat{\xi}$, there exists a point $\widetilde{x} \in \pi^{-1}(x)$ such that $u(\cdot)$ steers $\widehat{\xi}$ from $\widetilde{x}$ to $\widetilde{x}_a$. This control $u(\cdot)$ steers ξ from $\widetilde{x}$ to a point $\widetilde{x}_b \in \pi^{-1}(x_b)$. Notice that, denoting by $\overline{d}$ the sub-Riemannian distance defined by $\widehat{\xi}$, we have $\overline{d}(\widetilde{x}_a, \widetilde{x}) \leq \ell(u)$, and, from [1, Cor. 7.33], $\widetilde{d}(\widetilde{x}_a, \widetilde{x}) \leq 2\overline{d}(\widetilde{x}_a, \widetilde{x}) \leq 2\ell(u)$, provided that $\ell(u)$ is smaller than a continuous positive function of $\widetilde{x}_a$ (recall that $\widetilde{U}$ contains only regular points for ξ).

Using Lemma 2, there exist continuous positive functions $C_{\widetilde{e}}(\cdot)$ and $\varepsilon_{\widetilde{e}}(\cdot)$ on $\widetilde{U}$ such that $d(x_a, x) < \varepsilon_{\widetilde{e}}(x_a)$ implies

$$\begin{aligned} &\frac{1}{C_{\widetilde{e}}(\widetilde{x}_a)} \|\widetilde{z}(\widetilde{x})\|_{\widetilde{x}_a} \leq \widetilde{d}(\widetilde{x}_a, \widetilde{x}) \leq C_{\widetilde{e}}(\widetilde{x}_a)\, \|\widetilde{z}(\widetilde{x})\|_{\widetilde{x}_a} \\ \text{and} \quad &\|\widetilde{z}(\widetilde{x}_b)\|_{\widetilde{x}_a} \leq C_{\widetilde{e}}(\widetilde{x}_a) \max\left(\|\widetilde{z}(\widetilde{x})\|_{\widetilde{x}_a}, \ell(u)\right)\, \ell(u)^{1/(r+1)}, \end{aligned}$$

where $\widetilde{z} = \widetilde{\Phi}(\widetilde{x}_a, \cdot)$ are the privileged coordinates at $\widetilde{x}_a$ defined in the proof of Proposition 6.

In order to obtain a relation between $d(x_a, x_b)$ and $d(x_a, x)$, we establish three inequalities. First, (17) implies $\|\widetilde{z}(\widetilde{x}_b)\|_{\widetilde{x}_a} \geq D^{\mathcal{J}}_{x_a}(x_b)$, that is

$$d(x_a, x_b) \leq C_{\mathcal{J}}(x_a) \|\widetilde{z}(\widetilde{x}_b)\|_{\widetilde{x}_a}.$$

Second, since $\widetilde{d}(\widetilde{x}_a, \widetilde{x}) \leq 2\ell(u)$ there holds

$$\max(\|\widetilde{z}(\widetilde{x})\|_{\widetilde{x}_a}, \ell(u)) \leq 2C_{\widetilde{e}}(\widetilde{x}_a)\ell(u).$$

Third, it results from (18) and Proposition 6

$$\ell(u) \leq C_{\mathcal{J}}(x_a) C_{\ell}(x_a) d(x_a, x).$$

Grouping all together, we obtain that there exist a continuous positive functions $\mu(\cdot)$ on the neighborhood U of a such that, for any x_a, x in U satisfying $d(x_a, x) < \mu(x_a)$, we have

$$d(x_a, \mathrm{AppSteer}_{\mathcal{J}}(x_a, x)) \leq d(x_a, x)^{1+\frac{1}{2r}}.$$

Therefore $\mathrm{AppSteer}_{\mathcal{J}}$ is uniformly contractive on any compact subset of the neighborhood U of a. Since this property holds for every point $a \in \mathcal{V}_{\mathcal{J}}$, we get Proposition 7. □

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