IDENTIFICATION OF GENERALIZED IMPEDANCE BOUNDARY CONDITIONS IN INVERSE SCATTERING PROBLEMS

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Abstract. In the context of scattering problems in the harmonic regime, we consider the problem of identification of some Generalized Impedance Boundary Conditions (GIBC) at the boundary of an object (which is supposed to be known) from far field measurements associated with a single incident plane wave at a fixed frequency. The GIBCs can be seen as approximate models for thin coatings, corrugated surfaces or highly absorbing media. After pointing out that uniqueness does not hold in the general case, we propose some additional assumptions for which uniqueness can be restored. We also consider the question of stability when uniqueness holds. We prove in particular Lipschitz stability when the impedance parameters belong to a compact subset of a finite dimensional space.

1. Introduction

We address in this work uniqueness and stability issues related to the identification of a medium impedance from the knowledge of far measurements of a scattered wave at a given frequency. We restrict ourselves in these first investigations to the scalar case (the Helmholtz equation), that models either acoustic waves or two dimensional settings of electromagnetic problems. Assuming that the unknown medium occupies a domain $D$, the medium impedance is understood as a “local” operator that links the Cauchy data of the field $u$ on the medium boundary $\Gamma := \partial D$. More precisely we shall consider the cases where a boundary condition of the form

$$\frac{\partial u}{\partial \nu} + Zu = 0 \quad \text{on} \quad \Gamma$$

is satisfied, where $Z$ is a boundary operator and $\nu$ denotes the outward normal field on $\Gamma$.

The exact impedance operator $Z$ corresponds to the so-called Dirichlet-to-Neumann (DtN) map, i.e. $f \mapsto -\frac{\partial u}{\partial \nu}|_{\Gamma}$, where $u$ solves the Helmholtz equation inside $D$ and satisfies $u = f$ on $\Gamma$. Consequently determining this map is “equivalent” to identify the physical properties inside $D$, which is in general a severely ill-posed problem that requires more than a finite number of measurements.

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We are interested here in situations where the operator $Z$ is an approximation of the exact DtN map. In general these approximations correspond to asymptotic models associated with configurations that involve a small parameter. These cases include small amplitude roughness, thin coatings, periodic gratings, highly absorbing media, ... We refer to [18, 8, 2] for a review of these models.

The simplest form is the case where $Z$ is a scalar function, which corresponds in general to the lowest order (non trivial) approximations, for instance in the case of very rough surfaces of highly absorbing media (the Leontovich condition [15]). However, for higher order approximations or in other cases the operator $Z$ may involve boundary differential operators. For instance when the medium contains a perfect conductor coated with a thin layer of width $\delta$ then for TM polarization, the approximate boundary conditions of order 1 corresponds to $Z = 1/\delta$ while for the TE polarization it corresponds to $Z = \delta(\partial_s s + k^2 n)$ where $s$ denotes the curvilinear abscissa, $k$ the wave number and $n$ is the mean value of the thin coating index with respect to the normal coordinate. Higher order approximations would include curvature terms or even higher order derivatives [4, 10, 11]. This type of conditions will be referred to as Generalized Impedance Boundary Conditions.

We shall address in the present work the question of unique identification and stability of the reconstruction of the operator $Z$ from the knowledge of one scattered wave. One easily sees, from the given example, how the identification of the impedance would provide information on some effective properties of the medium (for instance, the thickness of the coating and the normal mean value of its index). Determining these effective properties would be less demanding in terms of measurements than solving the inverse problem with the exact DtN map (the unknown parameters have one dimension less) and we also expect that the inherent ill-posedness to be less severe. Motivated by the example above we shall consider generalized impedance boundary conditions of the form:

$$(1) \quad Z = \mu \Delta_G + \lambda,$$

where $\mu$ is a complex constant and $\lambda$ is a complex function and where $\Delta_G$ denotes the Laplace-Beltrami operator on $\Gamma$.

The case of standard impedance problems ($\mu = 0$) has been studied by several authors [14, 19], where for instance optimal logarithmic stability is obtained. Our analysis here is different and is rather motivated by numerical considerations. It also applies to the case $\mu \neq 0$. We shall prove injectivity of the Fréchet derivative (which is equivalent to local stability) of the inverse map by using the adjoint state technique. We also investigate the situations where Lipschitz stability can be obtained. In the same spirit as in [3] we shall prove a general result showing how Lipschitz stability holds when we restrict the impedance parameters to a compact subset of a finite dimensional space, assuming injectivity of the inverse map and of its derivative. Motivated by this result we investigate the uniqueness question from the knowledge of one scattered wave. We show that uniqueness when $\mu \neq 0$ cannot be true in general, but holds under further assumptions on the parameters (typically assuming that part of the parameters are known) and on the geometry.

The outline of our article is the following. We introduce in the next section the forward scattering problem and briefly recall the mathematical property of the solution in the case of GIBC. We then formulate the inverse problem. Section 3 is dedicated to the question of uniqueness from the knowledge of the far field associated to one incident wave. Section 4 deals with the question of stability by...
analyzing the injectivity of the Fréchet derivative. It also includes a general result on Lipschitz stability.

2. The forward and inverse problems

Let $D$ be an open bounded domain in $\mathbb{R}^3$, and $\Omega := \mathbb{R}^3 \setminus \overline{D}$. The domain $D$ will be referred to as the obstacle and a GIBC holds on its boundary $\Gamma$. An incident plane wave $u^i(x) := e^{ik \cdot x}$, with $k > 0$ denoting the wavenumber and $||d|| = 1$, is scattered by the obstacle $D$ and gives rise to a scattered field $u^s$. The governing equations for $u^s$ in $\Omega$ are

\[
\begin{cases}
\Delta u^s + k^2 u^s = 0 & \text{in } \Omega, \\
\frac{\partial u^s}{\partial \nu} + \mu \Delta u^s + \lambda u^s = f & \text{on } \Gamma, \\
\lim_{R \to +\infty} \int_{\partial B_R} |\partial u^s/\partial r - iku^s|^2 ds(x) = 0,
\end{cases}
\]

with

\[f := -\left( \frac{\partial u^i}{\partial \nu} + \mu \Delta u^i + \lambda u^i \right) |\Gamma|,\]

and where $\nu$ denotes the unit normal to $\Gamma$ oriented to the exterior of $D$. If we set $u := u^s + u^i$ as being the total field, then the second equation of (2) is equivalent to

\[
\partial u/\partial \nu + \mu \Delta u + \lambda u = 0 \quad \text{on } \Gamma.
\]

The last equation in (2) is the classical Sommerfeld radiation condition where $B_R$ denotes a ball of radius $R$.

We assume that the boundary $\Gamma$ is Lipschitz continuous, such that for a function $v \in H^1(\Gamma)$, the tangential gradient $\nabla_T v$ is defined in $(L^2(\Gamma))^3$, and $\Delta_T v$ is defined in $H^{-1}(\Gamma)$ by the identity

\[
\langle \Delta_T v, w \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = -\int_{\Gamma} \nabla_T v \cdot \nabla_T w \, ds, \quad \forall w \in H^1(\Gamma).
\]

The space of solutions to the forward problem differs between the cases $\mu = 0$ and $\mu \neq 0$.

1. In the case $\mu = 0$, problem (2) coincides with the classical impedance problem. It is uniquely solvable in $V_0 := \{ v \in D'(\Omega), \phi v \in H^1(\Omega), \forall \phi \in D(\mathbb{R}^3) \}$ provided $\lambda \in L^\infty(\Gamma)$ with $\text{Im}(\lambda) \geq 0$ (see for instance [17]). This result remains true for $f \in H^{-1/2}(\Gamma)$. In [7], it is proved that if moreover the boundary $\Gamma$ is $C^2$ and $\lambda \in C^0(\Gamma)$, then the solution $u^s$ is continuous up to the boundary $\Gamma$.

2. In the case $\mu \neq 0$, problem (2) will be referred to as the generalized impedance problem. It is uniquely solvable in $V := \{ v \in V_0, v|_{\Gamma} \in H^1(\Gamma) \}$ provided that $\lambda \in L^\infty(\Gamma)$ with $\text{Im}(\lambda) \geq 0$ as well as $\text{Re}(\mu) > 0$ and $\text{Im}(\mu) \leq 0$. This result remains true for $f \in H^{-1}(\Gamma)$.

For sake of completeness, we shall sketch the proof in the case of the generalized impedance problem which can be seen as a slight adaptation of the proof in [17] for classical impedance problems. We restrict the problem to a bounded domain $\Omega_R := \Omega \cap B_R$ with the help of the Dirichlet-to-Neumann map $S_R : H^{1/2}(\partial B_R) \rightarrow H^{-1/2}(\partial B_R)$, defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g = \partial_v u^s|_{\partial B_R}$, where $u^s$ is the solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{B_R}$ satisfying the Sommerfeld radiation condition and $u^s = g$ on $\partial B_R$. This operator satisfies in particular

\[
\text{Re} \langle S_R g, g \rangle \leq 0 \quad \text{and} \quad \text{Im} \langle S_R g, g \rangle \geq 0 \quad \forall g \in H^1(\partial B_R).
\]
where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-\frac{1}{2}}(\partial B_R)$ and $H^{\frac{1}{2}}(\partial B_R)$. Solving problem (2) is then equivalent to find $u^s$ in $V_R := \{ v \in H^1(\Omega_R); \; v|_{\Gamma} \in H^1(\Gamma) \}$ satisfying

$$
\begin{cases}
\Delta u^s + k^2 u^s = 0 & \text{in } \Omega_R, \\
\frac{\partial u^s}{\partial \nu} + \mu \Delta u^s + \lambda u^s = f & \text{on } \Gamma, \\
\frac{\partial u^s}{\partial r} = S_R(u^s|_{\partial B_R}) & \text{on } \partial B_R.
\end{cases}
$$

(4)

Problem (4) is then proved to be equivalent to the weak formulation: find $u^s$ in $V_R$ such that for all $v$ in $V_R,$

$$a(u^s, v) = l(v),$$

with

$$a(u^s, v) := \int_{\Omega_R} (\nabla u^s \cdot \nabla v - k^2 u^s v) \, dx + \int_{\Gamma} (\mu \nabla u^s \cdot \nabla v - \lambda u^s v) \, ds - \langle S_R u^s, v \rangle,$$

$$l(v) = \langle f, v \rangle_{H^{-1}(\Gamma), H^1(\Gamma)}.$$

It is easily seen that $V_R$ equipped with the scalar product $\langle \cdot, \cdot \rangle_{V_R} := \langle \cdot, \cdot \rangle_{H^1(\Omega_R)} + \langle \cdot, \cdot \rangle_{H^1(\Gamma)}$ is a Hilbert space. The weak formulation can then be written in the form: find $u^s$ in $V_R$ such that

$$(J_R + K_R)u^s = F_R \quad \text{in } V_R$$

where the operators $J_R : V_R \to V_R$ and $K_R : V_R \to V_R$ are uniquely defined by

$$(J_R u^s, v)_{V_R} = \langle u^s, v \rangle_{H^1(\Omega_R)} + \mu \int_{\Gamma} \nabla u^s \cdot \nabla v \, ds - \langle S_R u^s, v \rangle,$$

$$(K_R u^s, v)_{V_R} = - (1 + k^2) \langle u^s, v \rangle_{L^2(\Omega_R)} - \int_{\Gamma} \lambda u^s v \, ds,$$

and $F_R$ is uniquely defined by

$$l(v) = \langle F_R, v \rangle_{V_R}.$$

One then easily checks, by application of the Lax-Milgram theorem that $J_R$ is an isomorphism for $\text{Re}(\mu) > 0,$ and the application of the Rellich compact embedding theorem that $K_R$ is compact. Therefore, with the help of Fredholm alternative it is sufficient to prove the injectivity of the operator $J_R + K_R,$ which is equivalent to prove uniqueness of solutions to problem (2).

Following [7], a sufficient condition for uniqueness is that for all solution $u \in V$ of (2) with $f = 0,$

$$\int_{\Gamma} \text{Im}(\overline{u} \frac{\partial u}{\partial \nu}) \, ds \leq 0.$$

Since

$$\int_{\Gamma} \text{Im}(\overline{u} \frac{\partial u}{\partial \nu}) \, ds = \text{Im}(\mu) \int_{\Gamma} |\nabla u|^2 \, ds - \int_{\Gamma} \text{Im}(\lambda)|u|^2 \, ds$$

we have uniqueness in the case $\text{Im}(\mu) \leq 0$ and $\text{Im}(\lambda) \geq 0,$ which completes our sketch of proof.

Let us finally notice that both operators $J_R$ and $K_R$ depend continuously on $\lambda$ and $\mu.$
Formulation of the inverse problem. Following [7], the solution $u^s$ of problem (2) has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r}(u^\infty(\hat{x}) + O(\frac{1}{r})), \quad r \to +\infty,$$

uniformly in all directions $\hat{x} = x/r \in S^2$, where $r := |x|$ and $S^2$ denotes the unit sphere in $\mathbb{R}^3$. The far field $u^\infty \in L^2(S^2)$ is given by

$$u^\infty(\hat{x}) = \int_{\Gamma} \{u^s(y) \frac{\partial \Phi^\infty(y, \hat{x})}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y) \Phi^\infty(y, \hat{x})\} \, ds(y), \quad \forall \hat{x} \in S^2,$$

where $\Phi^\infty(y, \hat{x}) := \frac{1}{4\pi} e^{-ik\hat{x} \cdot y}$ and where the second integral has to be understood as a duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

The inverse problem we are interested in is to determine both $\mu$ and the function $\lambda$ on $\Gamma$ from the knowledge of the far field $u^\infty$ associated with one direction $d$ of the incident field. We shall address the questions of uniqueness and stability of this problem. For both questions we shall first consider the classical impedance problem ($\mu = 0$) and then the case of generalized impedance problem. Answering these questions amounts to study the properties of the nonlinear map $T : \lambda \mapsto u^\infty$ for the first case and $T : (\lambda, \mu) \mapsto u^\infty$ for the second one.

3. Uniqueness

This section is devoted to the investigation of the uniqueness for the inverse problem formulated above which is equivalent to analyzing the injectivity of the operator $T$.

3.1. The classical impedance problem. Analyzing the injectivity of $T$ as an operator acting on $L^\infty(\Gamma)$ seems to be challenging. We shall restrict ourselves to the subspace of $L^\infty(\Gamma)$ formed by piecewise-continuous functions. Let $I$ be a given integer and $\Gamma_i$, $i = 1, \ldots, I$, be open sets of $\Gamma$ such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, $\Gamma = \cup_{i=1}^I \Gamma_i$, and the sets $\partial \Gamma_i$ are negligible in the sense of the Lebesgue surface measure supported by $\Gamma$. We define the subspace $C_I(\Gamma)$ as the set of functions $u \in L^\infty(\Gamma)$ such that each restriction of $u$ to $\Gamma_i$ belongs to $C^0(\Gamma_i)$. The space $C_I(\Gamma)$ is obviously a closed subspace of $L^\infty(\Gamma)$.

We shall also use the subsets

$$L^\infty_+(\Gamma) := \{\lambda \in L^\infty(\Gamma) : \text{Im}(\lambda) \geq 0\} \text{ and } C_{I+}(\Gamma) := C_I(\Gamma) \cap L^\infty_+(\Gamma).$$

Proposition 1. The operator $T : C_{I+}(\Gamma) \to L^2(S^2)$ such that $\lambda \mapsto u^\infty$, where $u^\infty$ is defined from $u^s$ by using (2) and $u^s$ is defined from $\lambda$ by solving (3), is injective.

Proof. Assume that $T(\lambda_1) = T(\lambda_2) = u^\infty$ with $\lambda_1, \lambda_2 \in C_{I+}(\Gamma)$. From the Rellich Lemma and the unique continuation principle (see [7]), the function $u := u_1 - u_2 = u_1^s - u_2^s$ vanishes in $\Omega$. Since $u$ satisfies $\Delta u + k^2 u = 0$ in $\Omega$, the traces $(u_1 - u_2)|_{\Gamma}$ and $\partial_{\nu}(u_1 - u_2)|_{\Gamma}$, which are well defined in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively, also vanish.

Considering now the boundary condition on $\Gamma$,

$$\frac{\partial u_1}{\partial \nu} + \lambda_1 u_1 = \frac{\partial u_2}{\partial \nu} + \lambda_2 u_2 = 0,$$

we obtain on $\Gamma$,

$$\langle \lambda_1 - \lambda_2 \rangle u_1 = 0.$$
We shall prove that $\lambda_1 = \lambda_2$ almost everywhere on $\Gamma$. Assume that there exists $x_0 \in \Gamma$, where $x_0$ does not belong to some $\partial \Gamma_i$, such that $\lambda_1(x_0) \neq \lambda_2(x_0)$. Since $\lambda_1$ and $\lambda_2$ are continuous in a neighborhood of $x_0$, there exists $\eta > 0$ such that $|\lambda_1(x) - \lambda_2(x)| \geq c > 0$ for all $x$ in $B(x_0, \eta) \cap \Gamma$. It follows that $u_1 = 0$ on $B(x_0, \eta) \cap \Gamma$. The boundary condition on $\Gamma$ leads to $\partial_{\nu} u_1 = 0$ on $B(x_0, \eta) \cap \Gamma$, and unique continuation leads to $u_1 = 0$ in $\Omega$. Since the incident wave $u^i$ does not satisfy the radiation condition, while the scattered field $u^s_1$ does, we obtain a contradiction.

**Remark 1.** Let us notice that the proof remains valid even if $I$ and the partition $\Gamma_i, i = 1, \ldots, I$ are not a priori known.

### 3.2. The generalized impedance problem

We start this section by providing a counterexample showing that $T$ cannot be injective under the sole assumptions that guarantee well-posedness of problem (2), i.e.

$$\lambda \in L^\infty(\Gamma), \, \text{Im}(\lambda) \geq 0, \, \text{Re}(\mu) > 0 \text{ and } \text{Im}(\mu) \leq 0.$$  

The requirements of (6) will be referred to as assumption (H0) in the following.

Our counterexample is valid even if the boundary $\Gamma$ is $C^\infty$ and $\lambda$ is a $C^\infty$ function on $\Gamma$.

It is constructed as follows: one first chooses a constant impedance $\lambda_0 \in i\mathbb{R}$ with $\text{Im}(\lambda_0) > 0$ such that the corresponding solution $u_0$ of the classical impedance problem, which is $C^\infty$ up to the boundary $\Gamma$, does not vanish on $\Gamma$. This choice is possible as demonstrated in 2D by the case $D = B(0, 1), d = (1, 0), k = 1, \lambda_0 = i$ where the solution $u_0(1, \theta)$ on $\Gamma$ is given (after straightforward calculations) by

$$u_0(1, \theta) = \frac{2i}{\pi} \sum_{n \in \mathbb{Z}} \frac{e^{in\theta}}{H_n(1) + iH_n(1)},$$

where $H_n$ denotes the Hankel function of the first kind of order $n$. The modulus of the function $u_0(1, \cdot)$ is depicted in Figure 1. Let us denote by $u_0^\infty$ the far field associated with $u_0$. Since $u_0$ does not vanish on $\Gamma$, the function $\alpha = \Delta_\Gamma u_0 / u_0$ is a $C^\infty$ function on $\Gamma$, and therefore bounded on $\Gamma$. Now let $\mu_1 \neq \mu_2$ be any complex numbers such that

$$|\mu_i| \max_\Gamma |\alpha| \leq \text{Im}(\lambda_0), \quad \text{Re}(\mu_i) > 0, \quad \text{Im}(\mu_i) \leq 0, \quad i = 1, 2.$$
and let the associated $C^\infty$ functions $\lambda_1$ and $\lambda_2$ be defined on $\Gamma$ by

$$\lambda_i = \lambda_0 - \alpha \mu_i, \quad i = 1, 2.$$  

We have on $\Gamma$

$$\frac{\partial u_0}{\partial \nu} + \mu_i \Delta u_0 + \lambda_i u_0 = (-\lambda_0 + \alpha \mu_i + \lambda_i) u_0 = 0,$$

and

$$\text{Im}(\lambda_i) = \text{Im}(\lambda_0) - \text{Im}(\alpha \mu_i) \geq \text{Im}(\lambda_0) - |\mu_i| \max_{\Gamma} |\alpha| \geq 0.$$  

As a result, $u_\infty^0$ is the far field associated to the solution $u_0$ of the generalized impedance problem with both $(\lambda_1, \mu_1)$ and $(\lambda_2, \mu_2)$.

It should also be noted that even if $\mu$ is real (which would correspond to the thin coating case, as presented in the introduction, with $\mu = \delta$ and $\lambda = \delta k^2 n^2$), our counterexample shows that unique determination of $(\lambda, \mu)$ does not hold in general. The counterexample above justifies the need to consider restricted cases where some a priori knowledge is introduced in addition to the assumption $(H0)$ given by $[\text{6}]$.

We were able to recover injectivity in the following complementary cases:

1. Assumption $(H1)$: $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ are two constants.

Furthermore, we assume there exist $x_0 \in \Gamma$ and $\eta > 0$ such that $\Gamma_0 = \Gamma \cap B(x_0, \eta)$ is either a portion of a plane, or a portion of a cylinder, or a portion of a sphere, and such that the set $\{x + \gamma \nu(x), \ x \in \Gamma_0, \gamma > 0\}$ is included in $\Omega$.

2. Assumption $(H2)$: $\lambda \in C_I(\Gamma)$, $\mu \in \mathbb{C}$, and we assume that either of the following holds:

   - $(H2a)$ $\lambda$ is fixed and known, the unknown being $\mu$.
   - $(H2b)$ both $\text{Re}(\lambda)$ and $\text{Im}(\mu)$ are fixed and known, the unknown being $\text{Im}(\lambda)$ and $\text{Re}(\mu)$.
   - $(H2c)$ both $\text{Im}(\lambda)$ and $\text{Re}(\mu)$ are fixed and known, the unknown being $\text{Re}(\lambda)$ and $\text{Im}(\mu)$.
   - $(H2d)$ $\mu$ is fixed and known, the unknown being $\lambda$.

Furthermore, in the three cases $(H2a)$, $(H2b)$, $(H2c)$, we assume there is no constant $C$ such that on $\Gamma$:

$$\left( \begin{array}{c} u^* \cr \partial u^*/\partial \nu \end{array} \right) = CP \left( \begin{array}{c} 1 \\ -\lambda \end{array} \right)$$  

with $P := \begin{bmatrix} \frac{1}{2} - T & S \\ \mathcal{R} & \frac{1}{2} + T^* \end{bmatrix}$.

Here $S$ and $T$ respectively denote the traces of the single and double layer potentials on $\Gamma$, while $T^*$ and $-R$ denote the traces of the normal derivatives of the single and double layer potentials on $\Gamma$ (see equation $[\text{10}]$ below). This assumption will be referred to as $(HC)$ in the following. It should be noted that the matrix $P$ denotes the so-called Calderón projection for the interior problem (the operator $P$ actually satisfies $P^2 = P$, $[\text{15}]$). In particular, the first of the two above equations can be specified as

$$e^{ikx \cdot d} = \frac{C}{2} (1 - 2T(1)(x) - 2S(\lambda)(x)) \quad \forall x \in \Gamma.$$  

Assumption $(HC)$ may be impossible to verify in practice, this is why we here-after mention two particular situations where this condition is automatically satisfied (see Lemma $[\text{7}]$ below):
Figure 2. Illustration of assumption (H1).

- the function \( \lambda \) is real and \( k^2 \) is not a Dirichlet eigenvalue of the operator \(-\Delta \) in \( D \),
- the domain \( D \) is \( C^1 \) and both \( D \) and the function \( \lambda \) are assumed to be invariant by reflection against a plane which does not contain the direction \( d \) or by a rotation (different from identity) around an axis which is not directed by \( d \).

Let us remark that the proof under assumption (H2d) follows exactly the same lines as the proof in the case \( \mu = 0 \) given in the previous section. We therefore shall detail the proofs only for assumptions (H1), (H2a) – (HC), (H2b) – (HC), (H2c) – (HC). The results are formulated and proved in Propositions 2, 3 and 4.

**Proposition 2.** Under assumption (H1) the farfield associated with one incident plane wave uniquely determines the coefficients \( \lambda \) and \( \mu \).

**Proof.** Using the same arguments and notation as in the first paragraph of the proof of Proposition 1 we arrive in the present case at:

\[
\frac{\partial u_1}{\partial \nu} + \mu_1 \Delta_\Gamma u_1 + \lambda_1 u_1 = \frac{\partial u_1}{\partial \nu} + \mu_2 \Delta_\Gamma u_1 + \lambda_2 u_1 = 0
\]
on \( \Gamma \). In order to simplify the notation we set in the following \( u := u_1 \). If \( \mu_1 = \mu_2 \), then we easily conclude that \( \lambda_1 = \lambda_2 \) using the same arguments as in the proof of Proposition 1. If \( \mu_1 \neq \mu_2 \), then on \( \Gamma \)

\[
\Delta_\Gamma u = \alpha u, \quad \frac{\partial u}{\partial \nu} = \beta u,
\]

with

\[
\alpha = -\frac{\lambda_2 - \lambda_1}{\mu_2 - \mu_1}, \quad \beta = -\frac{\mu_2 \lambda_1 - \mu_1 \lambda_2}{\mu_2 - \mu_1}.
\]

We notice that \( \alpha = -k_\Gamma^2 \), where \( k_\Gamma^2 \) is an eigenvalue of the negative Laplace-Beltrami operator \(-\Delta_\Gamma\).

Let us consider first the case when \( \Gamma \) contains a portion of plane \( \Gamma_0 = \Gamma \cap B(x_0, \eta) \) of outward normal \( \nu \) such that the set \( Q_0 = \{ x + \gamma \nu, x \in \Gamma_0, \gamma > 0 \} \) is included in \( \Omega \). Then there exists a system of coordinates \( (x_1, x_2, x_3) \) such that \( x(0,0,0) = x_0 \) and

\[
Q_0 = \{ x(x_1, x_2, x_3), \sqrt{x_1^2 + x_2^2} < \eta, x_3 > 0 \},
\]
$\Gamma_0 = \{ x(x_1, x_2, x_3), \sqrt{x_1^2 + x_2^2} < \eta, x_3 = 0 \}$.

We now consider the function $\tilde{u}$ defined in $Q_0$ from $u$ by

$$\tilde{u}(x_1, x_2, x_3) = u(x_1, x_2, 0)c(x_3),$$

where the function $c$ is uniquely defined by

$$\frac{d^2c}{dx_3^2} + (k^2 - k_1^2)c = 0, \quad c(0) = 1, \quad \frac{dc}{dx_3}(0) = \beta.$$

In the domain $Q_0$ we obtain

$$\Delta \tilde{u} + k^2 \tilde{u} = \Delta_1 u(x_1, x_2, 0)c(x_3) + u(x_1, x_2, 0)\frac{d^2c}{dx_3^2} + k^2 u(x_1, x_2, 0)c(x_3)$$

$$= u(x_1, x_2, 0)\left( \frac{d^2c}{dx_3^2} + (k^2 - k_1^2)c \right)(x_3) = 0.$$ 

On the surface $\Gamma_0$ we obtain

$$\tilde{u}(x_1, x_2, 0) = u(x_1, x_2, 0), \quad \frac{\partial \tilde{u}}{\partial x_3}(x_1, x_2, 0) = \frac{\partial u}{\partial x_3}(x_1, x_2, 0).$$

The functions $\tilde{u}$ and $u$ are both solutions of the same Helmholtz equation in $Q_0$, and they satisfy $\tilde{u} = u$ and $\partial_\nu \tilde{u} = \partial_\nu u$ on $\Gamma_0$. Hence, unique continuation implies $\tilde{u} = u$ in $Q_0$.

Since $u^*$ satisfies the radiation condition when $||x|| \to +\infty$, we have in particular that $\lim_{x_3 \to +\infty} u^*(x_1, x_2, x_3) = 0$. Recalling that

$$u(x_1, x_2, x_3) = u^*(x_1, x_2, x_3) + e^{ik(d_1 x_1 + d_2 x_2 + d_3 x_3)},$$

we obtain

$$u(x_1, x_2, x_3) \sim e^{ik(d_1 x_1 + d_2 x_2 + d_3 x_3)}, \quad x_3 \to +\infty,$$

and in particular when $x_1 = x_2 = 0$,

$$u(0, 0, x_3) \sim e^{ikd_3 x_3}, \quad x_3 \to +\infty,$$

that is

$$c(x_3) \sim C e^{ikd_3 x_3}, \quad x_3 \to +\infty.$$ 

To see that the asymptotic behavior is impossible, we have to discuss separately the three cases $k_\Gamma < k$, $k_\Gamma = k$ and $k_\Gamma > k$. In the case $k_\Gamma < k$, we obtain from (10)

$$c(x_3) = \frac{1}{2}(1 - i\frac{\beta}{\sqrt{k^2 - k_1^2}})e^{\sqrt{k^2 - k_1^2} x_3} + \frac{1}{2}(1 + i\frac{\beta}{\sqrt{k^2 - k_1^2}})e^{-\sqrt{k^2 - k_1^2} x_3}.$$

This implies that either $\sqrt{k^2 - k_1^2} = kd_3$ or $-\sqrt{k^2 - k_1^2} = kd_4$. In both cases, $c(x_3) = e^{ikd_4 x_3}$ and by using again that $u(x_1, x_2, x_3) = u(x_1, x_2, 0)c(x_3)$ in $Q_0$, we conclude from (11) that $u(x_1, x_2, 0) = e^{ikd_1 x_1 + d_2 x_2} \text{ in } \Gamma_0$, whence $u = e^{ikd_3 x}$ in $Q_0$, and lastly $u = e^{ikd_4 x}$ in $\Omega$. Now we use $\partial u/\partial \nu = \beta u$ on $\Gamma$ to see that $d_\nu$ is constant on $\Gamma$. This is forbidden by the fact there exist at least 4 points $x \in \Gamma$ such that the 4 corresponding outward normals $\nu(x)$ are different.

In the case $k_\Gamma = k$, we have

$$c(x_3) = 1 + \beta x_3,$$

while in the case $k_\Gamma > k$, we have

$$c(x_3) = \frac{1}{2}(1 + \frac{\beta}{\sqrt{k_1^2 - k^2}})e^{\sqrt{k_1^2 - k^2} x_3} + \frac{1}{2}(1 - \frac{\beta}{\sqrt{k_1^2 - k^2}})e^{-\sqrt{k_1^2 - k^2} x_3},$$
which in both cases contradicts the asymptotic behavior \[(12)\].

We now briefly consider the cases when \( \Gamma_0 \) is a portion of cylinder or a portion of sphere. If \( \Gamma_0 \) is the portion of cylinder of radius \( R \), then in \( Q_0 \) we obtain with appropriate cylindrical coordinates \((r, \theta, z)\)

\[
u(r, \theta, z) = u(R, \theta, z)c(r),
\]

where the function \( c \) is uniquely defined, with \( \kappa^2 := k^2 - k_0^2 \), by

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dc}{dr} \right) + \kappa^2 c = 0, \quad c(R) = 1, \quad \frac{dc}{dr}(R) = \beta.
\]

Following the case \( \kappa = 0, \kappa^2 > 0 \) or \( \kappa^2 = (i \tilde{\kappa})^2 < 0 \), the function \( c \) is respectively a linear combination of 1 and \( \log r \), a linear combination of \( H_1^0(\kappa r) \) and \( H_2^0(\kappa r) \) (Hankel functions of the first and second kind), or a linear combination of \( I_0(\tilde{\kappa} r) \) and \( K_0(\tilde{\kappa} r) \) (Modified Bessel functions). Whatever the case, the behavior of \( c(r) \) when \( r \to +\infty \) is not consistent with

\[
u(r, \theta, z) \sim e^{ikr(d_1 + zd_2)}, \quad r \to +\infty.
\]

If \( \Gamma_0 \) is the portion of sphere of radius \( R \), then in \( Q_0 \) we obtain with appropriate spherical coordinates \((r, \theta, \phi)\)

\[
u(r, \theta, \phi) = u(R, \theta, \phi)c(r),
\]

where the function \( c \) is uniquely defined, with \( \kappa^2 := k^2 - k_0^2 \), by

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dc}{dr} \right) + \kappa^2 c = 0, \quad c(R) = 1, \quad \frac{dc}{dr}(R) = \beta.
\]

Following the case \( \kappa = 0, \kappa^2 > 0 \) or \( \kappa^2 = (i \tilde{\kappa})^2 < 0 \), the function \( c \) is respectively a linear combination of 1 and \( 1/r \), a linear combination of \( e^{ikr}/r \) and \( e^{-ikr}/r \), or a linear combination of \( e^{\tilde{\kappa}r}/r \) and \( e^{-\tilde{\kappa}r}/r \). Whatever the case, the behavior of \( c(r) \) when \( r \to +\infty \) is not consistent with

\[
u(r, \theta, \psi) \sim e^{ikrd_1}, \quad r \to +\infty,
\]

which completes the proof.

\[\square\]

**Remark 2.** The geometrical assumptions in (H1) are only technical and are dictated by the proof where an explicit solution, using separation of variables, is needed. It seems plausible that our technique can also be generalized to other geometries where explicit solutions that satisfy (8) can be obtained. However it does not apply to the general case.

**Proposition 3.** Under assumptions \((H2a)-(HC)\), the farfield associated with one incident plane wave uniquely determines \( \mu \).

**Proof.** The identity

\[
\frac{\partial u_1}{\partial \nu} + \mu_1 \Delta \Gamma u_1 + \lambda u_1 = \frac{\partial u_1}{\partial \nu} + \mu_2 \Delta \Gamma u_1 + \lambda u_1 = 0
\]

on \( \Gamma \) leads, by denoting again \( u_1 = u \) to simplify the notation, and if we assume \( \mu_1 \neq \mu_2 \), to

\[
u = C \quad \text{and} \quad \frac{\partial u}{\partial \nu} = -C \lambda \quad \text{on} \quad \Gamma,
\]

\[\square\]
where $C \in \mathbb{C}$ is a constant.

We now recall the following classical representation formulas for $u^s$ and $u^i$ (see for example [16]).

\begin{align}
(14) & \quad u^s(x) = DL(u^s)(x) - SL(\frac{\partial u^s}{\partial \nu})(x), \quad \forall x \in \Omega, \\
(15) & \quad u^i(x) = -DL(u^i)(x) + SL(\frac{\partial u^i}{\partial \nu})(x), \quad \forall x \in D,
\end{align}

where the single- and double-layer potentials $SL$ and $DL$ are defined for $\phi \in H^{-1/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$ by

\begin{align*}
SL(\phi) = \langle \Phi(x, \cdot), \phi(\cdot) \rangle_{H^{1/2}(\Gamma),\overline{H^{-1/2}(\Gamma)}}, & \quad DL(\psi) = \left\langle \frac{\partial \Phi(x, \cdot)}{\partial \nu}, \psi(\cdot) \right\rangle_{H^{-1/2}(\Gamma),\overline{H^{1/2}(\Gamma)}}.
\end{align*}

Here $\Phi(x, y) := e^{ik|x-y|}/4\pi ||x-y||$ denotes the fundamental solution of the Helmholtz equation that satisfies the Sommerfeld radiation condition.

Let $\gamma_0^-$ (resp. $\gamma_0^+$) be the interior (resp. exterior) trace application, $\gamma_1^-$ (resp. $\gamma_1^+$) be the interior (resp. exterior) trace of the normal derivative application. Following [16], we define

\begin{align}
(16) & \quad \left\{ \begin{array}{l}
S = \gamma_0^- SL = \gamma_0^+ SL : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\
T = \frac{1}{2}(\gamma_0^+ DL + \gamma_0^- DL) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\
R = -\gamma_1^- DL = -\gamma_1^+ DL : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\
T^* = \frac{1}{2}(\gamma_1^+ SL + \gamma_1^- SL) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),
\end{array} \right.
\end{align}

and we recall the jump relations

\begin{align*}
\gamma_0^\pm DL \psi = \frac{1}{2}(\pm \psi + 2T \psi), & \quad \forall \psi \in H^{1/2}(\Gamma), \\
\gamma_1^\pm SL \phi = \frac{1}{2}(\pm \phi + 2T^* \phi), & \quad \forall \phi \in H^{-1/2}(\Gamma).
\end{align*}

First, by passing to the limit $x \to \Gamma$ in (14) and (15), it follows that in the sense of trace on $\Gamma$,

\begin{align}
(17) & \quad \frac{1}{2} u^s(x) = T(u^s)(x) - S(\frac{\partial u^s}{\partial \nu})(x), \\
(18) & \quad \frac{1}{2} u^i(x) = -T(u^i)(x) + S(\frac{\partial u^i}{\partial \nu})(x).
\end{align}

Using now $u = u^s + u^i$, the boundary conditions [13] and the equation (17), we obtain that

\begin{align}
\frac{1}{2} u^s(x) = C T(1)(x) - T(u^i)(x) + CS(\lambda)(x) + S(\frac{\partial u^i}{\partial \nu})(x).
\end{align}

By subtracting (15) to the above equation we obtain

\begin{align}
\frac{1}{2} (u^s(x) - u^i(x)) = C T(1)(x) + CS(\lambda)(x),
\end{align}

so that by using once again $u^s + u^i = C$ on $\Gamma$,

\begin{align}
(19) & \quad u^i(x) = C \left( \frac{1}{2} - T(1)(x) - S(\lambda)(x) \right)
\end{align}
Secondly, by taking the trace on $\Gamma$ of the normal derivative in (14) and (15) and by repeating the same calculations as above, we obtain

\begin{equation}
\frac{\partial u_1}{\partial \nu} = C(R(1) - \frac{\lambda}{2} - T^*(\lambda))
\end{equation}

in the sense of $H^{-1/2}(\Gamma)$. Equations (19) and (20) are equivalent to (7), which is forbidden by assumption $(HC)$.

**Remark 3.** In view of lemma 3.1, we point out that since $\lambda \in L^\infty(\Gamma)$, $S(\lambda)$ reduces on $\Gamma$ to $S(\lambda)(x) = \int_{\Gamma} \lambda(y) \Phi(x,y) \, ds(y)$ and, if the boundary $\Gamma$ is of class $C^1$ or is a polyhedral, $T(1)$ reduces on $\Gamma$ to $T(1)(x) = \int_{\Gamma} \frac{\partial \Phi}{\partial \nu_y}(x,y) \, ds(y)$.

**Remark 4.** One can easily show that if assumption $(HC)$ is not satisfied, i.e. one can find a constant $C$ and a field $\lambda$ which satisfy (7), then one can reconstruct a total field $u$ associated to the incident field $u^i$ that satisfies the generalized impedance boundary conditions for any value of $\mu$. In other words, assumption $(HC)$ is necessary to have uniqueness.

**Proposition 4.** Under assumptions $(H2b) - (HC)$ or $(H2c) - (HC)$, the farfield associated with one incident plane wave uniquely determines the missing parts of $\lambda$ and $\mu$.

**Proof.** The identity

$$\frac{\partial u_1}{\partial \nu} + \mu_1 \Delta u_1 + \lambda_1 u_1 = \frac{\partial u_1}{\partial \nu} + \mu_2 \Delta u_1 + \lambda_2 u_1 = 0$$

on $\Gamma$ leads, by denoting again $u_1 = u$ to simplify the notation, and if we assume $\mu_1 \neq \mu_2$, to

$$-\Delta u = \frac{1}{\mu_2 - \mu_1} (\lambda_2 - \lambda_1) u.$$

By multiplying the above equation by $u$ and by integrating by part over $\Gamma$, we obtain

$$\int_{\Gamma} |\nabla u|^2 \, ds = \frac{1}{\mu_2 - \mu_1} \int_{\Gamma} (\lambda_2 - \lambda_1) |u|^2 \, ds.$$

The assumption $(H2b)$ implies that $\lambda_2 - \lambda_1 \in i\mathbb{R}$ and $\mu_2 - \mu_1 \in \mathbb{R}$, while the assumption $(H2c)$ implies that $\lambda_2 - \lambda_1 \in \mathbb{R}$ and $\mu_2 - \mu_1 \in i\mathbb{R}$. In both cases $(\lambda_2 - \lambda_1)/(\mu_2 - \mu_1) \in i\mathbb{R}$. By taking the real part of the above equation we obtain

$$\int_{\Gamma} |\nabla u|^2 \, ds = 0,$$

whence there exists a constant $C$ such that $u = C$ on $\Gamma$. This implies that $\Delta u = 0$ and by using the same arguments as in the proof of Proposition 1, one obtains in particular that $\lambda_1 = \lambda_2 := \lambda$. We are then in the same configuration as in (14), and therefore one can complete the proof as for Proposition 3.

Now we specify two particular situations where the condition $(HC)$ is automatically satisfied.
Lemma 3.1. Let $\lambda \in L^\infty(\Gamma)$ and $d$ be a unitary vector of $\mathbb{R}^3$ and assume that one of the following holds:

1. The function $\lambda$ is real and $k^2$ is not a Dirichlet eigenvalue of the operator $-\Delta$ in $D$.
2. The domain $D$ is $C^1$ and both $D$ and the function $\lambda$ are assumed to be invariant by reflection against a plane which does not contain the direction $d$ or by a rotation (different from identity) around an axis which is not directed by $d$.

Then, there exists no constant $C$ such that

$$e^{ikx \cdot d} = C(1 - 2T(1)(x) - 2S(\lambda)(x)) \quad \forall x \in \Gamma.$$  

Proof. Let us consider first the case when $\lambda$ is real and $k^2$ is not a Dirichlet eigenvalue of the operator $-\Delta$ in $D$. Taking the imaginary part of (21), we obtain on $\Gamma$

$$\text{Im}\left(\frac{e^{ikx \cdot d}}{2C}\right) = -\text{Im}(T)(1)(x) - \text{Im}(S)(\lambda)(x).$$  

Let us define the two functions on $D \cap \Omega$,

$$f_1(x) = \text{Im}\left(\frac{e^{ikx \cdot d}}{2C}\right), \quad f_2(x) = -\text{Im}(DL)(1)(x) - \text{Im}(SL)(\lambda)(x).$$

Considering the regularity of

$$\text{Im}(DL)(1)(x) = \int_\Gamma \frac{\partial}{\partial y_q}(\text{Im}(\Phi(x, y))) \, ds(y), \quad x \in D \cap \Omega$$

and

$$\text{Im}(SL)(\lambda)(x) = \int_\Gamma \text{Im}(\Phi(x, y))\lambda(y) \, ds(y), \quad x \in D \cap \Omega,$$

since $\text{Im}(\Phi(x, y)) = f(||x - y||)$ where $f(r) = \sin(kr)/(4\pi r)$ is a $C^\infty$ function, it is readily shown by using differentiation under the integral sign that for general Lipschitz domain $D$, the function $f_2$ is in $C^1(\mathbb{R}^3)$. First, this implies that the traces on $\Gamma$ of functions $f_1$ and $f_2$ coincide with the left-hand side and the right-hand side of equation (22), respectively. Secondly, $f_1$ and $f_2$ are functions in $C^1(\mathbb{R}^3)$ which both satisfy the Helmholtz equation in $D$ and $\Omega$. Hence they satisfy the Helmholtz equation in $\mathbb{R}^3$. Since their traces on $\Gamma$ coincide, and since $k^2$ is not a Dirichlet eigenvalue of the negative laplacian in $D$, $f_1$ and $f_2$ coincide in $D$, and then in $\mathbb{R}^3$ by unique continuation. But $f_2$ tends to 0 when $||x|| \to +\infty$, while $f_1$ does not, which completes the proof for the first case.

Let us now consider the second case where $D$ and $\lambda$ are assumed to be invariant by a transformation $S$ which is either a reflection against a plane $P$ with $d \notin P$ or a rotation (different from identity) around an axis $A$ with $d \notin A$.

Without loss of generality, we assume that the plane $P$ and the axis $A$ contains $O$. We have

$$T(1)(x) = \int_\Gamma \frac{\partial}{\partial y_q}(x, y) \, ds(y) = \int_\Gamma \frac{\partial}{\partial y_q}(x - y) \, ds(y), \quad \phi(z) := \frac{e^{ik||z||}}{4\pi ||z||},$$

$$S(\lambda)(x) = \int_\Gamma \lambda(y)\Phi(x, y) \, ds(y) = \int_\Gamma \lambda(y)\phi(x - y) \, ds(y).$$
We use the new variables $x = Sx'$ and $y = Sy'$, and the new function $φ'$ defined by $φ(x) = φ'(x') = φ(Sx')$. We have for some $x ∈ Γ$,

$$T(1)(x) = \int \nabla_y φ(x - y).ν_y \, ds(y) = \int \nabla_y φ(S(x' - y')).ν_y \, ds(y).$$

Using the chain rule $\nabla_y φ'(x' - y') = \nabla_y φ(S(x' - y')).S$ and $ν_y = Sν_y'$, we obtain

$$T(1)(x) = \int \nabla_y φ'(x' - y').ν_y' \, ds(y).$$

By using the change of variable $y = Sy'$ in the integral, we obtain

$$T(1)(x) = \int \nabla_y φ'(x' - y').ν_y' \, ds(y').$$

Finally, since $∥Sx′∥ = ∥x∥$ and given the form of $φ$, $φ'(x') = φ(x')$, and thus $T(1)(x) = T(1)(x')$. As a result $T(1)(Sx) = T(1)(x)$ for all $x ∈ Γ$. We prove the same way that $S(λ)(Sx) = S(λ)(x)$ for all $x ∈ Γ$, and if (21) holds for some constant $C$, then for all $x ∈ Γ$

$$e^{ikSx \cdot d} = e^{ikx \cdot d},$$

that is for some $n ∈ Z$

$$Sx \cdot d = x \cdot d + \frac{2nπ}{k}.$$
Proof. We denote \( u^* \) and \( u^*_h \), the solutions of (2) which are associated to \( \lambda \) and \( \lambda + h \) in \( L^\infty(\Gamma) \). In order to prove differentiability, we first prove continuity. As we already remarked at the end of section 2, if we denote \( K = -J^{-1}_R K_R \) and \( F = J^{-1}_R F_R \) in order to simplify notations, \( u^* \) and \( u^*_h \) are respectively solutions in \( H^1(\Omega_R) \) of
\[
(I - K)u^* = F, \quad (I - K_h)u^*_h = F_h,
\]
where \( K_h \) and \( F_h \) satisfy \( K_0 = K \), \( F_0 = F \) and depend continuously on \( h \). By denoting \( A = I - K \) and \( A_h = I - K_h \), we already know that \( A^{-1} \) and \( A_h^{-1} \) exist and are bounded. Then the family of operators \( A_h \) meet the assumptions of Theorem 10.1 in [13], and provided that \( |||A^{-1}(A_h - A)||| < 1 \), which happens when \( ||h||_{L^\infty(\Gamma)} \) is sufficiently small, we obtain the error estimate
\[
||u^*_h - u^*||_{H^1(\Omega_R)} \leq \frac{|||A^{-1}|||}{1 - |||A^{-1}(A_h - A)|||} (||(A_h - A)u^*||_{H^1(\Omega_R)} + ||F_h - F||_{H^1(\Omega_R)}) .
\]
The continuity of the operator \( T \) follows.
Now let us denote \( e^*_h = u^*_h - u^* \). The function \( e^*_h \) satisfies the boundary condition on \( \Gamma \)
\[
\frac{\partial e^*_h}{\partial \nu} + \lambda e^*_h = -h(u^*_h + u^*).
\]
This leads to the fact that there exists \( C > 0 \) (independent of \( h \)) such that
\[
||e^*_h||_{H^1(\Omega_R)} = ||u^*_h - u^*||_{H^1(\Omega_R)} \leq C||h||_{L^\infty(\Gamma)}.
\]
Finally, let us define \( v^*_h \) as the far field associated to the scattered field \( v^*_h \), where \( v^*_h \) is the solution of problem (2) with data \( f = -h(u^*_h + u^*) \). The mapping \( h \to v^*_h \) is clearly a linear continuous operator \( L^\infty(\Gamma) \to L^2(S^2) \). We obtain
\[
\frac{\partial(e^*_h - v^*_h)}{\partial \nu} + \lambda(e^*_h - v^*_h) = -he^*_h.
\]
It follows that \( u^*_h - u^* - v^*_h \) is the solution of problem (2) with data \( f = -he^*_h \). By using (24), it is clear that \( T \) is differentiable in the sense of Fréchet with \( dT_h(h) = v^*_h \). We have from (4),
\[
v^*_h(\hat{x}) = \int_\Gamma \left\{ v^*_h(y) \frac{\partial \Phi^\infty(y, \hat{x})}{\partial \nu(y)} - \frac{\partial v^*_h(y)}{\partial \nu(y)} \Phi^\infty(y, \hat{x}) \right\} ds(y), \quad \forall \hat{x} \in S^2.
\]
Using the identity
\[
\int_\Gamma \left( \frac{\partial p^s}{\partial \nu} + \lambda p^s \right) v^*_h ds(y) = \int_\Gamma \left( \frac{\partial v^*_h}{\partial \nu} + \lambda v^*_h \right) df ds(y),
\]
the definitions of \( p^s \) and \( v^*_h \) lead to
\[
\int_\Gamma \left( \frac{\partial \Phi^\infty}{\partial \nu} + \lambda \Phi^\infty \right) v^*_h ds(y) = \int_\Gamma h(u^* + u^')p^s ds(y),
\]
As a result, we obtain
\[
v^*_h(\hat{x}) = \int_\Gamma \left\{ (u^* + u^')(p^s + \Phi^\infty)h \right\} ds(y),
\]
which is the result since \( u = u^* + u^t \) and \( p = p^* + \Phi^\infty \).

It remains to prove that \( dT_\lambda : C_I(\Gamma) \rightarrow L^2(S^2) \) is injective. Assume that for 
\( h \in C_I(\Gamma) \), we have
\[
\int_\Gamma p(y, \hat{x}) u(y) h(y) \, ds(y) = 0, \quad \forall \hat{x} \in S^2.
\]

By using lemma 4.1 there exists a sequence \( \hat{x}_n, n \in \mathbb{N} \), with \( p(., \hat{x}_n)|_{\Gamma} \rightarrow u|_{\Gamma} h \) in \( L^2(\Gamma) \). It follows that \( u|_{\Gamma} h = 0 \) on \( \Gamma \), and by using the same arguments as in the proof of proposition 1, we conclude that \( h = 0 \). \( \square \)

The proof of the above proposition requires the following density lemma.

**Lemma 4.1.** If \( f \in L^2(\Gamma) \) satisfies
\[
\int_\Gamma p(y, \hat{x}) f(y) \, ds(y) = 0, \quad \forall \hat{x} \in S^2,
\]
then \( f = 0 \).

**Proof.** For \( f \in L^2(\Gamma) \), we hence assume that
\[
(25) \quad \int_\Gamma p(y, \hat{x}) f(y) \, ds(y) = 0, \quad \forall \hat{x} \in S^2.
\]

Let \( u^* \) be the solution of (2) which is associated to data \( f \). We have
\[
\int_\Gamma (p^* \partial u^* - \partial p^* u^*) \, ds(y) = 0,
\]
which leads by using the boundary condition for \( u^* \), i.e. \( \partial u^*/\partial \nu + \lambda u^* = f \) on \( \Gamma \),
\[
\int_\Gamma (\partial p^* + \lambda p^*) u^* \, ds(y) = \int_\Gamma f p^* \, ds(y).
\]

Using now the boundary condition for \( p^* \), i.e. \( \partial p^*/\partial \nu + \lambda p^* = -(\partial \Phi^\infty/\partial \nu + \lambda \Phi^\infty) \) on \( \Gamma \),
\[- \int_\Gamma (\partial \Phi^\infty + \lambda \Phi^\infty) u^* \, ds(y) = \int_\Gamma f p^* \, ds(y).
\]

From (25) we have
\[
\int_\Gamma f p^* \, ds(y) = - \int_\Gamma f \Phi^\infty \, ds(y).
\]

By using the two previous equalities and once again the boundary condition for \( u^* \), it follows that
\[
\int_\Gamma (u^* \partial \Phi^\infty - \partial u^* \Phi^\infty) \, ds(y) = 0,
\]
which is exactly \( u^\infty(\hat{x}) = 0 \), for all \( \hat{x} \in S^2 \). We conclude from Rellich’s lemma and unique continuation that \( u^* = 0 \), and hence \( f = \partial u^*/\partial \nu + \lambda u^* = 0 \). \( \square \)

We now introduce \( D_I(\Gamma) \) a finite dimensional subspace of \( C_I(\Gamma) \). We have the following result of local Lipschitz stability in \( D_I(\Gamma) \cap L^\infty_I(\Gamma) \).

**Lemma 4.2.** For each \( \lambda \in D_I(\Gamma) \cap L^\infty_I(\Gamma) \) there exist \( \eta(\lambda) > 0 \) and \( C(\lambda) > 0 \) such that for all \( h \in L^\infty(\Gamma) \) verifying \( \lambda + h \in D_I(\Gamma) \cap L^\infty_I(\Gamma) \) and \( ||h||_{L^\infty(\Gamma)} \leq \eta(\lambda) \),
\[
||h||_{L^\infty(\Gamma)} \leq C(\lambda)||u_h^\infty - u^\infty||_{L^2(S^2)},
\]
where \( u^\infty = T(\lambda) \) and \( u_h^\infty = T(\lambda + h) \).
Proof. We denote \( v_h^\infty = dT_\lambda(h) \) and \( w_h^\infty = ||h||_{L^\infty(\Gamma)}\varepsilon(h) \) in \( D_\lambda \), so that
\[
u_h^\infty - u^\infty = v_h^\infty + w_h^\infty.
\]
From proposition \( \delta \) we deduce that \( dT_\lambda : D_\lambda(\Gamma) \rightarrow R(D_\lambda(\Gamma)) \) is injective, whence it is of continuous inverse since \( D_\lambda(\Gamma) \) is finite dimensional. Precisely, for all \( \lambda \in D_\lambda(\Gamma) \cap L_+^\infty(\Gamma) \) there exists \( c(\lambda) > 0 \) such that for all \( h \in L^\infty(\Gamma) \) with \( \lambda + h \in D_\lambda(\Gamma) \cap L_+^\infty(\Gamma) \),
\[
||h||_{L^\infty(\Gamma)} \leq c(\lambda)||v_h^\infty||_{L^2(S^2)}.
\]
Setting \( \varepsilon = ||u_h^\infty - u^\infty||_{L^2(S^2)} \),
\[
||v_h^\infty||_{L^2(S^2)} \leq ||v_h^\infty + w_h^\infty||_{L^2(S^2)} + ||w_h^\infty||_{L^2(S^2)} = \varepsilon + ||h||_{L^\infty(\Gamma)}||\varepsilon(h)||_{L^2(S^2)}.
\]
From the two above estimates we conclude that
\[
\left( \frac{1}{c(\lambda)} - ||\varepsilon(h)||_{L^2(S^2)} \right) ||h||_{L^\infty(\Gamma)} \leq \varepsilon.
\]
The fact that \( ||\varepsilon(h)||_{L^2(S^2)} \rightarrow 0 \) when \( ||h||_{L^\infty(\Gamma)} \rightarrow 0 \) completes the proof.

Lastly, \( K_I(\Gamma) \) denotes a compact subset of \( D_\lambda(\Gamma) \cap L_+^\infty(\Gamma) \). In the same spirit as in \( \delta \), we obtain the following result of stability in \( K_I(\Gamma) \), which is Lipschitz stability for the considered inverse impedance problem.

**Theorem 4.3.** For all \( \lambda \in K_I(\Gamma) \), there exists a positive constant \( C(\lambda) \) such that for all \( \tilde{\lambda} \in K_I(\Gamma) \),
\[
||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} \leq C(\lambda)||u^\infty - \tilde{u}^\infty||_{L^2(S^2)},
\]
where \( u^\infty = T(\lambda) \) and \( \tilde{u}^\infty = T(\tilde{\lambda}) \).

**Proof.** For \( \lambda \in K_I(\Gamma) \), we have for all \( \tilde{\lambda} \in K_I(\Gamma) \) such that \( ||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} \leq \eta(\lambda) \),
\[
||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} \leq C(\lambda)||u^\infty - \tilde{u}^\infty||_{L^2(S^2)},
\]
where \( \eta(\lambda) \) and \( C(\lambda) \) are defined as in Lemma 4.2.

It remains to prove that \( (26) \) is still valid for \( ||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} > \eta(\lambda) \), perhaps with another constant \( C(\lambda) \). Assume that for all \( n \in \mathbb{N} \), there exists \( \lambda_n \) such that
\[
||\lambda - \lambda_n||_{L^\infty(\Gamma)} > \eta
\]
and
\[
||\lambda - \lambda_n||_{L^\infty(\Gamma)} \geq n||u^\infty - \tilde{u}^\infty||_{L^2(S^2)}.
\]
where \( T(\lambda_n) := \tilde{u}_n^\infty \). It follows that
\[
||u^\infty - \tilde{u}_n^\infty||_{L^2(S^2)} \leq \frac{2M}{n},
\]
where \( M > 0 \) is such that \( K_I(\Gamma) \subset B(0, M) \) in \( L^\infty(\Gamma) \). Since the sequence \( (\lambda_n) \) belongs to the compact set \( K_I(\Gamma) \), it follows that there exists a sub-sequence, still denoted \( (\lambda_n) \), which converges to \( \tilde{\lambda} \). By continuity of operator \( T \), it follows that \( \tilde{u}_n^\infty \) converges to \( T(\tilde{\lambda}) := \tilde{u}^\infty \). From the previous inequality it follows that \( u^\infty = \tilde{u}^\infty \), and from the injectivity of the restriction of \( T \) to \( C_I(\Gamma) \), it follows that \( \lambda = \tilde{\lambda} \), which is in contradiction with \( ||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} > \eta \). The proof is complete.

**Remark 6.** Let us remark that similar Lipschitz stability results were already established in \( \tilde{\delta} \) (see Theorem 2.4) for the Laplace equation and standard impedance problem \( (\mu = 0) \) with a piecewise-constant impedance. In \( \tilde{\delta} \), the Lipschitz constant is proved to be independent of \( \lambda \), with the help of a more complex technique based on a quantification of the unique continuation principle. One can also find in
a quantification of the exponential blowing up of the Lipschitz stability constant with respect to the space dimension of the parameters. For other results related to the impedance problem for the Laplace equation one can refer to [10, 11, 12, 13] and the references therein.

Remark 7. Let $M > 0$ be a given real. A typical example of subsets $K_I(\Gamma)$ is the set of functions $\lambda(x) = \sum_{i=1}^I \alpha_i\varphi_i(x)$ where $\varphi_i, i = 1, \ldots I$ are given continuous non negative real functions on $\Gamma$ and the $\alpha_i$ satisfy: $|\alpha_i| \leq M$ and $\text{Im}(\alpha_i) \geq 0$ for $i = 1, \ldots I$. This would correspond for instance to a discretization of the problem using a finite element method.

4.2. The generalized impedance problem. We analyze the particular case of assumption $(H2b) - (HC)$ or $(H2c) - (HC)$, the other cases can be studied similarly. The analysis of stability for the inverse problem here is based on the Frechét derivative of operator $T : V(\Gamma) \subset L^\infty(\Gamma) \times \mathbb{C} \rightarrow L^2(S^2)$, where $V(\Gamma)$ is defined by assumptions $(H0) - (H2b) - (HC)$ or $(H0) - (H2c) - (HC)$ (that guarantee well posedness of the forward scattering problem as well as uniqueness for the inverse problem). We have the following proposition.

Proposition 6. The operator $T$ is differentiable in $V(\Gamma)$ and its Fréchet derivative is the operator $dT_{\lambda,\mu} : L^\infty(\Gamma) \times \mathbb{C} \rightarrow L^2(S^2)$ which maps $(h, l)$ to $v^\infty_h$ such that

$$v^\infty_h(\hat{x}) = \langle p(\cdot, \hat{x}), l\Delta_\Gamma u + uh \rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)}, \quad \forall \hat{x} \in S^2,$$

where

$$p(\cdot, \hat{x}) = \Phi^\infty(\cdot, \hat{x}) + p^\ast(\cdot, \hat{x}),$$

$p^\ast(\cdot, \hat{x})$ is the solution of problem (3) in which $u^\ast$ is replaced by $\Phi^\infty(\cdot, \hat{x})$.

Moreover, the operator $dT_{\lambda,\mu} : C_I(\Gamma) \rightarrow L^2(S^2)$ is injective under assumption $(H2b) - (HC)$ or $(H2c) - (HC)$.

Proof. By reasoning exactly as in the proof of proposition 5 we obtain that $T$ is differentiable and that $dT_{\lambda,\mu}(h, l)$ coincide with the far field $v^\infty_h$ associated to the scattered field $v^s_{h,l}$, where $v^s_{h,l}$ is the solution of problem (2) with data $f = -l\Delta_\Gamma(u^s + u^\ast) - h(u^s + u^\ast) = -l\Delta_\Gamma u - hu$. From (3), we have for all $\hat{x} \in S^2$

$$v^\infty_{h,l}(\hat{x}) = \left<v^s_{h,l}, \frac{\partial \Phi^\infty(\cdot, \hat{x})}{\partial \nu} \right>_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} - \left<\frac{\partial v^s_{h,l}}{\partial \nu}(\hat{x}), \Phi^\infty(\cdot, \hat{x}) \right>_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

Since on $\Gamma$

$$\frac{\partial v^s_{h,l}}{\partial \nu} + \mu \Delta_\Gamma v^s_{h,l} + \lambda v^s_{h,l} = -l\Delta_\Gamma u - hu,$$

we obtain that

$$v^\infty_{h,l}(\hat{x}) = \left<v^s_{h,l}, \frac{\partial \Phi^\infty(\cdot, \hat{x})}{\partial \nu} \right>_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} + \left<\mu \Delta_\Gamma v^s_{h,l} + \lambda v^s_{h,l} + l\Delta_\Gamma u + hu, \Phi^\infty \right>_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

By using the fact that

$$\langle \Delta_\Gamma v^s_{h,l}, \Phi^\infty \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = -\int_\Gamma \nabla v^s_{h,l} \cdot \nabla \Phi^\infty \, ds = \langle v^s_{h,l}, \Delta_\Gamma \Phi^\infty \rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)},$$

$$v^\infty_{h,l}(\hat{x}) = \left<v^s_{h,l}, \frac{\partial \Phi^\infty}{\partial \nu} + \mu \Delta_\Gamma \Phi^\infty + \lambda \Phi^\infty \right>_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} + \left<\Phi^\infty, l\Delta_\Gamma u + hu \right>_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)}.$$
Using now the definition of $p^s$, we have on $\Gamma$
\[
\frac{\partial p^s}{\partial \nu} + \mu \Delta p^s + \lambda p^s = -(\mu \Phi^\infty + \lambda \Phi^\infty).
\]
Thus
\[
\left\langle v^s_{h,l}, \frac{\partial \Phi^\infty}{\partial \nu} + \mu \Delta \Phi^\infty + \lambda \Phi^\infty \right\rangle_{H^1(\Gamma), H^{-1}(\Gamma)} = -\left\langle v^s_{h,l}, \frac{\partial p^s}{\partial \nu} + \mu \Delta p^s + \lambda p^s \right\rangle_{H^1(\Gamma), H^{-1}(\Gamma)}.
\]
Finally we obtain
\[
v^\infty_{h,l}(\hat{x}) = \langle p', I \Delta u + hu \rangle_{H^1(\Gamma), H^{-1}(\Gamma)},
\]
which is the desired result.

It follows from Lemma 4.4 that $I \Delta u + hu = 0$ on $\Gamma$, and by using the same arguments as in the proof of proposition 4, we conclude that $h = 0$ and $l = 0$.

The proof of the above proposition requires the following lemma, which is the analogous of Lemma 4.3.

**Lemma 4.4.** If $f \in H^{-1}(\Gamma)$ satisfies
\[
\langle p(\cdot, \hat{x}), f \rangle_{H^1(\Gamma), H^{-1}(\Gamma)} = 0, \quad \forall \hat{x} \in S^2,
\]
then $f = 0$.

**Proof.** Let $u^s$ be the solution of (2) which is associated to data $f$. We have
\[
\left\langle p^s, \frac{\partial u^s}{\partial \nu} \right\rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} = \left\langle \frac{\partial p^s}{\partial \nu} + u^s \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0,
\]
which leads by using the boundary condition for $u^s$, i.e. $\partial u^s / \partial \nu + \Delta u^s + \lambda u^s = f$ on $\Gamma$,
\[
\left\langle u^s, \frac{\partial p^s}{\partial \nu} + \mu \Delta p^s + \lambda p^s \right\rangle_{H^1(\Gamma), H^{-1}(\Gamma)} = \langle p^s, f \rangle_{H^1(\Gamma), H^{-1}(\Gamma)}.
\]
Using now the boundary condition for $p^s$, i.e. $\partial p^s / \partial \nu + \mu \Delta p^s + \lambda p^s = -(\partial \Phi^\infty / \partial \nu + \mu \Delta \Phi^\infty + \lambda \Phi^\infty)$ on $\Gamma$,
\[
- \left\langle u^s, \frac{\partial \Phi^\infty}{\partial \nu} + \mu \Delta \Phi^\infty + \lambda \Phi^\infty \right\rangle_{H^1(\Gamma), H^{-1}(\Gamma)} = \langle p^s, f \rangle_{H^1(\Gamma), H^{-1}(\Gamma)}.
\]
From (27) we have
\[
\langle p^s, f \rangle_{H^1(\Gamma), H^{-1}(\Gamma)} = -\langle \Phi^\infty, f \rangle_{H^1(\Gamma), H^{-1}(\Gamma)}.
\]
By using the two previous equalities and once again the boundary condition for $u^s$, it follows that
\[
\left\langle u^s, \frac{\partial \Phi^\infty}{\partial \nu} \right\rangle_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} = \left\langle \frac{\partial u^s}{\partial \nu}, \Phi^\infty \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0,
\]
which is exactly $u^\infty(\hat{x}) = 0$, for all $\hat{x} \in S^2$. We conclude from Rellich’s lemma and unique continuation that $u^s = 0$, and hence $f = \partial u^s / \partial \nu + \Delta u^s + \lambda u^s = 0$. \qed

By using the same arguments as for the classical impedance problem, we obtain the following Lipschitz stability for the inverse problem when the parameters are restricted to the subset $K(\Gamma)$ defined as follows. Let $D(\Gamma)$ be a finite dimensional subspace of $C_f(\Gamma) \times \mathbb{C}$, $K(\Gamma)$ a compact subset of $D(\Gamma) \cap V(\Gamma)$. 
Theorem 4.5. For all \((\lambda, \mu) \in K(\Gamma)\), there exists a positive constant \(C(\lambda, \mu)\) such that for all \((\tilde{\lambda}, \tilde{\mu}) \in K(\Gamma)\),
\[
||\lambda - \tilde{\lambda}||_{L^\infty(\Gamma)} + |\mu - \tilde{\mu}| \leq C(\lambda, \mu) \||u^\infty - \tilde{u}^\infty||_{L^2(S^2)},
\]
where \(u^\infty = T(\lambda, \mu)\) and \(\tilde{u}^\infty = T(\tilde{\lambda}, \tilde{\mu})\).

References