Energy preserving scheme for non linear systems of wave equations.
Application to piano strings.

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Abstract

The linear wave equation does not describe the complexity of the piano strings vibration enough for physics based sound synthesis. The nonlinear coupling between transversal and longitudinal modes has to be taken into account, as does the "geometrically exact" model. This system of equations can be classified among a general energy preserving class of systems. We present an implicit, centered, second order accurate, numerical scheme that preserves a discrete energy, leading to unconditional stability of the numerical scheme. The complete model takes into account the bridge coupling the strings, and the hammer non linear attack on the strings.

1 Non linear string vibrations

Observations on piano strings spectra have shown that the linear wave equation is not enough to describe the physical phenomenon of piano strings vibration. Unusual frequency series have been noticed, which Conklin [2] called "phantom partials". Using
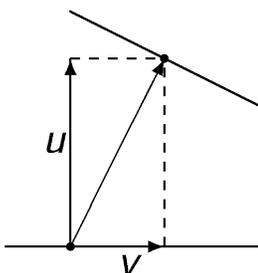

Figure 1: Displacement of one string point : transverse motion u and longitudinal motion v.

a non linear geometric description of the transversal and longitudinal motions of the string leads to the "geometrically exact" model (see Morse & Ingard [5]). We use adimensionned variables, let E be the Young's modulus, A the section and T_0 the rest tension of the string, and define $\alpha = \frac{EA - T_0}{EA}$.

The system can be written:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial x} - \alpha \dfrac{\frac{\partial u}{\partial x}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(1 + \frac{\partial v}{\partial x}\right)^2}} \right] = 0 \\[2em] \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial}{\partial x}\left[\dfrac{\partial v}{\partial x} - \alpha \dfrac{1 + \frac{\partial v}{\partial x}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(1 + \frac{\partial v}{\partial x}\right)^2}} \right] = 0 \end{cases}$$

This system is a particular case of:

$$\begin{cases} \text{Find } \mathbf{u} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N \\[1em] \dfrac{\partial^2 \mathbf{u}}{\partial t^2} - \dfrac{\partial}{\partial x}\left[\nabla H\left(\dfrac{\partial \mathbf{u}}{\partial x}\right) \right] = 0 \end{cases} \quad (1)$$

where $H : \mathbb{R}^N \to \mathbb{R}$. String model corresponds to $N = 2$, $\mathbf{u} = (u, v)$ and

$$H(u, v) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - \alpha[\sqrt{u^2 + (1+v)^2} - (1+v)] \quad (2)$$

This quasilinear system is hyperbolic when H is convex, which is the case with (2) for small values of (u, v). The theory for these equations is now well known, see Li TaTsien [4]. An important result is the energy conservation for this general type of systems:

Theorem 1. *Let $\mathbf{u}$ be a sufficiently smooth solution of (1). Then it satisfies :*

$$\frac{d}{dt} \int_\Omega \left\{ \frac{1}{2} \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 + H\left(\frac{\partial \mathbf{u}}{\partial x}\right) \right\} \, dx = 0 \quad (3)$$

Energy preservation can lead to H^1 stability on the solution, under a coercivity condition on H.

2 Energy preserving scheme

A classic way of building stable numerical schemes for PDEs is to preserve, on a discrete level, a continuous conserved quantity, like an energy for instance. If this is relatively easy and well-known for scalar equations ($N = 1$, see Furihata [3]), this is less obvious for systems for which existing solutions that we found in the litterature (see for instance Bilbao [1]) seem

to be limited to specific hamiltonians. In the general case, we developed an implicit, energy preserving scheme which is also centered in time. The scheme can be written for any value of N and any function H. It is second order accurate, unconditionnally stable and its algebraic complexity grows as 2^{N-1}. The scheme is based on a variational approach which can lead to finite elements for space variable. The main difficulty remains in the time discretisation, which we based on finite differences. In the following, V_h is a finite dimensional subspace of H_0^1, $u_{k,h}$ denotes the kth coordinate of the discrete solution u_h ($k \in [1, N]$). For simplicity, we will write abusively $H(u_k, u_{l \neq k})$ for $H(\mathbf{u})$. Next, we define the directional finite difference

$$\delta_k H(u_{k,1}, u_{k,2}; u_{l \neq k}) := \frac{H(u_{k,1}, u_{l \neq k}) - H(u_{k,2}, u_{l \neq k})}{u_{k,1} - u_{k,2}}$$

This will be used in the discretisation of the kth equation of (1) with $u_{k,1} = u_{k,h}^{n+1}$ and $u_{k,2} = u_{k,h}^{n-1}$. The difficult point is to decide a which times the $N - 1$ other quantities $u_{l \neq k}$ are taken. The choice is made in order to preserve a discrete energy.

Let us introduce

$$\Sigma_k = \big\{ \sigma : J_k \to \{-1, +1\}, \ J_k = \{1, \cdots, N\} \setminus \{k\} \big\}$$

and for $\sigma \in \Sigma_k$, define

$$\theta(\sigma) = \mu(\sigma)! \ (N - 1 - \mu(\sigma))! \ / \ N!$$

where $\mu(\sigma) = \ \# \ \big\{ l \in J_k, \ \sigma(l) = +1 \big\}$.

The scheme can be written, for any $v_h \in V_h$:

$$\begin{cases} \displaystyle\oint_0^L \frac{u_{k,h}^{n+1} - 2u_{k,h}^n + u_{k,h}^{n-1}}{\Delta t^2} \ v_h + \\[2em] \displaystyle\sum_{\sigma \in \Sigma_k} \oint_0^L \theta_\sigma \, \delta_k H(\partial_x u_{k,h}^{n+1}, \partial_x u_{k,h}^{n-1}; \partial_x u_{l \neq k,h}^{n+\sigma(l)}) \, \partial_x v_h = 0 \end{cases}$$

where $\oint$ refers to any chosen numerical integration.

The main theoretical property of this scheme is:

Theorem 2. *Our scheme preserves the energy:*

$$\begin{aligned} \mathcal{E}^{n+\frac{1}{2}} \ &= \ \sum_{k=1}^N \Big[\ \frac{1}{2} \oint_0^L \Big[\frac{u_{k,h}^{n+1} - u_{k,h}^n}{\Delta t} \Big]^2 \Big] + \\ &\qquad \oint_0^L \frac{H(\partial_x \mathbf{u}_h^{n+1}) + H(\partial_x \mathbf{u}_h^n)}{2} \end{aligned}$$

3 Coupled strings and hammer

Piano strings are mostly grouped by three in order to produced one note. The coupling point is the bridge at which there is actually a displacement, that is the same for all the strings. For now, we modelled this motion by a simple damped oscillating movement, expecting a better modelisation in the future. The hammer has also been taken into account, coupled with strings by a damped non linear law with hysteresis. Figure (2) shows the conservation of total energy (full black line) without any damping, and the evolution of internal energies (hammer, string, bridge) according to the time steps.
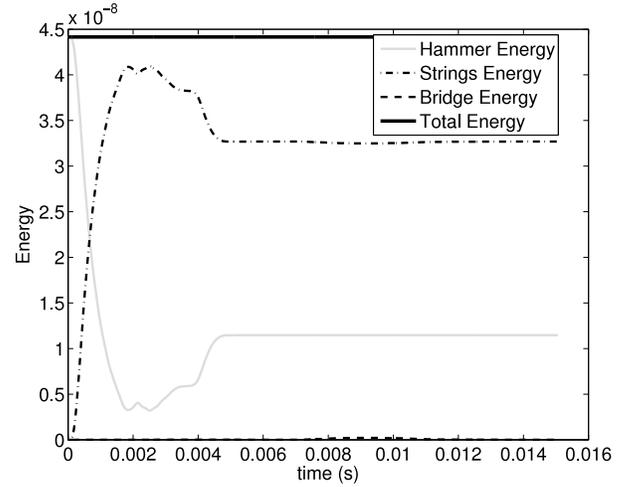

Figure 2: Energy values according to time steps.

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