Electromagnetic wave propagation at classical material/metamaterial interfaces - Mathematical aspects

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Setting of the problem

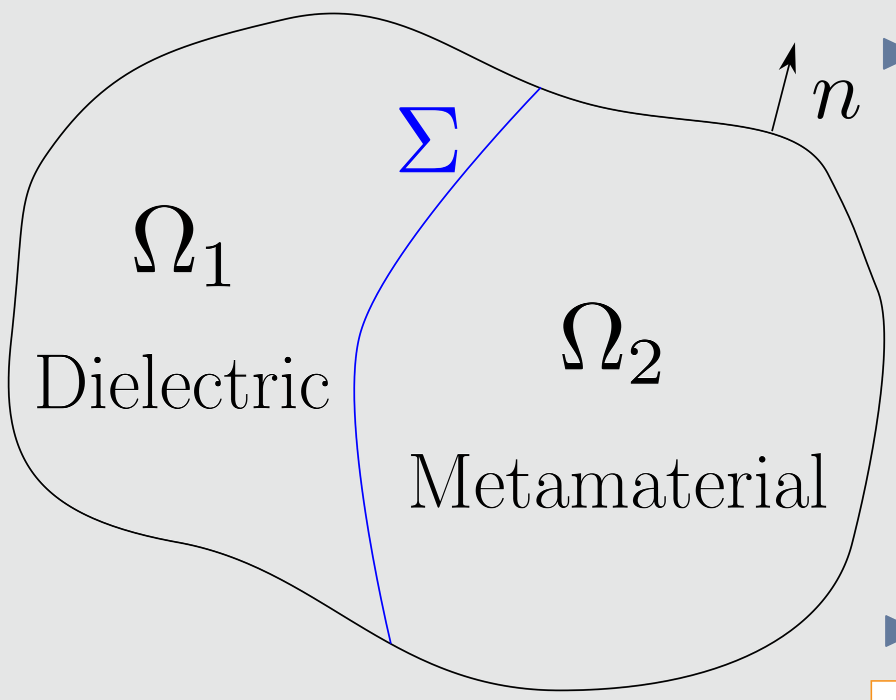
- Maxwell time-harmonic problem (electric field) set in a heterogeneous medium Ω like below (2D example).

▷ At a given frequency, Metamaterial = structure s.t.

$$\Re e(\mu_2^\eta) = \mu_2 < 0$$
$$\Re e(\epsilon_2^\eta) = \epsilon_2 < 0$$

▷ Dissipation modeled by η

$$\mu^\eta = \mu\,(1 + i\,\text{sign}(\mu)\,\eta)$$
$$\epsilon^\eta = \epsilon\,(1 + i\,\text{sign}(\epsilon)\,\eta)$$



- Define $X_N(\Omega, \epsilon^\eta) = \{E \in L^2(\Omega)^3 \,|\, \textbf{curl}\, E \in L^2(\Omega)^3,\ \text{div}\, \epsilon^\eta E \in L^2(\Omega) \text{ and } E \times n = 0 \text{ on } \partial\Omega\}$

$(\mathcal{P}^\eta)$ Find $E \in X_N(\Omega, \epsilon^\eta)$ such that :

$$\textbf{curl}\left(\frac{1}{\mu^\eta}\textbf{curl}\, E\right) - \omega^2 \epsilon^\eta E = F \quad \text{in } \Omega$$

- Augmented variational formulation (continuous Galerkin FE) :

$(\mathcal{P}_V^\eta)$ Find $E \in X_N(\Omega, \epsilon^\eta)$ such that :

$$\int_\Omega \frac{1}{\mu^\eta}\textbf{curl}\, E \cdot \textbf{curl}\, \overline{V} + s\, \text{div}\, \epsilon^\eta E\, \text{div}\, \overline{\epsilon^\eta V}\, d\Omega$$
$$-\omega^2 \int_\Omega \epsilon^\eta\, E \cdot \overline{V} d\Omega = \int_\Omega F \cdot \overline{V} d\Omega, \ \forall V \in X_N(\Omega, \epsilon^\eta)$$

Questions :

▷ Is the problem to be solved well-posed ?
▷ How to compute a numerical approximation of the solution ?
▷ Influence of dissipation ?

Well - posedness ?

- Coerciveness over $X_N(\Omega, \epsilon^\eta)$ of

$$a(E, V) = \int_\Omega \frac{1}{\mu^\eta}\textbf{curl}\, E \cdot \textbf{curl}\, \overline{V} + s\, \text{div}\, \epsilon^\eta E\, \text{div}\, \overline{\epsilon^\eta V}\, d\Omega$$

✔ Ok provided $\eta \neq 0$ (take $s = (|\epsilon^\eta|^2\, \mu^\eta)^{-1}$)

- Compactness of the term

$$b(E, V) = -\omega^2 \int_\Omega \epsilon^\eta\, E \cdot \overline{V} d\Omega$$

✔ Ok as the canonical embedding of $X_N(\Omega, \epsilon^\eta)$ into $L^2(\Omega)^3$ is compact when $\eta \neq 0$ (extension of [Weber'80] result)

If $\eta \neq 0$ (dissipative case)

▷ Coercive+compact framework $\Rightarrow$ problem $(\mathcal{P}_V^\eta)$ is well-posed
▷ Numerical convergence (assumption : no singular electric fields)

What if $\eta = 0$?

- EXAMPLE

Consider a symmetric domain Ω with
$\kappa_\mu = \mu_1/\mu_2 = -1$
$\kappa_\epsilon = \epsilon_1/\epsilon_2 = -1$

Dielectric	Metamaterial	Dielectric
$\mu_1^0 > 0$	$\mu_2^0 = -\mu_1^0 < 0$	$\mu_1^0 > 0$
l	$2l$	l

▷ Infinite dimensional kernel $\Rightarrow$ scalar problem ill-posed
▷ The embedding of $X_N(\Omega, \epsilon)$ into $L^2(\Omega)^2$ is not compact

- In general, FE error estimation

$$\|E^\eta - E_h^\eta\|_{X_N} \leq \frac{C\, h^{s-1}}{\eta} \|E^\eta\|_{PH^s(\Omega)}$$

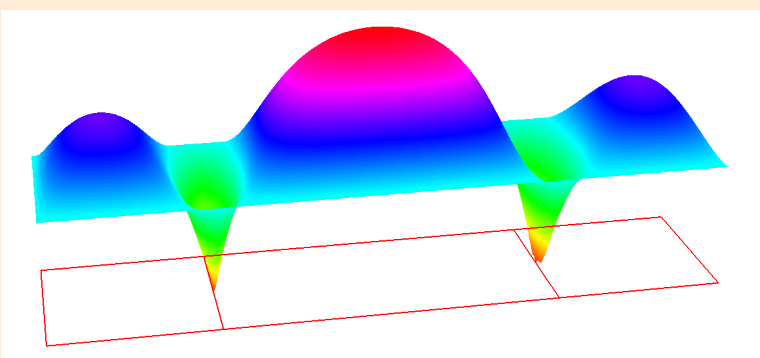
When $\eta \to 0$, not an optimal estimation.

If $\eta = 0$ (dissipationless case)

⚠ Usual techniques fail and classical results no longer hold

Results and conjectures for the dissipationless Maxwell problem

If Σ is smooth with $\kappa_\mu \neq -1$ and $\kappa_\epsilon \neq -1$



✔ The embedding of $X_N(\Omega, \epsilon)$ into $L^2(\Omega)^3$ is still compact
✔ $a(\cdot,\cdot)$ is stable (T-coercive) when $\kappa_\mu \notin I_\mu = \left[\kappa_\mu^{inf}; \kappa_\mu^{sup}\right]$ with $-1 \in I_\mu$
[Bonnet-Ciarlet Jr.-Zwölf'09], [Chesnel, work in progress]

☛ Conjecture : $(\mathcal{P}_V^0)$ is well-posed and standard numerical methods converge

If Σ is piecewise–smooth (with corners) with $\kappa_\mu \neq -1$ and $\kappa_\epsilon \neq -1$

✔ The embedding of $X_N(\Omega, \epsilon)$ into $L^2(\Omega)^2$ (2D) is compact except for a finite set of κ_ϵ (Mellin techniques) [Bonnet-Dauge-Ramdani'99], [Chesnel, work in progress]

- If $\kappa_\mu \notin I_\mu = \left[\kappa_\mu^{inf}; \kappa_\mu^{sup}\right]$ with $-1 \in I_\mu$

☛ Conjecture : $(\mathcal{P}_V^0)$ is well-posed and standard numerical methods converge

- If $\kappa_\mu \in I_\mu$

☛ Conjecture : $(\mathcal{P}_V^0)$ is ill-posed (solution with infinite energy)

Question : Are the models derived from physics still relevant ?