RESONANCES OF AN ELASTIC PLATE COUPLED WITH A COMPRESSIBLE CONFINED FLOW

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Summary

The theoretical study of the resonances of an elastic plate in a compressible flow in a 2D duct is presented. Due to the fluid-structure coupling a quadratic eigenvalue problem is involved, in which the resonance frequencies k solve the equations $\lambda(k) = k^2$ where γ are the eigenvalues of a self-adjoint operator of the form $A + kB$. In a previous paper we have already proved that a linear eigenvalue problem can be recovered if the plate is rigid or the fluid at rest. We focus here on the general problem for which elasticity and flow are jointly present and derive a lower bound for the number of resonances. The expression of this bound, based on the solution of two linear eigenvalues problems, points out that the coupling between elasticity and flow generally reduces the number of resonances. This estimate is validated numerically.

Introduction

We consider a 2D duct, with rigid walls, containing a compressible fluid and in its middle, an elastic plate. Our aim is to determine the resonance frequencies of the plate when the fluid is moving, with a uniform flow. This study is motivated by experimental results obtained for a plate with a clamped leading edge and a free trailing edge: when the flow velocity reaches a critical value, the plate initially at rest starts to oscillate with large amplitudes (**1**). Our hypothesis is that the oscillation frequency of the plate at threshold is a resonance frequency of the fluid-structure coupling problem.

Studies of resonances have been widely performed for a rigid plate and a fluid at rest (**2**, **3**, **4**). In these cases the resonant modes are rather called trapped modes. When the plate is elastic the problem cannot be written as a classical eigenvalue problem: a resonance frequency k satisfies a problem of the general form $A_0X + kA_1X - k^2X = 0$ with A_0 and A_1 self-adjoint operators. A general method consists in solving a non-linear eigenvalue problem: find the eigenvalues $\lambda_n(k)$ of $A(k) = A_0 + kA_1$ and solve the fixed point equations $\lambda_n(k) = k^2$.

When the fluid is at rest or when the plate is rigid, we showed in a previous article that existence results of resonances can be obtained by solving a linear eigenvalue problem (**5**). More precisely we proved that the number of resonances is exactly the number of eigenvalues of $A(k_c)$ where k_c is the cut-off value of the guide (defined when restricting to skew-symmetric guide modes). Then comparing to two simpler problems, a rigid plate in a

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fluid or an elastic plate in vacuum, we derived a lower bound for the number of resonances. We proved in particular that this number is an increasing function of the length of the plate, of the fluid-structure coupling and of the Mach number of the flow, and a decreasing function of the rigidity.

In the general case where the fluid is in flow and the plate is elastic, the previous results are no longer true and the extension of the method described just above is not straightforward. In particular, the combined effects of the flow and of the elasticity of the plate, which generate resonance effects when considered separately, can surprisingly reduce the number of resonances. The understanding of this phenomenon is the aim of this paper.

To our knowledge, no study of trapped modes in the presence of an elastic obstacle and of a compressible flow in a duct has been led but we can mention two kinds of related studies. For a confined flow, the stability of a finite elastic plate in a waveguide has been studied but in the case of an incompressible flow (**6**). When the fluid is not confined in a duct, the influence of the elastic deformations of a plate on a flow has been considered for 2D problems: when the fluid is compressible (without flow (**7**) and with flow (**8**)) or when the fluid is incompressible (plate with free ends (**9**) and plate with fixed ends (**10**)). Recently the 3D incompressible case has been addressed for clamped-free or pinned-free plates (**11**). In all cases no trapped mode can exist, but the coupling between an elastic plate and a flow can lead to instabilities.

In the first section, we present the mathematical framework and the self-adjoint non-linear eigenvalue problem. In section 2, we focus on the spectral study, determine the essential spectrum and recall the Min-Max principle to determine the eigenvalues. In section 3 we present a way to deal with the non-linear eigenvalue problem: the main result is that a lower bound for the number of resonances is given by the difference between the number $\mathcal{N}\left(A(k_c)\right)$ of eigenvalues of $A(k)$ at $k = k_c$ and the number $\mathcal{N}^{neg}\left(A(0)\right)$ of negative eigenvalues at $k = 0$. This last number takes positive values (so that the number of resonances is reduced) only in the completely coupled case (flow and elastic plate): it is related to the phenomenon already mentioned. We derive a lower bound $\mathcal{N}^+$ of $\mathcal{N}\left(A(k_c)\right)$ in section 4 and an upper bound $\mathcal{N}^-$ for $\mathcal{N}^{neg}\left(A(0)\right)$ in section 5, leading finally to the lower bound $(\mathcal{N}^+ - \mathcal{N}^-)$ for the number of resonances. The dependence of the number of resonances versus the different parameters is discussed in section 6 and validated numerically in section 7.

1. The Mathematical Framework

1.1 *The equations for the resonant modes*

We consider a two-dimensional infinite duct $\tilde{D} = \{(x, y); -h/2 < y < h/2\}$ of height h and of boundary $\partial\tilde{D}$. The elastic plate is defined at rest by $\Gamma = \{(x, y); y = 0 \text{ and } 0 < x < L\}$. The propagation domain is noted $\tilde{\Omega}$ (see Fig. 1): $\tilde{\Omega} = \tilde{D}\backslash\Gamma$. The velocity of the uniform flow is noted U ($U \geqslant 0$).

Concerning the fluid, in time harmonic regime of frequency ω, Euler's equations lead to the "convected" Helmholtz equation for the velocity potential ϕ (**12**) ($\mathbf{v} = \boldsymbol{\nabla}\phi$ and the factor $e^{-i\omega t}$ is omitted):

$$H_M(k)\phi = (1 - M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + 2ikM\frac{\partial\phi}{\partial x} + k^2\phi = 0 \quad \text{in} \quad \tilde{\Omega}. \tag{1.1}$$

c is the sound velocity and $M = U/c$ is the Mach number. The acoustic wavenumber

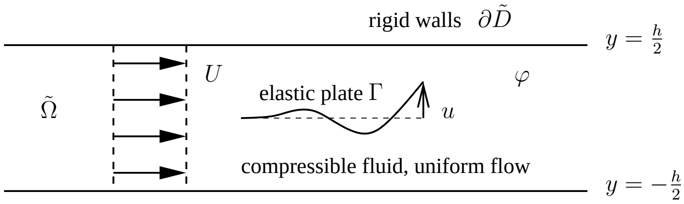

Fig. 1 Geometry of the problem

$k = \omega/c$ will be called frequency in the sequel. The acoustic pressure p is linked to the velocity potential through the relation

$$p + \rho\left(U\frac{\partial}{\partial x} - i\omega\right)\phi = 0,$$

where ρ is the fluid density. In the following we will restrict ourselves to the subsonic case $0 \leqslant M < 1$. Note that the supersonic case $M > 1$ is not a simple extension of the case $M < 1$ since the partial differential equation (1.1), elliptic in the subsonic case becomes hyperbolic in the supersonic case. For a trapped mode, the potential Φ should belong to $L^2(\tilde{\Omega})$. As we will see later, this implies an exponential decay at infinity.

Concerning the elastic plate, the vertical displacement of the plate $\mathrm{Re}\,[w(x)e^{-i\omega t}]$ follows the equation:

$$D\,\frac{d^4 w}{dx^4} - \rho_p a\omega^2 w = -[p], \tag{1.2}$$

where ρ_p is the mass density of the plate, a its thickness, $D = Ea^3/12(1-\nu^2)$ its flexural rigidity with E the Young modulus and ν the Poisson ratio. $[p]$ designs the pressure jump through the plate, i.e. $[p] = p(x, 0^+) - p(x, 0^-)$, where $p(x, 0^{\pm}) = \lim_{\epsilon \overset{>}{\to} 0} p(x, \pm\epsilon)$.

Concerning the boundary conditions, on the guide walls $\partial\tilde{D}$ the slip condition reads $\partial\phi/\partial y = 0$. On the plate, if we let $\boldsymbol{\zeta}$ denote the displacement in the fluid, the continuity of the displacement at the fluid/plate interface reads $w = \zeta_y$ on both sides of Γ. Using the link between the displacement and the fluid velocity $\mathbf{v} = D\boldsymbol{\zeta}/Dt$ where $D/Dt = U\partial/\partial x - i\omega$ is the convective operator, we get easily that $v_y = Dw/Dt$ and therefore,

$$\frac{\partial\phi}{\partial y} = \left(U\frac{d}{dx} - i\omega\right)w.$$

Last for the plate, we consider the case of a clamped leading edge and a free trailing edge

$$\begin{aligned} w(0) = \quad 0 \quad &= \frac{dw}{dx}(0), \\ \frac{d^2 w}{dx^2}(L) = \quad 0 \quad &= \frac{d^3 w}{dx^3}(L). \end{aligned}$$

1.2 *The generalized eigenvalue problem*

We restrict ourselves to the case of a plate placed in the center of the duct and thus we take into account symmetry considerations in order to reduce the problem to a half-guide: every velocity potential ϕ can be decomposed in the form $\phi_s + \phi_a$ where ϕ_s (respectively ϕ_a) is symmetric (respectively skew-symmetric) with respect to the middle plane $y = 0$. We get that the two fields satisfy decoupled problems: ϕ_s is solution of a problem without plate (since $[\phi_s] = 0 = \partial\phi_s/\partial y$ on $y = 0$) whereas (ϕ_a, u) is a resonant mode (then $[\phi_a] = 2\phi_a$ on $y = 0$). Therefore the symmetric problem can not give raise to trapped mode and our study will concern only the skew-symmetric problem.

In the following only the upper half guide defined by $\Omega = \mathbb{R} \times]0, h/2[$ will be considered and ϕ_a will be called ϕ. The upper rigid wall of Ω is noted ∂D and $\Gamma^+ = \{(x, 0); x > L\}$ and $\Gamma^- = \{(x, 0); x < 0\}$ are introduced (Fig. 2). We define the parameters $\eta^2 = 2\rho/a\rho_p$ and

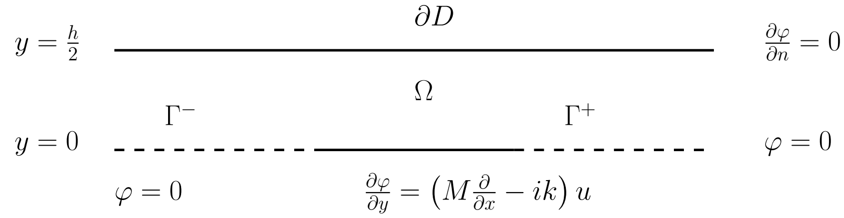

Fig. 2 Half duct

$\mu^4 = D/\rho_p ac^2$. η measures the fluid-structure coupling whereas μ measures the elasticity of the plate. μ and η are not non-dimensional parameters and they are introduced just to simplify the writings. Finally thanks to the change of unknowns $\varphi = \phi/c$ and $u = w/\eta$ and if we define the operator

$$-J_M(k)\varphi = (1 - M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + 2ikM\frac{\partial\varphi}{\partial x},$$

then looking for resonance frequencies is equivalent to solve the following problem:

For $M \in [0, 1[$, find $k \in \mathbb{R}$ such that there exist $(\varphi, u) \neq 0$, $\varphi \in L^2(\Omega)$ satisfying

$$\begin{array}{rcl}
J_M(k)\varphi &=& k^2\varphi \text{ in } \Omega,\\
\mu^4\dfrac{d^4u}{dx^4} - \eta\left(M\dfrac{d}{dx} - ik\right)\varphi &=& k^2 u \text{ on } \Gamma,\\
\varphi &=& 0 \text{ on } \Gamma^+ \cup \Gamma^-,\\
\dfrac{\partial\varphi}{\partial y} &=& 0 \text{ on } \partial D,\\
\dfrac{\partial\varphi}{\partial y} &=& \eta\left(M\dfrac{d}{dx} - ik\right)u \text{ on } \Gamma,\\
u = 0 = \dfrac{du}{dx} && \text{at } x = 0,\\
\dfrac{d^2u}{dx^2} = 0 = \dfrac{d^3u}{dx^3} && \text{at } x = L.
\end{array} \tag{1.3}$$

k^2 appears as an eigenvalue in the right hand side of the equations, but k appears also in the left hand side. Thus we are led to solve a generalized eigenvalue problem which is a classical situation in fluid-structure problems. If (k, φ, u) is solution, $(-k, \overline{\varphi}, \overline{u})$ is also solution, and thus we consider $k \geqslant 0$.

1.3 *Variational formulation, operator form and self-adjointness*

We will first derive the variational formulation associated to equations (1.3). Let us set

$$\Phi = \left\{\varphi \in H^1(\Omega); \varphi = 0 \text{ on } \Gamma^+ \cup \Gamma^-\right\}, \tag{1.4}$$

and

$$U = \left\{u \in H^2(\Gamma); u(0) = 0 = \frac{du}{dx}(0)\right\}, \tag{1.5}$$

where the Sobolev spaces $H^1(\Omega)$ and $H^2(\Gamma)$ are defined by $H^1(\Omega) = \left\{\varphi \in L^2(\Omega); \boldsymbol{\nabla}\varphi \in L^2(\Omega)\right\}$ and $H^2(\Gamma) = \left\{u \in L^2(\Gamma); du/dx \text{ and } d^2u/dx^2 \in L^2(\Gamma)\right\}$. We introduce $W = \Phi \times U$ and we consider the following bilinear forms for (φ, u) and $(\psi, v) \in W$:

$$\begin{aligned}
a_{fluid}(k; \varphi, \psi) &= \int_\Omega \left[(1 - M^2)\frac{\partial\varphi}{\partial x}\frac{\partial\bar{\psi}}{\partial x} - 2ikM\frac{\partial\varphi}{\partial x}\bar{\psi} + \frac{\partial\varphi}{\partial y}\frac{\partial\bar{\psi}}{\partial y}\right], \\
a_{plate}(u, v) &= \mu^4 \int_\Gamma \frac{d^2u}{dx^2}\overline{\frac{d^2v}{dx^2}}, \\
a_{coupling}(k; (\varphi, u), (\psi, v)) &= \int_\Gamma \bar{\psi}\left(M\frac{d}{dx} - ik\right)u + \varphi\left(M\frac{d}{dx} + ik\right)\bar{v},
\end{aligned}$$

and we define the main bilinear form

$$a(k; (\varphi, u), (\psi, v)) = a_{fluid}(k; \varphi, \psi) + a_{plate}(u, v) + \eta\, a_{coupling}(k; (\varphi, u), (\psi, v)).$$

With these notations the problem to solve has the following variational formulation:

> For $M \in [0, 1[$, find $k \in \mathbb{R}^+$ such that there exist $(\varphi, u) \in W, (\varphi, u) \neq 0$ satisfying
> $a(k; (\varphi, u), (\psi, v)) = k^2 \left(\int_\Omega \varphi\bar{\psi} + \int_\Gamma u\bar{v}\right), \forall(\psi, v) \in W.$

This bilinear form $a(k)$ is continuous on $W \times W$ and symmetric. Note that $a(k)$ is not positive even for the static case ($k = 0$) because of the coupling.

Remark 1.1. $a_{coupling}(k)$ is built thanks to the integration by parts:

$$\forall v \in U, \qquad -\int_\Gamma \bar{v}\left(M\frac{d}{dx} - ik\right)\varphi = \int_\Gamma \varphi\left(M\frac{d}{dx} + ik\right)\bar{v}.$$

This relation is immediately satisfied in the case of a clamped trailing edge, whereas in the case of a free trailing edge it is obtained from $\varphi(L, 0) = 0$. Notably it is false if a wake is introduced behind the plate. Then the continuity of pressure through the wake would lead to $[\varphi](x > L, 0) = Fe^{ikx/M}$, where F can be determined thanks to the Kutta condition at the trailing edge (finite velocity) (**13**).

In presence of a wake (as soon as $M \neq 0$) and for a rigid plate it has been found numerically that no real resonance can exist (**14**). Indeed a vortex uses up energy for its formation leading to a lossy system. Since our aim is to derive existence results for resonances, we prefered not to take into account a wake to remain with a self-adjoint problem for which classical tools of spectral theory can be used. In spite of this approximation we think that our results for a problem without wake give relevant informations about the problem with a wake : since the resonance frequencies, real when $F = 0$, move continuously in the complex plane when $F \neq 0$ (**14**), we conjecture that the real parts of the frequencies with a wake are close from the frequencies without wake we consider in this paper. This is particularly true if F is small. We have checked numerically this assertion for $L = 4$ and $L = 6$ and $M = 0$, 0.3, 0.5 and 0.7 and good agreement has been found.

Concerning the operator form of the eigenvalue problem, we denote by $A(k)$ the operator associated to $a(k)$. It is defined as follows:

$$D(A(k)) = \Big\{(\varphi, u) \in W; J_M(k)\varphi \in L^2(\Omega), \mu^4 \frac{d^4 u}{dx^4} - \eta\left(M\frac{d}{dx} - ik\right)\varphi \in L^2(\Gamma),$$

$$\frac{\partial \varphi}{\partial y} = 0 \text{ on } \partial D, \frac{\partial \varphi}{\partial y} = \eta\left(M\frac{d}{dx} - ik\right) u \text{ on } \Gamma, \frac{d^2 u}{dx^2}(L) = 0 = \frac{d^3 u}{dx^3}(L).\Big\}$$

and

$$\forall(\varphi, u) \in D(A(k)), \qquad A(k)\begin{pmatrix}\varphi \\ u\end{pmatrix} = \begin{pmatrix} J_M(k) & 0 \\ -\eta\left(M\dfrac{d}{dx} - ik\right) & \mu^4\dfrac{d^4}{dx^4}\end{pmatrix}\begin{pmatrix}\varphi \\ u\end{pmatrix},$$

where a matricial notation has been used for clarity. Then for every $(\varphi, u) \in D(A(k))$ and $(\psi, v) \in W$, $(A(k)(\varphi, u), (\psi, v)) = a\left(k; (\varphi, u), (\psi, v)\right)$. Note that all the boundary conditions have to be understood in the weak sense. Now the problem of finding resonant modes can be written:

> For $M \in [0, 1[$, find $k \in \mathbb{R}^+$ such that there exists $(\varphi, u) \in D(A(k)), (\varphi, u) \neq (0, 0)$ satisfying : $A(k)(\varphi, u) = k^2(\varphi, u)$.

Therefore we have to study the eigenvalues of the operator $A(k)$. First of all, proceeding as in (**5**) we can prove the

Lemma 1.2. For $k \in \mathbb{R}$, the operator $A(k)$ is self-adjoint.

2. Spectral study

Let us denote by $\sigma(k)$ the spectrum (real) of the operator $A(k)$ which is the union of the discrete spectrum $\sigma_{disc}(k)$ (which contains all isolated eigenvalues with finite multiplicity) and the essential spectrum $\sigma_{ess}(k)$. Our purpose is now to determine $\sigma_{ess}(k)$.

2.1 *The essential spectrum*

Let us introduce the space

$$T = \left\{\varphi \in H^1(\Omega) / \varphi = 0 \text{ on } \Gamma^D\right\}, \tag{2.1}$$

where

$$\Gamma^D = \{(x, 0) \in \partial\Omega\}, \tag{2.2}$$

is the lower boundary of Ω. Now we can find a lower bound for $a_{fluid}(k)$:

LEMMA 2.1. For every φ in T defined in (2.1):

$$a_{fluid}\,(k;\varphi,\varphi) \geqslant \left[\left(\frac{\pi}{h}\right)^2 - \frac{k^2M^2}{1-M^2}\right]\int_\Omega |\varphi|^2.$$

PROOF. Let us denote by $\hat{\varphi}(s,y)$ the partial Fourier transform of $\varphi(x,y)$ with respect to x:

$$\hat{\varphi}(s,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x,y)e^{-isx}dx.$$

By Plancherel's theorem we have for all φ in T:

$$a_{fluid}\,(k;\varphi,\varphi) = \int_0^{h/2}\int_{-\infty}^{\infty}\left\{\left[(1-M^2)s^2 - 2kMs\right]|\hat{\varphi}|^2 + \left|\frac{\partial\hat{\varphi}}{\partial y}\right|^2\right\}dsdy.$$

Since

$$\inf_{s\in\mathbb{R}}\left[(1-M^2)s^2 - 2kMs\right] = -\frac{k^2M^2}{1-M^2},$$

we deduce thanks to lemma A.2 of Appendix A that

$$a_{fluid}\,(k;\varphi,\varphi) \geqslant \left[\left(\frac{\pi}{h}\right)^2 - \frac{k^2M^2}{1-M^2}\right]\int_\Omega |\hat{\varphi}|^2.$$

□

This leads to the

THEOREM 2.2. $\sigma_{ess}(k) = [\gamma^\infty(k), +\infty[$ where

$$\gamma^\infty(k) = \left(\frac{\pi}{h}\right)^2 - \frac{k^2M^2}{1-M^2}. \tag{2.3}$$

The proof is presented in Appendix B.

2.2 *The Min-Max principle*

Here we are interested in the point spectrum of the operator $A(k)$. The eigenvalues λ such that $\lambda < \gamma^\infty(k)$ form the discrete spectrum; they can be studied by means of the Min-Max principle as will be developed in the following. The question of the existence of eigenvalues above $\gamma^\infty(k)$ (embedded eigenvalues) is still open. Note that the eigenvalues in the discrete spectrum associated to the skew-symmetric problem become embedded when considering the general problem (problem without any restriction on the symmetry of the velocity potential).

A main difficulty of our study is that $A(k)$ depends on k (contrary for instance to the simpler case of a rigid plate and a fluid at rest (**5**)). When the fluid is in motion or when the plate is elastic, we have to solve a "non-linear" eigenvalue problem. The procedure to find the resonance frequencies is the following:

- use the Min-Max principle (**17**) to prove the existence of isolated eigenvalues of $A(k)$. We note $\lambda_n(k)$ the n^{th} eigenvalue, ordered by increasing values.

- solve the fixed-point equations $\lambda_n(k) = k^2$ for $k \geqslant 0$ and $n \geqslant 1$.

$\mathcal{N}$ denotes the number of resonance frequencies below k_c. These resonance frequencies k are such that $k^2 < \left(\frac{\pi}{h}\right)^2 - \frac{k^2 M^2}{1-M^2}$, which is equivalent to

$$k < k_c = \frac{\pi}{h}\sqrt{1-M^2}. \tag{2.4}$$

This frequency k_c has a physical interpretation: it is the cut-off frequency of the half wave guide (no wave can propagate in the half duct when $k < k_c$). Indeed looking to the duct modes defined in Appendix A we get easily that if $k < k_c$, $\beta_n^{\pm} \notin \mathbb{R}$ for all n, and all the duct modes are evanescent.

The mathematical tool we will use to study the discrete spectrum of the operator $A(k)$ is the Min-Max principle. Let us recall it.

For every integer $n \geqslant 1$, we set:

$$\gamma_n(k) = \inf_{F \in \mathcal{V}_n(W)} \sup_{(\varphi,u)\in F, (\varphi,u)\neq 0} \mathcal{R}(k;\varphi,u), \tag{2.5}$$

where $\mathcal{V}_n(W)$ denotes the set of all n-dimensional subspaces of W and $\mathcal{R}(k;\varphi,u)$ is the Rayleigh quotient defined as follows:

$$\mathcal{R}(k;\varphi,u) = \frac{a(k;(\varphi,u),(\varphi,u))}{\int_\Omega |\varphi|^2 + \int_\Gamma |u|^2}.$$

In the following we denote $\mathcal{N}(A(k))$ the number of eigenvalues of $A(k)$ located strictly below $\gamma^\infty(k)$ and counted with their multiplicity. For the problem we are interested in here, the Min-Max principle reads as follows:

THEOREM 2.3. The sequence $(\gamma_n(k))_{n\geqslant 1}$ is increasing and converges to $\gamma^\infty(k)$. Moreover, the following alternative holds:

- $\gamma_m(k) < \gamma^\infty(k)$. Then $A(k)$ has at least m eigenvalues (counted with their multiplicity) below $\gamma^\infty(k)$ (i.e. $\mathcal{N}(A(k)) \geqslant m$) and these eigenvalues are the m first Min-Max values $\gamma_1(k), \gamma_2(k), \cdots, \gamma_m(k)$.
- $\gamma_m(k) = \gamma^\infty(k)$ and then $A(k)$ has at most $m-1$ eigenvalues strictly lower than $\gamma^\infty(k)$ (i.e. $\mathcal{N}(A(k)) \leqslant m-1$).

See for example Reed and Simon (**17**) for a proof. To summarize:

$$\mathcal{N}(A(k)) = \max\{n \in \mathbb{N}; \gamma_n(k) < \gamma^\infty(k)\}, \tag{2.6}$$

where we set for convenience $\gamma_0(k) = -\infty$. This convention will be used throughout all the paper.

3. The fixed point equations

Remember that the resonance frequencies are the solutions of $\lambda_n(k) = k^2$ where λ_n is an eigenvalue. The most complicated part is the resolution of these fixed-point equations.

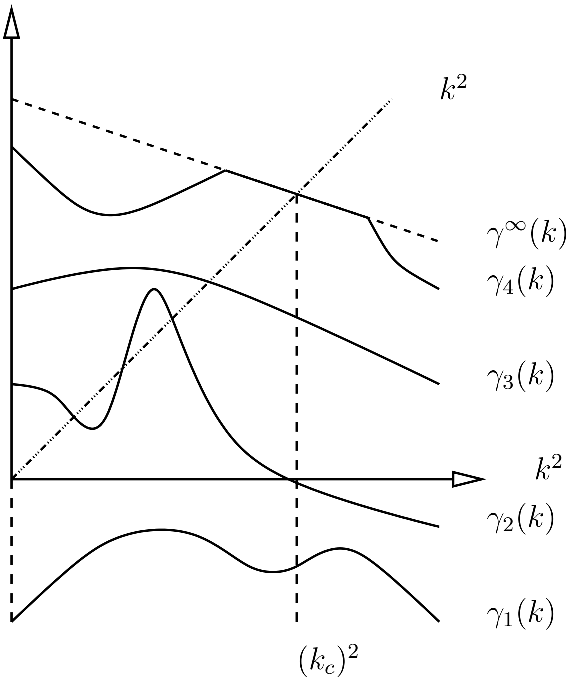

Fig. 3 Different possible behaviors of the eigenvalues

Indeed the existence of an eigenvalue of $A(k)$ does not imply the existence of a solution for the fixed point equation, even if the eigenvalue exists for all the frequencies $k \in [0, k_c[$. This is illustrated on Fig. 3. The solid lines represent the Min-Max values γ_n versus k^2. The lower bound of the essential spectrum $\gamma^\infty(k)$ is drawn as a dashed line whereas the straight dot-dashed line stands for k^2. It is clear that an eigenvalue negative in $k = 0$ might never cross k^2. But it is not sufficient for an eigenvalue to be positive in $k = 0$ to cross k^2 (see γ_4). To ensure the existence of a crossing between an eigenvalue and k^2 we need to know the behaviors of the eigenvalues both in $k = 0$ and $k = k_c$. More precisely we have the

LEMMA 3.1. For all $n \geqslant 1$, if $\gamma_n(0) \geqslant 0$ and $\gamma_n(k_c) < (k_c)^2$ then $\gamma_n(k) = k^2$ has at least one solution $k \in [0, k_c[$.

PROOF. This is a simple consequence of the fact that the functions $k \to \gamma_n(k)$ are continuous. Indeed for any $(\varphi, u) \in W$ the function $k \to \mathcal{R}(k; \varphi, u)$ is continuous and this property is valid for the Min-Max values (**18**). The lemma follows remembering that $\gamma^\infty(k_c) = (k_c)^2$. □

Now we will estimate the total number of resonances $\mathcal{N}$. From the previous lemma we deduce that we need to count the number of Min-Max values lower than $(k_c)^2$ in $k = k_c$ and to determine how many are negative in $k = 0$. We note $\mathcal{N}^{neg}(A(0))$ the number of strictly negative eigenvalues in $k = 0$. We have the

THEOREM 3.2. If $\mathcal{N}^{neg}(A(0)) < \mathcal{N}(A(k_c))$ then $\mathcal{N} \geqslant \mathcal{N}(A(k_c)) - \mathcal{N}^{neg}(A(0))$.

Proof. For all m such that $\mathcal{N}^{neg}(A(0)) < m \leqslant \mathcal{N}(A(k_c))$ we have $\gamma_m(0) \geqslant 0$ and $\gamma_m(k_c) < (k_c)^2$ by definition (2.6) of $\mathcal{N}(A(k_c))$ and since $\gamma^\infty(k_c) = (k_c)^2$. The result is immediately deduced from lemma 3.1. □

Of course this theorem just gives a lower bound of the number of resonances, for two reasons illustrated on Fig. 4:

- a Min-Max value satisfying the hypothesis of lemma 3.1 can cross several times the line k^2,
- a Min-Max value negative in $k = 0$ (and thus not taken into account in theorem 3.2) can cross k^2.

For this example, the estimation is $\mathcal{N} \geqslant 3 - 2 = 1$, the true value being $\mathcal{N} = 5$. We will

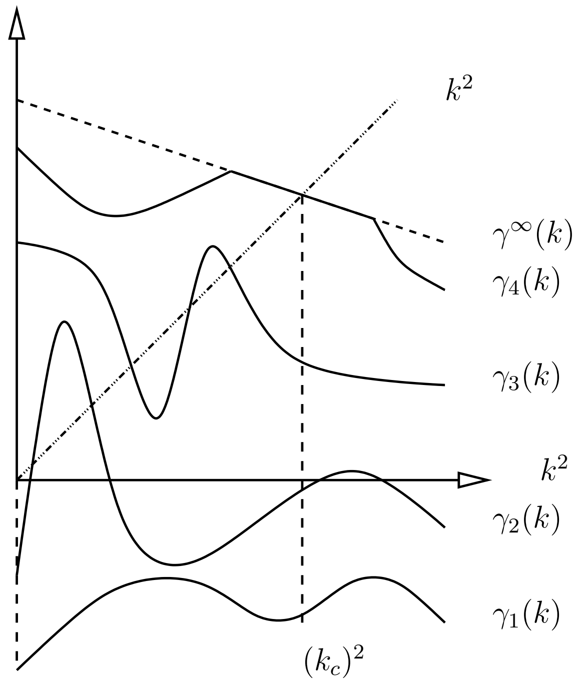

Fig. 4 Possible behaviors of the eigenvalues

now demonstrate that in many situations lower bounds for $\mathcal{N}$ can be found. The procedure to get estimates on the number of resonances will be the following:

1. find a lower bound $\mathcal{N}^+$ for $\mathcal{N}(A(k_c))$,
2. find an upper bound $\mathcal{N}^-$ for the number of negative Min-Max values in $k = 0$.

The final result will be that the number of resonances satisfies $\mathcal{N} \geqslant \mathcal{N}^+ - \mathcal{N}^-$.

4. Number of eigenvalues at $k = k_c$

First we will concentrate on the discrete spectrum in $k = k_c$ defined by (2.4). Some particular situations are simpler and it is why they have been considered in a previous article (**5**): when the plate is rigid (fluid at rest or in motion) or when the plate is elastic in a fluid at rest. In both these cases:

- the functions $k \to \gamma_n(k)/k$ are decreasing. Therefore if $\gamma_n(k)$ crosses k^2 (or equivalently $\gamma_n(k)/k$ crosses k), the intersection is unique,
- the spectrum in $k = 0$ is positive.

Consequently for these cases we have exactly $\mathcal{N} = \mathcal{N}(A(k_c))$. To get lower bounds for $\mathcal{N}(A(k_c))$ in the general case, we will compare an elastic plate in a flow to these two particular configurations and we will prove that

$$\mathcal{N}(A(k_c)) \geqslant \mathcal{N}^+ \equiv \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate}), \tag{4.1}$$

where $\mathcal{N}_{rigid}$ is defined in (4.3) and $\mathcal{N}_{plate}$ is defined in (4.5).

4.1 *Comparison with a rigid plate in a fluid in flow*

When the plate is rigid ($u = 0$) the Rayleigh quotient $\mathcal{R}(k; \varphi, 0)$ reduces to $\mathcal{R}_{rigid}(k; \varphi) = a_{fluid}(k; \varphi, \varphi)/ \|\varphi\|^2_{L^2(\Omega)}$ and the Min-Max values are

$$\gamma_n^{rigid}(k) = \inf_{F \in \mathcal{V}_m(\Phi)} \sup_{\varphi \in F, \varphi \neq 0} \mathcal{R}_{rigid}(k; \varphi), \tag{4.2}$$

where Φ is defined by (1.4). We note

$$\mathcal{N}_{rigid} = \max \left\{ n \in \mathbb{N} \ : \gamma_n^{rigid}(k_c) < (k_c)^2 \right\}, \tag{4.3}$$

and k_n^{rigid} the n^{th} resonance frequency, which is the unique solution of $\gamma_n^{rigid}(k) = k^2$. We have proved in (**5**) that if $m < L/h\sqrt{1 - M^2} \leqslant m + 1$ then $\mathcal{N}_{rigid} = m$ or $m + 1$ (the number of resonance frequencies varies like $L/h\sqrt{1 - M^2}$). For small plates $L < h\sqrt{1 - M^2}$ we have $\mathcal{N}_{rigid} = 1$ (**2**).

Let us prove now that $\mathcal{N}(A(k_c)) \geqslant \mathcal{N}_{rigid}$.

Lemma 4.1. For every integer $n \geqslant 1$ and every real k,

$$\gamma_n(k) \leqslant \gamma_n^{rigid}(k),$$

where $\gamma_n(k)$ is defined by (2.5) and $\gamma_n^{rigid}(k)$ is defined by (4.2).

Proof. Since we have

$$\mathcal{R}_{rigid}(k; \varphi) = \mathcal{R}(k; \varphi, 0) \quad \forall \varphi \in \Phi,$$

the lemma follows from the Min-Max principle and the inclusion $\{(\varphi, 0); \varphi \in \Phi\} \subset W$. □

We deduce from this lemma an existence result and estimates for the eigenvalues of $A(k_c)$:

Corollary 4.2. The operator $A(k_c)$ has at least $\mathcal{N}_{rigid}$ eigenvalues below its essential spectrum where $\mathcal{N}_{rigid}$ has been defined in (4.3).

PROOF.
By the Min-Max principle, $A(k_c)$ has at least m eigenvalues below its essential spectrum if and only if

$$\gamma_m(k_c) < \gamma^\infty(k_c) = (k_c)^2.$$

By the previous lemma, this inequality is a fortiori true if

$$\gamma_m^{rigid}(k_c) < (k_c)^2,$$

which holds for $m \leqslant \mathcal{N}_{rigid}$ (by definition of $\mathcal{N}_{rigid}$). □

4.2 *Comparison with a plate in vacuum*
When the plate is in vacuum ($\varphi = 0$) the Rayleigh quotient $\mathcal{R}(k; \varphi, u)$ reduces to $\mathcal{R}_{plate}(u) = a_{plate}(u,u)/\left\|u\right\|^2_{L^2(\Gamma)}$. We note γ_n^{plate} the n^{th} Min-Max value for an elastic plate in vacuum (see Appendix C). It is defined as

$$\gamma_n^{plate} = \min_{F \in \mathcal{V}_n(U)} \max_{u \in F, u \neq 0} \mathcal{R}_{plate}(u), \tag{4.4}$$

LEMMA 4.3. For every integer $n \geqslant 1$ and every real k,

$$\gamma_n(k) \leqslant \gamma_n^{plate},$$

where $\gamma_n(k)$ is defined by (2.5) and γ_n^{plate} is defined by (4.4).

PROOF. Thanks to the following identity:

$$\mathcal{R}_{plate}(u) = \mathcal{R}(k; 0, u) \quad \forall u \in U,$$

the lemma follows from the inclusion $\{(0, u);\ u \in U\} \subset W$. □

As previously we deduce from this lemma new existence results and estimates for the eigenvalues of $A(k_c)$ (we recall that α_n defined in (C.1) is the square root of the n^{th} eigenfrequency of a plate of rigidity $\mu = 1$ and of length $L = 1$):

COROLLARY 4.4. The operator $A(k_c)$ has at least $\mathcal{N}_{plate}$ eigenvalues below its essential spectrum where:

$$\mathcal{N}_{plate} = \max\left\{ n \in \mathbb{N} \ : \gamma_n^{plate} = \left(\frac{\mu \alpha_n}{L}\right)^4 < \left(\frac{\pi}{h}\right)^2 (1 - M^2) \right\}. \tag{4.5}$$

PROOF. We proceed as in the proof of corollary 4.2. □

REMARK 4.5. $\mathcal{N}_{plate}$ is the number of eigenfrequencies of the plate in vacuum $k_n^{plate} = \sqrt{\gamma_n^{plate}}$ (see (C.2)) below the cut-off frequency k_c. Thus this is the number of frequencies of the plate which cannot " radiate away " in the fluid (remember that above k_c no wave can propagate in the duct, see Appendix A).

REMARK 4.6. It is easy to prove that $(n-1)\pi < \alpha_n < n\pi$ for all $n \in \mathbb{N}^*$ (α_n is defined in Appendix C), and also that $\alpha_n \sim \left(n - \frac{1}{2}\right)\pi$ when $n \to \infty$. Therefore using $k_n^{plate} = (\mu\alpha_n/L)^2$ (C.1) we get that if $m < \dfrac{L\left(1-M^2\right)^{1/4}}{\mu\left(\pi h\right)^{1/2}} \leqslant m+1$ then $\mathcal{N}_{plate} = m$ or $m+1$ ($\mathcal{N}_{plate}$ behaves like $L\left(1-M^2\right)^{1/4}/\sqrt{h}$). In particular $\mathcal{N}_{plate}$ is a decreasing function of M, contrary to $\mathcal{N}_{rigid}$.

5. Number of negative eigenvalues at $k = 0$

We need now to count the number of negative eigenvalues of $A(k)$ in $k = 0$. As already mentioned, this is immediate is some particular situations: if

i) the fluid is at rest,
ii) or the plate is rigid,
iii) or the plate and the fluid are uncoupled,

then all the eigenvalues are positive in $k = 0$. Indeed

$$\mathcal{R}(0;\varphi,u) = \frac{\displaystyle\int_\Omega \left[(1-M^2)\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2\right] + \mu^4 \int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2 + 2M\eta\,\mathrm{Re}\left(\int_\Gamma \bar\varphi\,\frac{du}{dx}\right)}{\displaystyle\int_\Omega |\varphi|^2 + \int_\Gamma |u|^2}$$

and $\mathcal{R}(0;\varphi,u) > 0$ as soon as $M = 0$ or $u = 0$ or $\eta = 0$. In these situations it is easy to conclude that $\gamma_n(0) > 0$ for all n since

$$\gamma_n(0) = \inf_{F\in\mathcal{V}_n(W)} \sup_{(\varphi,u)\in F,(\varphi,u)\neq 0} \mathcal{R}(0;\varphi,u).$$

In the general case we will derivate a condition for $\gamma_n(0)$ to be positive. To do this we establish the following inequality:

Lemma 5.1. For every integer $n \geqslant 1$:

$$(1-\varepsilon)\,\sigma_n - \frac{8}{h^2}\varepsilon \leqslant \gamma_n(0),$$

where $\varepsilon = \dfrac{M\eta L\sqrt{h}}{2\mu^2}$ and

$$\sigma_n = \inf_{F\in\mathcal{V}_n(W)} \sup_{(\varphi,u)\in F,(\varphi,u)\neq 0} \frac{\displaystyle\int_\Omega \left[(1-M^2)\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2\right] + \mu^4 \int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2}{\displaystyle\int_\Omega |\varphi|^2 + \int_\Gamma |u|^2}. \tag{5.1}$$

Proof. Let us consider the splitting $a(0;(\varphi,u),(\varphi,u)) = b(\varphi,u) + c(\varphi,u)$ where we have introduced the positive part of $a(0)$

$$b(\varphi,u) = \int_\Omega \left[(1-M^2)\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2\right] + \mu^4 \int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2,$$

and the coupling term

$$c(\varphi,u) = 2M\eta\,\mathrm{Re}\left(\int_\Gamma \bar\varphi\,\frac{du}{dx}\right).$$

We get immediately:

$$a(0;(\varphi,u),(\varphi,u)) \geqslant (1-\varepsilon)b(\varphi,u) + \varepsilon\left(b(\varphi,u) - \frac{|c(\varphi,u)|}{\varepsilon}\right).$$

Now we will find a lower bound for $b(\varphi,u) - |c(\varphi,u)|/\varepsilon$. Thanks to Poincaré's inequality : $\forall u \in U$ defined in (1.5),

$$\int_\Gamma \left|\frac{du}{dx}\right|^2 \leqslant \frac{L^2}{2} \int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2,$$

we deduce that:

$$\begin{aligned} \frac{|c(\varphi,u)|}{\varepsilon} &\leqslant 2\left(\sqrt{\frac{2}{h}}\mu^2 \,||\varphi||_{L^2(\Gamma)}\right) \left|\left|\frac{d^2u}{dx^2}\right|\right|_{L^2(\Gamma)}, \\ &\leqslant \frac{2}{h}\int_\Gamma |\varphi|^2 + \mu^4 \int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2, \end{aligned}$$

thanks to Young's inequality. Then using the trace inequality : $\forall \varphi \in \Phi$ defined in (1.4)

$$\int_\Gamma |\varphi|^2 \leqslant \frac{4}{h}\int_\Omega |\varphi|^2 + \frac{h}{2}\int_\Omega \left|\frac{\partial \varphi}{\partial y}\right|^2,$$

we get

$$b(\varphi,u) - \frac{|c(\varphi,u)|}{\varepsilon} \geqslant -\frac{8}{h^2}\int_\Omega |\varphi|^2 .$$

The trace inequality is derived by using an alternative of the Poincaré's inequality for a fixed value of x, and then by integrating along the x axis on the plate Γ. Finally is deduced the inequality

$$\frac{a(0;(\varphi,u),(\varphi,u))}{\displaystyle\int_\Omega |\varphi|^2 + \int_\Gamma |u|^2} \geqslant (1-\varepsilon)\frac{b(\varphi,u)}{\displaystyle\int_\Omega |\varphi|^2 + \int_\Gamma |u|^2} - \frac{8}{h^2}\varepsilon,$$

which extends to the min-max values. □

Note that $\sigma_n \neq 0$ since $\forall(\varphi,u) \in W$ with $\varphi \neq 0$ and $u \neq 0$,

$$\begin{aligned} \frac{b(\varphi,u)}{\displaystyle\int_\Omega |\varphi|^2 + \int_\Gamma |u|^2} &\geqslant \min\left\{\frac{\displaystyle\int_\Omega (1-M^2)\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2}{\displaystyle\int_\Omega |\varphi|^2}, \mu^4\frac{\displaystyle\int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2}{\displaystyle\int_\Gamma |u|^2}\right\}, \\ &\geqslant \min\left\{\gamma_1^{rigid}(0), \gamma_1^{plate}\right\} \end{aligned}$$

Corollary 5.2. If

$$\eta \leqslant g_n(M,\mu,L,h), \tag{5.2}$$

defined as

$$g_n(M,\mu,L,h) \equiv \frac{2}{L\sqrt{h}}\frac{\mu^2}{M^2}\frac{\sigma_n}{\sigma_n + \dfrac{8}{h^2}},$$

then $A(0)$ has at most $n-1$ strictly negative eigenvalues.

Proof. By lemma 5.1, $\gamma_n(0) \geqslant 0$ (or equivalently $\mathcal{N}^{neg}(A(0)) \leqslant n-1$) as soon as

$$\varepsilon \leqslant \frac{\sigma_n}{\sigma_n + \dfrac{8}{h^2}}.$$

Using the definition of ε and as σ_n is independent of η, this can be written equivalently $\eta \leqslant g_n(M, \mu, L, h)$. □

As the function $\gamma \to \gamma/(\gamma + 8/h^2)$ is increasing and as we can prove that $0 < \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \pi^2/h^2$, one has

$$g_1(M, \mu, L, h) \leqslant g_2(M, \mu, L, h) \leqslant \cdots \leqslant g_\infty \equiv \frac{2}{L\sqrt{h}} \frac{\mu^2}{M^2} \frac{\pi^2}{\pi^2 + 8}.$$

In particular, if $\eta > g_\infty$, no integer satisfy (5.2) : our approach does not provide in this case an upper bound for the negative eigenvalues of $A(0)$.

This leads us finally to define an integer $\mathcal{N}^-$ as follows:

$$\mathcal{N}^- = n - 1, \tag{5.3}$$

where n is the smallest integer such that $\eta \leqslant g_n(M, \mu, L, h)$. If no integer satisfy this inequality, we set $\mathcal{N}^- = +\infty$.

6. Existence results

We can now deduce a lower bound for the number of resonances. We know from corollaries 4.2 and 4.4 that the operator $A(k_c)$ has at least $\mathcal{N}^+$ (defined in (4.1)) eigenvalues. On the other hand, $A(0)$ has at most $\mathcal{N}^-$ negative eigenvalues. Theorem 3.2 gives the conclusion:

$$\mathcal{N} \geqslant \mathcal{N}^+ - \mathcal{N}^-, \tag{6.1}$$

where $\mathcal{N}^+$ is defined by (4.1) and $\mathcal{N}^-$ is defined by (5.3). It is in general not possible to deduce from (6.1) an explicit lower bound for $\mathcal{N}$, because, contrary to $\mathcal{N}^+$, $\mathcal{N}^-$ is not explicit. However one case is particularly simple: when η is small.

6.1 *An existence criterium for weakly coupled systems*

Lemma 6.1. If $\eta \leqslant g_1(M, \mu, L, h)$ then $\mathcal{N} \geqslant \mathcal{N}^+$.

Proof. This is a consequence of the fact that if $\eta \leqslant g_1(M, \mu, L, h)$, then $\mathcal{N}^- = 0$. □

As $g_1(M, \mu, L, h) \to +\infty$ when $M \to 0$ or when $\mu \to +\infty$, this holds not only for small values of η but also for small values of M or for large values of μ.

As it could be expected, the results for weak coupling are very similar to the cases treated in (**5**). Some general behaviors are recovered: there are more resonances when the plate is elastic than rigid. Moreover the number of resonances is at least equal to the number of eigenfrequencies of the plate in vacuum below the cut-off frequency of the waveguide.

6.2 *Dependance versus the parameters in the general case*

Inequality (6.1) provides a lower bound for the number of resonances. We are now interested in its dependence with respect to the different parameters η, M, μ and L. Note that $\mathcal{N}^+ = \mathcal{N}^+(M, \mu, L, h)$ depends on the variables M, μ, L and h whereas $\mathcal{N}^- = \mathcal{N}^-(\eta, M, \mu, L, h)$ depends on the same variables and also on the coupling parameter η. The case of the coupling parameter η is quite simple:

Lemma 6.2.

- $\mathcal{N}^- = \mathcal{N}^-(\eta, M, \mu, L, h)$ is an increasing function of η and therefore $\mathcal{N}^+ - \mathcal{N}^- = \mathcal{N}^+(M, \mu, L, h) - \mathcal{N}^-(\eta, M, \mu, L, h)$ is a decreasing function of η.
- If
$$\eta > g_\infty = \frac{2}{L\sqrt{h}} \frac{\mu^2}{M^2} \frac{\pi^2}{\pi^2 + 8},$$
then $\mathcal{N}^- = +\infty$ and therefore no lower bound for $\mathcal{N}$ is obtained.

The dependence versus other parameters is more complicated.

Let us first notice that $\mathcal{N}^+$ and $\mathcal{N}^-$ have a monotonic dependence versus some of them.

- By (4.5), $\mathcal{N}_{plate}$ is a decreasing function of μ/L and M. On the other hand, as proved in (**5**) and in accordance with estimate (4.3), $\mathcal{N}_{rigid}$ is an increasing function of L and M. We deduce that $\mathcal{N}^+$ is an increasing function of L and a decreasing function of μ since $\mathcal{N}_{rigid}$ is independent of μ. The behavior versus M is unknown.
- For any functions $(\varphi, u) \in W$, $(M, \mu) \to b(\varphi, u)$ is an increasing function of μ and a decreasing function of M, and this remains true for the Min-Max values γ_n. As a consequence, $\mathcal{N}^-$ is an increasing function of M and a decreasing function of μ. Moreover, $\mathcal{N}^-$ is an increasing function of L (the proof is similar to that of lemma 3.5 in (**5**)).

Summing up, both $\mathcal{N}^+(M, \mu, L, h)$ and $\mathcal{N}^-(\eta, M, \mu, L, h)$ are increasing functions of L and decreasing functions of μ so that the behavior of $\mathcal{N}^+ - \mathcal{N}^-$ with respect to these parameters is unknown.

Finally $\mathcal{N}^-(\eta, M, \mu, L, h)$ is an increasing function of M and $\mathcal{N}^+(M, \mu, L, h)$ is generally not a monotonic function of M. This particular behavior will be illustrated numerically in section 7, where it will be shown that $\mathcal{N}$ is also generally not a monotonic function of M.

As already mentioned in lemma 6.1, the only simple situation corresponds to parameters satisfying $\eta \leqslant g_1(M, \mu, L, h)$ which holds not only for small values of η but also for small values of M or for large values of μ. This will be illustrated in the next section.

Remark 6.3. In fact, we have not only lower estimates for the number of resonances but also upper estimates for the resonance frequencies themselves. Indeed, from lemmas 4.1 and 4.3, denoting by k_n the n^{th} resonance frequency, we deduce that :

$$\begin{cases} k_n \leqslant k^{rigid}_{n+\mathcal{N}^-} & \text{for } n = 1 \text{ to } \mathcal{N}_{rigid} - \mathcal{N}^-, \\ k_n \leqslant k^{plate}_{n+\mathcal{N}^-} & \text{for } n = 1 \text{ to } \mathcal{N}_{plate} - \mathcal{N}^-. \end{cases}$$

In particular
$$k_n \leqslant \min(k^{rigid}_{n+\mathcal{N}^-}, k^{plate}_{n+\mathcal{N}^-})$$
for $n = 1$ to $\min(\mathcal{N}_{rigid}, \mathcal{N}_{plate}) - \mathcal{N}^-$.

7. Numerical study

In order to illustrate our theoretical results, we have determined numerically the resonances of an elastic plate in presence of a flow. For a given frequency k, the eigenvalues of the operator $A(k)$ are computed numerically thanks to a finite element method.

For the comparison with the theoretical results, we need to calculate explicitly $\mathcal{N}_{rigid}$ and $\mathcal{N}_{plate}$. $\mathcal{N}_{plate}$ is the number of eigenfrequencies of the plate below the cut-off frequency k_c (remark 4.5). Since we know explicitly these frequencies (see Appendix C), $\mathcal{N}_{plate}$ is very easy to determine. Concerning $\mathcal{N}_{rigid}$, let us recall that it is defined as the integer part of $L/h\sqrt{1-M^2}$ or this number plus one. In the following we will take $\mathcal{N}_{rigid} = m$ if $m < L/h\sqrt{1-M^2} < m + 1/2$ and $\mathcal{N}_{rigid} = m + 1$ if $m + 1/2 \leqslant L/h\sqrt{1-M^2} < m + 1$ (when the plate is rigid and the fluid in flow, we have found numerically that this is exactly the right number of resonances (**5**)).

Before determining numerically the number of resonances for various limit cases, let us present some general results. First we will illustrate that $\mathcal{N}^+ - \mathcal{N}^-$ is not the number of resonances but just a lower bound. Then we will illustrate that the dependance of $\mathcal{N}^+$ versus M is complicated. For a duct of height $h = 4$ and an elastic plate of length $L = 10$ (as for all the numerical studies presented hereafter), Fig. 5 illustrates a typical result for $\mu = 0.5$.

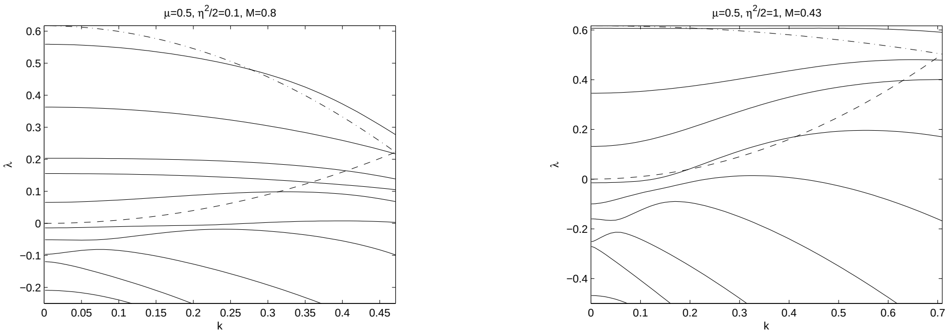

Fig. 5 For $\mu = 0.5$, $-$: $\lambda_n(k)$, $--$: k^2 and $.-$: $\gamma^\infty(k)$

Fig. 5 (a) gives the variations of the first ten eigenvalues $\lambda_n(k)$ versus the frequency $k \leqslant k_c$ for $M = 0.8$ and $\eta^2 = 0.2$. The lower bound of the essential spectrum $\gamma^\infty(k)$ is represented as a dot-dashed line. We find that $\mathcal{N}^+ = 9$ eigenvalues are below the essential spectrum in $k = k_c$. $\mathcal{N}^- = 5$ eigenvalues are negative in $k = 0$ and thus we know that $\mathcal{N} \geqslant 4$. Counting the number of intersections of $\lambda_n(k)$ with k^2 represented as a dashed line, we find that $\mathcal{N} = 4$.

Fig. 5 (b) concerns the case $M = 0.43$ and $\eta^2 = 2$. Here $\mathcal{N}^+ = 8$ and $\mathcal{N}^- = 6$ and thus $\mathcal{N} \geqslant 2$. Since the sixth eigenvalue, negative in $k = 0$, crosses two times k^2, we find $\mathcal{N} = 4$ intersections of $\lambda_n(k)$ with k^2, confirming that $\mathcal{N}^+ - \mathcal{N}^-$ is only a lower bound of the number of resonances.

On Fig. 6 are represented the first twelve eigenvalues λ_n in $k = k_c$ for $M = 0.4$ and $\eta^2 = 0.02$. The lower bound of the essential spectrum $\gamma^\infty(k)$ is represented as a dashed line. Here we focus on $\mathcal{N}^+(M)$ the number of eigenvalues below this line. We see that the ninth eigenvalue crosses several times $\gamma^\infty(k)$, these intersections being marked by a vertical dot-dashed line. Therefore $\mathcal{N}^+(M)$ oscillates between eight and nine and no simple dependence versus M appears obviously.

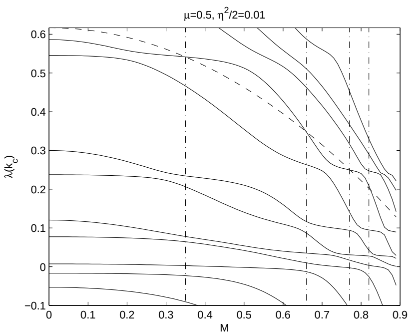

Fig. 6 For $\mu = 0.5$, $\eta^2/2 = 0.01$ and $M = 0.4$, $-$: $\lambda_n(k_c)$, $--$: k_c^2.

7.1 *Influence of the coupling parameter* η

The results are presented on Fig. 7 for $\mu = 0.5$, $M = 0.6$ and $M = 0.8$. The calculated numbers of resonances are represented as circles. The solid lines represent $\mathcal{N}_{rigid}$ and the dashed lines stand for $\mathcal{N}_{plate}$ (when $M = 0.8$ both values are equal). All these lines are horizontal because neither $\mathcal{N}_{rigid}$ nor $\mathcal{N}_{plate}$ depends of η. We observe that in both cases, $\mathcal{N}$ is a decreasing function of η, as could be expected from the mononicity of its lower bound $\mathcal{N}^+ - \mathcal{N}^-$ (lemma 6.2). This confirms that an increasing coupling between an elastic plate and a flow reduces the number of resonances. Also as expected from the same lemma, $\mathcal{N} \geqslant \mathcal{N}^+ = \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate})$ for η small enough ($\eta^2/2 \leqslant \eta_c^2/2 = 0.1$), when the fluid and the plate become uncoupled enough. For smaller values of η, $\eta \leqslant \eta_0$ the fluid and the plate become entirely uncoupled and $\mathcal{N} = \mathcal{N}_{rigid} + \mathcal{N}_{plate}$. Note that when $M = 0$ we always have $\mathcal{N} = \mathcal{N}_{rigid} + \mathcal{N}_{plate}$ whatever are the values of η and μ (see also Fig. 9) but we are unable to prove this result. Here $\eta_0^2/2 = 10^{-4}$ for $M = 0.6$ and $\eta_0^2/2 = 10^{-3}$ for $M = 0.8$. η_0 seems to be an increasing function of M.

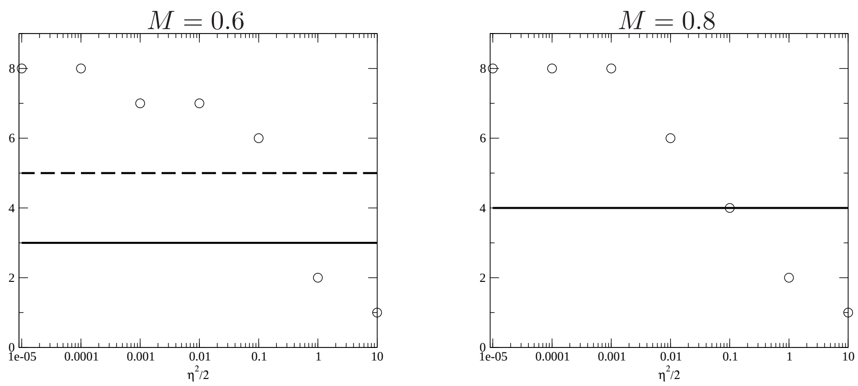

Fig. 7 For $\mu = 0.5$ and $M = 0.8$, $-$: $\mathcal{N}_{rigid}$, $--$: $\mathcal{N}_{plate}$ and $\circ$: $\mathcal{N}$

7.2 *Influence of the rigidity* μ

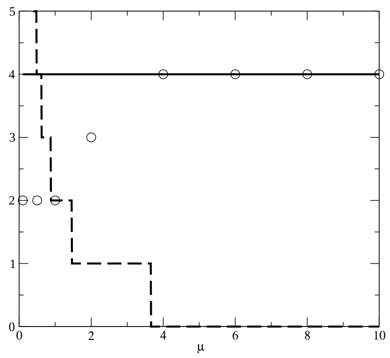

Fig. 8 For $\eta^2/2 = 1$ and $M = 0.8$, $-$: $\mathcal{N}_{rigid}$, $--$: $\mathcal{N}_{plate}$ and $\circ$: $\mathcal{N}$

On Fig. 8 are represented as circles the calculated numbers of resonances versus μ in the case $\eta^2 = 2$ and $M = 0.8$. The solid line representing $\mathcal{N}_{rigid}$ is horizontal because this number is independant of μ. The dashed line representing $\mathcal{N}_{plate}$ tends to infinity when $\mu \to 0$ (see (4.5)) and is equal to zero as soon as $\gamma_1^{plate} = (\mu\alpha_1/L)^4 > (\pi/h)^2 (1-M^2)$. The numerical study confirms that $\mathcal{N} \geqslant \mathcal{N}_{rigid}$ for μ large enough ($\mathcal{N} \geqslant \mathcal{N}^+$ and $\mathcal{N}^+ = \mathcal{N}_{rigid}$ because $\mathcal{N}_{plate} = 0$ for large rigidity). More precisely we find that $\mathcal{N} = \mathcal{N}_{rigid}$ for μ large enough (rigid plate) as could be expected (it is rigorously proved for $\mu = \infty$ (**5**)). We observe here that $\mathcal{N}$ is an increasing function of μ, contrary to $\mathcal{N}^+$.

7.3 *Influence of the Mach number M*

The number of resonance frequencies for a plate corresponding to $\mu = 0.5$ and for the values of the fluid-plate coupling $\eta^2/2 = 10,\ 1,\ 0.1$ and 0.01 is drawn on Fig. 9 versus M. For $\eta^2/2 \geqslant 0.1$ the lower estimate $\mathcal{N} \geqslant \mathcal{N}^+$ becomes valid only for M small enough as expected ($M \leqslant M_c$) and the critical Mach number M_c is found to be a decreasing function of η. Moreover for $\eta^2/2 = 10^{-2}$ (and this remains true for $\eta^2/2 \leqslant 10^{-2}$), we get $M_c = 1$ (which means that the estimate is valid everywhere).

Let us mention that for very small values of η, we get numerically $\mathcal{N} = \mathcal{N}_{rigid} + \mathcal{N}_{plate}$ for all $M \in [0, 1[$. This is rigorously valid in the uncoupled case $\eta = 0$. This is illustrated by Fig. 10 where the dot-dashed lines correspond to $\mathcal{N}_{rigid} + \mathcal{N}_{plate}$. Finally, we notice that $\mathcal{N}$ is generally not a monotonic function of M and can even be non convex. However, $\mathcal{N}$ seems to be always decreasing for small values of M : roughly speaking, adding a slow flow reduces the number of resonances.

8. Conclusion

We have determined the trapped modes for a fluid-structure problem: an elastic plate in a confined uniform compressible flow. We showed that this problem imposes to solve a non-linear eigenvalue problem. We derived a lower estimate for the number of resonances

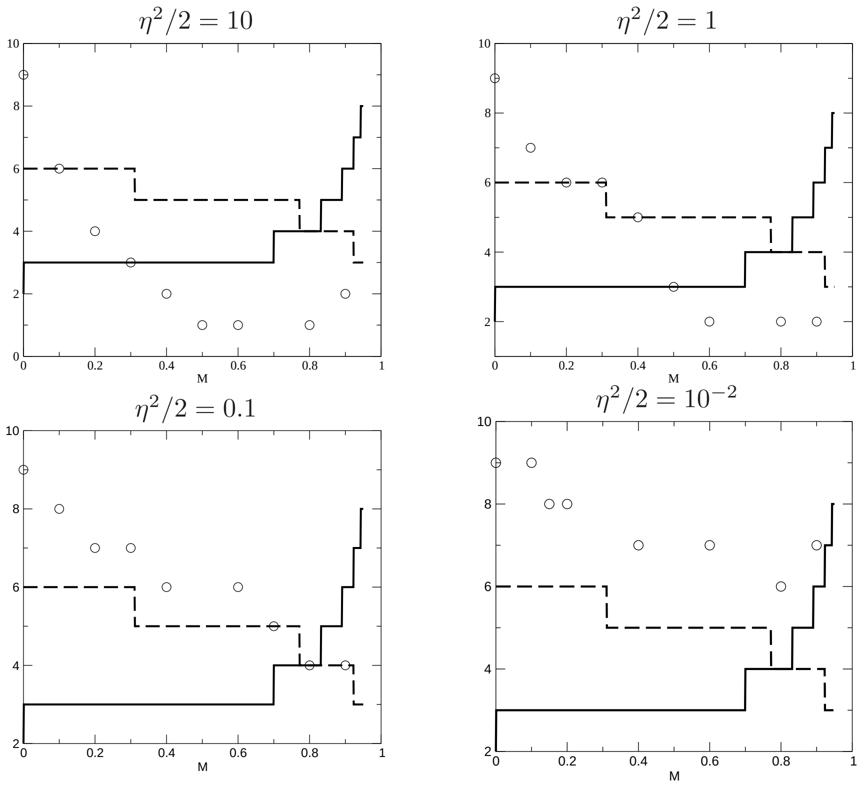

Fig. 9 For $\mu = 0.5$, $-$: $\mathcal{N}_{rigid}$, $--$: $\mathcal{N}_{plate}$ and $\circ$: $\mathcal{N}$

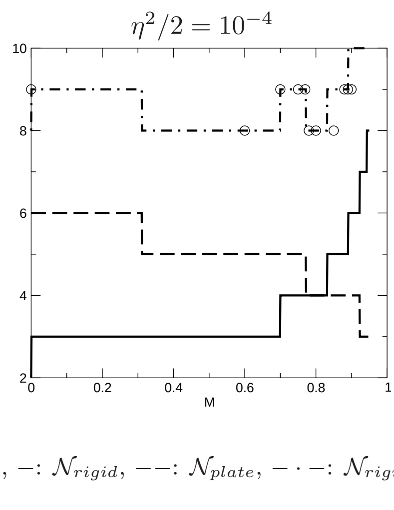

Fig. 10 For $\mu = 0.5$, $-$: $\mathcal{N}_{rigid}$, $--$: $\mathcal{N}_{plate}$, $-\cdot-$: $\mathcal{N}_{rigid} + \mathcal{N}_{plate}$ and $\circ$: $\mathcal{N}$

as the difference between the number of eigenvalues and the number of negative eigenvalues of two linear self-adjoint problems. The dependence of this lower bound with respect to the different parameters of the problem is discussed. In particular, we proved that if the coupling parameter η is small enough, or if the Mach number M is small enough, or if the rigidity of the plate μ is large enough, then the number of resonances satisfies $\mathcal{N} \geqslant \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate})$ and this lower bound can be explicitly determined : $\mathcal{N}_{rigid} \sim L/h\sqrt{1-M^2}$ and $\mathcal{N}_{plate} \sim L\left(1-M^2\right)^{1/4}/\mu\,(\pi h)^{1/2}$ are the number respectively of resonances for a rigid plate and of eigenfrequencies below the cut-off frequency for an elastic plate in vacuum.

These estimates have been confirmed numerically. In particular the numerical experiments confirmed the hypothesis deduced theoretically : the coupling effects of the flow and the elasticity of the plate reduce the number of resonances. Moreover for small values of η we found numerically the more precise estimate $\mathcal{N} = \mathcal{N}_{rigid} + \mathcal{N}_{plate}$.

In the case of a clamped trailing edge $(u(L) = 0 = u'(L))$, all the results presented are valid if the space U for the plate deformations is replaced with $V = \{u \in H^2(\Gamma)/u(0) = 0 = u'(0)$ and $u(L) = 0 = u'(L)\}$ and if the m^{th} resonance frequency for a clamped plate of unit length α_m is sought as a solution of $\cos\alpha_m \cosh\alpha_m = 1$. Thanks to the min-max principle we get easily that $\mathcal{N}_{plate}$ is smaller in the case of a clamped trailing edge.

In fluid-structure problems it is usual to look for real resonances (self-adjoint problems). In our case, the addition of a flow to the fluid-structure problem can lead to complex resonances $k \in \mathbb{C}\backslash\mathbb{R}$: these correspond to instabilities (exponential growing in time $\sim e^{\mathrm{Im}\,(k)t}$ of the associated eigenmode). Indeed suppose a resonant mode $(\varphi, u) \neq 0$ exists, satisfying $a\,(k; (\varphi, u), (\psi, v)) = k^2\left((\varphi, u), (\psi, v)\right), \forall(\psi, v) \in W$ with $k \in \mathbb{C}$. Taking $(\psi, v) = (\varphi, u)$, multiplying by $-\bar{k}$ and taking the imaginary part leads to

$$\mathrm{Im}\,(k)\left[\int_\Omega (1-M^2)\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2 + |k\varphi|^2 + \int_\Gamma \mu^4\left|\frac{d^2u}{dx^2}\right|^2 + |ku|^2 + 2M\eta\,\mathrm{Re}\left(\bar{\varphi}\frac{du}{dx}\right)\right] = 0.$$

In some particular situations only real resonances exist: indeed if the fluid is at rest $(M = 0)$ or the plate is rigid $(u = 0)$, then $\mathrm{Im}\,(k) \neq 0$ implies $(\varphi, u) = 0$. It is no longer the case when the plate is elastic and in presence of a flow: the coupling of elasticity and flow can create instabilities. As a first step we have focused in this paper on the determination of real resonances $k \in \mathbb{R}$. We are currently searching numerically unstable frequencies (we found some for particular values of the problem's parameters) which will be presented in a forthcoming article. Note that these unstable states are different from the resonant states as defined in (**20**): although the later states are associated to a complex k they do not correspond to trapped modes (since $\varphi \notin L^2(\Omega)$). Also resonant states exist when the fluid is at rest or the plate is rigid.

The difficulty of the present study comes from the presence of both the duct and the plate. For the simpler problems of either the duct, without the plate, or the elastic plate in the vacuum, analytical calculations can be easily made. We recall here some classical results, which will be useful in the following.

APPENDIX A

Modes of the duct

The acoustic propagation in the duct, in presence of a uniform flow, can be completely described by computing the *modes* of the duct. These modes are the solutions of (1.1) with separated variables, belonging to T defined in (2.1). Let us point out that every fluid vibration can de decomposed on these modes. The modes read $\varphi_n^\pm = e^{i\beta_n^\pm x}\sin(\zeta_n y)$, $n \in \mathbb{N}$, where $\zeta_n = \pi(2n+1)/h$,

$n \geqslant 0$. A finite number of propagative modes exists, corresponding to $2n+1 < kh/\pi\sqrt{1-M^2}$, associated to the wave numbers $\beta_n^\pm = \left[-kM \pm \sqrt{k^2-(1-M^2)\zeta_n^2}\right]/1-M^2$. The + modes correspond to a positive group velocity and thus propagate downstream, whereas the − modes propagate upstream. Note that the + propagative modes for $kh/\pi < 2n+1$ are particular: they are associated to a negative phase speed and thus they are called the inverse upstream modes.

The other modes for $2n+1 > kh/\pi\sqrt{1-M^2}$ are evanescent, associated to the wave numbers $\beta_n^\pm = \left[-kM \pm i\sqrt{(1-M^2)\zeta_n^2-k^2}\right]/1-M^2$.

The following mathematical result, related to the $(\zeta_n)_{n\geqslant 0}$, will be used in the following:

Lemma A.1.

$$\int_0^{\frac{h}{2}} \left|\frac{d\varphi}{dy}\right|^2 \geqslant \left(\frac{\pi}{h}\right)^2 \int_0^{\frac{h}{2}} |\varphi|^2 ,$$

for every $\varphi \in X$ where

$$X = \left\{\varphi \in H^1\left(0, \frac{h}{2}\right); \varphi(0) = 0\right\}. \tag{A.1}$$

Proof. Let us consider the operator C_{fluid} defined as follows:

$$D(C_{fluid}) = \left\{\varphi \in H^2\left(0, \frac{h}{2}\right); \varphi(0) = 0 \text{ and } \frac{d\varphi}{dy}\left(\frac{h}{2}\right) = 0\right\}$$

and

$$C_{fluid}\,\varphi = -\frac{d^2\varphi}{dy^2} \text{ for } \varphi \in D(C_{fluid}).$$

The operator C_{fluid} is self-adjoint and has a compact resolvent. As a consequence, its spectrum consists in a sequence of eigenvalues which are the $(\zeta_n)_{n\geqslant 0}$ defined above, the eigenfunctions $\theta_n(y) = 2\sin(\zeta_n y)/\sqrt{h}$ form an orthonormal basis of $L^2(0, h/2)$ and one has:

$$\zeta_0 = \min_{\varphi\in X, \varphi\neq 0} \frac{\displaystyle\int_0^{\frac{h}{2}} \left|\frac{d\varphi}{dy}\right|^2}{\displaystyle\int_0^{\frac{h}{2}} |\varphi|^2}.$$

This proves the lemma since $\zeta_0 = (\pi/h))^2$. □

This lemma can be extended to the whole domain Ω:

Lemma A.2. For all φ in T, where T is defined by (2.1), $\displaystyle\int_\Omega \left|\frac{\partial\varphi}{\partial y}\right|^2 \geqslant \left(\frac{\pi}{h}\right)^2 \int_\Omega |\varphi|^2$.

Proof. Let $\varphi \in Y = \{\varphi \in C^\infty(\overline{\Omega}) \text{ with } \varphi = 0 \text{ on } \Gamma^D\}$, where Γ^D has been defined by (2.2). For all $x \in \mathbb{R}$ the function $y \to \varphi(x, y)$ belongs to X defined in (A.1). Thus thanks to lemma A.1 we can write

$$\forall x \in \mathbb{R}, \int_0^{\frac{h}{2}} \left|\frac{\partial\varphi}{\partial y}\right|^2 dy \geqslant \left(\frac{\pi}{h}\right)^2 \int_0^{\frac{h}{2}} |\varphi|^2\, dy.$$

The lemma follows thanks to an integration with respect to x and to the density of Y in T.

□

APPENDIX B

Essential spectrum

We give here the proof of theorem 2.2 indicating that the essential spectrum is $\sigma_{ess}(k) = \left[(\pi/h)^2 - \left(k^2M^2/1-M^2\right), +\infty\right[$.

We define the scalar product on $L^2(\Omega) \times L^2(\Gamma)$ by

$$((\varphi, u), (\psi, v)) = \int_\Omega \varphi\bar{\psi} + \int_\Gamma u\bar{v},$$

and the associated norm is noted $||\cdot||$. We recall that $W = \Phi \times U$ where Φ is defined by (1.4) and U is defined by (1.5).

B.1 *A lower bound for the essential spectrum*

First we prove that $\sigma_{ess}(k) \subset \left[(\pi/h)^2 - \left(k^2M^2/1-M^2\right), +\infty\right[$. In this aim suppose $\gamma \in \sigma_{ess}(k)$ and consider a singular sequence $(\varphi_n, u_n) \in D(A(k))$ (as defined in (**16**)) such that:

$$||(\varphi_n, u_n)|| = 1, \tag{B.1}$$

$$||(A(k)-\gamma)(\varphi_n, u_n)|| \to 0, \tag{B.2}$$

$$(\varphi_n, u_n) \rightharpoonup 0 \text{ weakly in } L^2(\Omega) \times L^2(\Gamma). \tag{B.3}$$

By (B.1) and (B.2):

$$\gamma = \lim_{n\to\infty} a(k; (\varphi_n, u_n), (\varphi_n, u_n)). \tag{B.4}$$

To prove the lemma we will prove that

$$\lim_{n\to\infty} a(k; (\varphi_n, u_n), (\varphi_n, u_n)) \geqslant \left(\frac{\pi}{h}\right)^2 - \frac{k^2M^2}{1-M^2}.$$

First by (B.4) and thanks to the characterization of lemma 1.2 ($\exists\lambda > 0$ such that $a(k; (\varphi, u), (\psi, v)) + \lambda((\varphi, u), (\psi, v))$ is coercive on $W \times W$), (φ_n, u_n) is bounded in W. Consequently, by (B.3),

$$(\varphi_n, u_n) \rightharpoonup 0 \text{ in } W. \tag{B.5}$$

By compact injection of $H^{\frac{1}{2}}(\Gamma)$ in $L^2(\Gamma)$ and of $H^2(\Gamma)$ in $H^1(\Gamma)$,

$$a_{coupling}(k; (\varphi_n, u_n), (\varphi_n, u_n)) \to 0$$

and, on the other hand, $a_{plate}(u_n, u_n) \geqslant 0$.

The difficulty to obtain a lower bound for $a_{fluid}(k; \varphi_n, \varphi_n)$ comes from the fact that φ_n does not belong to T ($\varphi_n \neq 0$ on Γ), and thus lemma A.2 can not be applied directly to conclude. To overcome this difficulty we introduce the truncating function θ, a real $\mathcal{C}^\infty$ function of x such that $\theta(x) = 1$ for $0 < x < L$ and $\theta(x) = 0$ for $x < -1$ and $x > L+1$. Also we define the subdomain $\Omega_0 = \{(x, y) \in \Omega; -1 < x < L+1\}$ outside which $\theta = 0$ and we consider the two functions $f_n = (1-\theta)\varphi_n \in T$ and $g_n = \theta\varphi_n \in H^1(\Omega_0)$. Let us split $a_{fluid}(k)$ in four parts: $a_{fluid}(k; \varphi_n, \varphi_n) = a_{fluid}(k; f_n, f_n) + a_{fluid}(k; g_n, g_n) + a_{fluid}(k; f_n, g_n) + a_{fluid}(k; f_n, g_n)$. We will prove that

$$\lim_{n\to\infty} a(k; (\varphi_n, u_n), (\varphi_n, u_n)) \geqslant \lim_{n\to\infty} a_{fluid}(k; f_n, f_n). \tag{B.6}$$

From (B.5) we get that $g_n \rightharpoonup 0$ in $H^1(\Omega_0)$, $g_n \to 0$ in $L^2(\Omega_0)$, and even $g_n \to 0$ in $L^2(\Omega)$ since g_n vanishes outside Ω_0. Thus we get from (B.1)

$$||f_n||_{L^2(\Omega)} = ||\varphi_n - g_n||_{L^2(\Omega)} \to 1, \tag{B.7}$$

since $u_n \to 0$ in $H^1(\Gamma)$. Then from lemma 2.1 we get

$$\lim_{n\to\infty} a_{fluid}(k; f_n, f_n) \geqslant \left(\frac{\pi}{h}\right)^2 - \frac{k^2M^2}{1-M^2}. \tag{B.8}$$

To prove (B.6) we decompose $a_{fluid}(k; g_n, g_n)$ in two parts, $a_{fluid}(k; g_n, g_n) = a^1_{fluid}(k; g_n, g_n) + a^2_{fluid}(k; g_n, g_n)$, where

$$a^1_{fluid}(k; g_n, g_n) = \int_\Omega (1 - M^2)\left|\frac{\partial g_n}{\partial x}\right|^2 + \left|\frac{\partial g_n}{\partial y}\right|^2 \geqslant 0,$$

and

$$a^2_{fluid}(k; g_n, g_n) = -\int_\Omega 2ikM\frac{\partial g_n}{\partial x}\bar{g}_n \to 0.$$

In a same way we write $a_{fluid}(k; f_n, g_n) = a^3_{fluid}(k; f_n, g_n) + a^4_{fluid}(k; f_n, g_n)$, where

$$a^3_{fluid}(k; f_n, g_n) = \int_\Omega \theta(1-\theta)\left[(1 - M^2)\left|\frac{\partial \varphi_n}{\partial x}\right|^2 + \left|\frac{\partial \varphi_n}{\partial y}\right|^2\right] \geqslant 0,$$

and $a^4_{fluid}(k; f_n, g_n) \to 0$. Finally, we deduce from (B.4, B.6, B.7 and B.8) that $\gamma \geqslant \left(\frac{\pi}{h}\right)^2 - \frac{k^2M^2}{1-M^2}$.

B.2 *The singular sequences*

We will now construct singular sequences to prove that the inclusion $\sigma_{ess}(k) \subset [\gamma^\infty(k), +\infty[$, where $\gamma^\infty(k)$ is defined by (2.3), is in fact an equality.

Let $\gamma \geqslant \gamma^\infty(k)$ and consider the sequence $(\varphi_n, 0)_{n\geqslant 1}$ of $D(A(k))$ defined by

$$\varphi_n(x, y) = \frac{1}{\sqrt{n}}\psi\left(\frac{x}{n}\right)e^{i\delta x}\theta_1(y),$$

where $\theta_1(y) = 2\sin(\pi y/h)/\sqrt{h}$ (cf. Appendix A) and ψ is a $\mathcal{C}^\infty$ compactly supported function such that $\psi(x) = 0$ for $0 < x < L$. δ is chosen as the positive root of $(1-M^2)\delta^2 + 2kM\delta + (\pi/h)^2 - \gamma = 0$ (which exists thanks to the hypothesis on γ). Thanks to ψ, the sequence $(\varphi_n, 0)$ belongs to $D(A(k))$ since $\partial\varphi_n/\partial y = 0$ on Γ. It has also the following properties:

(a) $||\varphi_n||_{L^2(\Omega)}$ is independent of n. Indeed:

$$||\varphi_n||^2_{L^2(\Omega)} = \int_{\mathbb{R}} |\psi(s)|^2 ds.$$

(b) φ_n converges weakly to 0 in $L^2(\Omega)$. Indeed by Lebesgue's theorem, $(\varphi_n, \varphi)_{L^2(\Omega)} \to 0$ for every $\varphi \in \mathcal{D}(\Omega)$ which remains true, by the density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$, for every $\varphi \in L^2(\Omega)$.

(c) Finally, since $[-J_M(k) + \gamma][e^{i\delta x}\theta_1(y)] = 0$, we have:

$$[-J_M(k) + \gamma]\,\varphi_n = \left[(1 - M^2)\left(\frac{1}{n^2}\psi''\left(\frac{x}{n}\right) + 2i\delta\frac{1}{n}\psi'\left(\frac{x}{n}\right)\right) + 2ikM\frac{1}{n}\psi'\left(\frac{x}{n}\right)\right]\frac{1}{\sqrt{n}}e^{i\delta x}\theta_1(y),$$

and thus $||[A(k) - \gamma](\varphi_n, 0)|| = ||[-J_M(k) + \gamma]\varphi_n||_{L^2(\Omega)} \to 0.$

Consequently $\gamma \in \sigma_{ess}(k)$.

APPENDIX C

Eigenmodes of the plate

The problem of finding the eigenmodes of a plate in vacuum clamped at one end and free at the other end reads as follows:

$$\begin{aligned}\mu^4 \frac{d^4u}{dx^4} &= k^2 u \text{ on } \Gamma,\\ u = 0 = \frac{du}{dx} & \qquad \text{in } x = 0,\\ \frac{d^2u}{dx^2} = 0 = \frac{d^3u}{dx^3} & \qquad \text{in } x = L.\end{aligned}$$

A simple calculation shows that there is an infinite sequence of eigenfrequencies tending to infinity. Moreover the n^{th} frequency, $n = 1, 2...$, is given by

$$k_n^{plate} = \left(\frac{\mu\alpha_n}{L}\right)^2, \tag{C.1}$$

where α_n is the n^{th} real positive root of $\cos\alpha_n \cosh\alpha_n = -1$. The eigenmodes are

$$u_n(x) = A_n \left[\sinh\left(\frac{\alpha_n}{L}x\right) - \sin\left(\frac{\alpha_n}{L}x\right)\right] + B_n \left[\cosh\left(\frac{\alpha_n}{L}x\right) - \cos\left(\frac{\alpha_n}{L}x\right)\right],$$

where

$$B_n = -A_n \frac{\sinh(\alpha_n) + \sin(\alpha_n)}{\cosh(\alpha_n) + \cos(\alpha_n)},$$

and A_n can be chosen such that $\displaystyle\int_\Gamma |u_n|^2 = 1$.

The following mathematical characterization of the eigenfrequencies will be used in the following:

Lemma C.1. Let

$$\gamma_n^{plate} = \min_{F\in\mathcal{V}_n(V)} \max_{u\in F, u\neq 0} \mathcal{R}_{plate}(u),$$

where $\mathcal{V}_n(V)$ denotes the set of all n-dimensional subspaces of U, U defined in (1.5) and

$$\mathcal{R}_{plate}(u) = \mu^4 \frac{\displaystyle\int_\Gamma \left|\frac{d^2u}{dx^2}\right|^2}{\displaystyle\int_\Gamma |u|^2}.$$

Then

$$\gamma_n^{plate} = \left(k_n^{plate}\right)^2 \quad \forall n \geqslant 1. \tag{C.2}$$

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