Non-reflecting boundary conditions for acoustic propagation in ducts with acoustic treatment and mean flow

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SUMMARY

We consider a time-harmonic acoustic scattering problem in a 2D infinite waveguide with walls covered with an absorbing material, in the presence of a mean flow assumed uniform far from the source. To make this problem suitable for a finite element analysis, the infinite domain is truncated. This paper concerns the derivation of a non-reflecting boundary condition on the artificial boundary by means of a Dirichlet-to-Neumann (DtN) map based on a modal decomposition. Compared with the hard-walled guide case, several difficulties are raised by the presence of both the liner and the mean flow. In particular, acoustic modes are no longer orthogonal and behave asymptotically like the modes of a soft-walled guide. However, an accurate approximation of the DtN map can be derived using some bi-orthogonality relations, valid asymptotically for high-order modes. Numerical validations show the efficiency of the method. The influence of the liner with or without mean flow is illustrated.

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1. INTRODUCTION

In this paper, we are interested in the numerical simulation of 2D acoustic scattering in locally perturbed infinite lined guides with or without mean flow in the frequency domain. Our aim is to model the infinite domain by transparent boundary conditions suitable for use in a finite element scheme. We focus here on the method based on a Dirichlet-to-Neumann (DtN) boundary condition. This method requires the knowledge of the transverse modes of the guide: an important part of this paper is then devoted to the calculation and the description of duct modes and we expect such results to be useful for other applications or other modal methods [1]. This type of boundary condition is used to reduce as much as possible the computational domain, including cases with significant attenuation for which one could use simpler conditions instead (rigid artificial boundaries for instance) but far enough from the source.

Solving such scattering problems can be very helpful to better understand physical properties of the acoustical propagation in a focused part of a complex configuration. Indeed, many industrial applications, such as turbofan aircraft engine, car mufflers, and ventilations shafts, contain ducts that present some localized inhomogeneity related to their shape, their wall properties, or the presence...
of sources and obstacles. Because a complex system can be often separated into elementary systems connected by uniform waveguides, the use of transparent boundary conditions appears to be very suitable for determining transfer functions. In the above-cited applications, the fluid is often in motion and the walls of the guide can be covered by an acoustical treatment in order to reduce the noise radiation.

The purpose of this paper is to propose an expression of the DtN operator adapted to a 2D straight lined guide with a uniform mean flow. Let us point out that our method applies to quite general configurations (see Figure 1): the only restriction is to have a straight guide with a uniform mean flow outside a bounded complex region. This will be illustrated in Section 5.

Modeling acoustic propagation in an unbounded domain requires the use of specific methods, such as boundary integral methods [2, 3], infinite elements [4–6], absorbing layers [7, 8], or non-reflecting boundary condition (NRBC) which may be local [9, 10] or non-local [11–15]. Among infinite domain problems, a distinction must be made between scattering problems around a body in a free space and problems in semi-infinite or infinite waveguides that make up the issue of the present work. This last case raises specific difficulties such as the multimodal nature of the far field and the occurrence of inverse modes (unstable in Perfectly Matched Layers (PMLs)). NRBC is the commonly used method for ducts: it consists of specifying a boundary condition (a DtN condition, for instance) on a transverse artificial boundary. Such condition is exact but non-local: all degrees of freedom on the artificial boundary are coupled, leading to a full block matrix and requiring therefore a specific treatment in the finite element (FE) scheme.

The DtN approach, which is easy to carry out in a hard-walled guide, raises several difficulties in presence of an absorbing lining. The determination of DtN operators requires the calculation of the transverse modes of the guide, which satisfy an eigenvalue problem. For the no-flow case, in presence of an absorbent treatment, the modal eigenvalue problem is not self-adjoint. However, it has been proved in a different context in [14] that, except for a countable set of critical impedance values, completeness of the modes holds and a diagonal expression of the modal DtN map can be derived, using a suitable bi-orthogonality relation. Extending this approach to the case of a uniform mean flow is not straightforward: the scalar model of the Helmholtz-convected equation (for the pressure or the velocity potential) leads to a modal eigenvalue problem which is quadratic (instead of linear in the no-flow case). Consequently, the derivation of an exact bi-orthogonality relation would require the introduction of additional unknowns (like for instance the acoustic velocity) [15]. In order to keep a scalar model, we exploit an asymptotic bi-orthogonality relation, valid for high-order modes and leading to a non-diagonal expression of the DtN map. Let us mention that contrary to the case of a fluid at rest, the study of the modal problem in the presence of a mean flow still raises many open mathematical questions. In particular, completeness of the modes, which is required in our method, is not established.

Another difficulty is the possible presence of instabilities in the presence of uniform mean flow and absorbent material. The existence of unstable modes is a difficult problem that goes beyond our work about defining DtN operators. The problem is not to include these modes in the definition of the DtN operators but rather to determine the relevance of unstable modes in the presence of the Myers boundary condition. This is still an open question in particular since a recent paper
[16] that shows that the classical Briggs–Bers criterium, to separate stable to unstable modes induced by Myers condition, is not theoretically justified. To avoid this difficulty, we consider here configurations where the influence of unstable modes on the near field is neglectful.

The outline of the paper is the following. The setup of the problem and governing equations are presented in the next section. The DtN method is presented in Section 3 for a lined guide without flow. Except for a countable sequence of critical values of the impedance $Z$, a DtN operator is derived thanks to a bi-orthogonality relation. Section 4 is devoted to the generalization of the method to a lined guide with a uniform flow. This is possible with the definition of a pseudo-inner product deduced from an asymptotic orthogonality relation. Numerical results are presented in Section 5. The method is first validated in the case of a straight lined guide with a uniform flow. Generalization to a non-uniform lined guide with a potential mean flow shows the efficiency of the approach. All the results are compared with those obtained using PMLs.

2. GEOMETRY AND GOVERNING EQUATIONS

We consider an infinite 2D duct of height $h$ containing a fixed acoustic source and a perfect compressible fluid in subsonic uniform flow of horizontal speed $v_0 = u_0 e_x$. The problem is posed in the $Oxy$ plane, where the $x$-axis is parallel to the walls of the guide (see Figure 2). The upper wall $\Gamma_{Z_\infty}$ ($y = h$) is covered with an absorbing material characterized by an impedance $Z (Z \in \mathbb{C})$. For the sake of simplicity, the lower horizontal boundary $\Gamma_{\infty}$ ($y = 0$) of the duct is supposed to be perfectly rigid. In order to use an FE method, the computational domain containing the source should be truncated by two artificial boundaries $\Sigma_-$ and $\Sigma_+$ at $x = 0$ and $x = L$, respectively (see Figure 2(b)). For the sake of clarity, the method is presented here for a straight guide. It is also possible to consider a problem with some irregularities (such as non-uniform cross-section, inhomogeneities, obstacle). In this case, artificial boundaries should be located beyond these irregularities as presented in Section 5.

With a uniform flow of Mach number $M = u_0/c$, where $c$ is the speed of sound, according to the classical hypothesis of linear acoustics, the time-harmonic acoustic wave propagation (assuming an $e^{-i\omega t}$ time dependence) is described by the convected Helmholtz equation:

$$
(1 - M^2) \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + 2ikM \frac{\partial p}{\partial x} + k^2 p = f \quad \text{in } \Omega_{\infty},
$$

(1)

where $p$ denotes the acoustic pressure, $k = \omega/c$ is the wavenumber and $f$ is a source with compact support. The boundary condition on the rigid wall is

$$
\frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma_{\infty},
$$

(2)

where $\partial p/\partial n = \nabla p \cdot n$ and $n$ is the unit exterior normal vector.

Figure 2. Problem geometry: (a) Infinite waveguide and (b) truncated waveguide.
The proper boundary condition associated with a locally reacting liner of impedance \( Z_0 \) (\( Z_0 \) is a complex number with \( \Re(Z_0) > 0 \)), expressed with the quantities continuous at the interface liner/fluid \( \xi \) and the normal displacement \( \xi n = \xi n \), reads:

\[
p = Z_0 \frac{\partial \xi n}{\partial t} = -i\omega Z_0 \xi n. \tag{3}
\]

Then, for a uniform mean flow parallel to the \( x \) coordinate, the normal component of the acoustic velocity can be expressed in terms of the convective derivative of the normal displacement:

\[
v_n = c_0 \left( M \frac{\partial}{\partial x} - ik \right) \xi n. \tag{4}
\]

The normal derivative of the pressure is deduced from the momentum equation in the \( x \)-direction:

\[
-\frac{\partial p}{\partial n} = \rho_0 c_0 \left( M \frac{\partial}{\partial x} - ik \right) v_n. \tag{5}
\]

Using (3) and (4), and introducing the dimensionless impedance \( Z = Z_0 / \rho_0 c_0 \), we obtain finally the so-called Myers boundary condition:

\[
\frac{\partial p}{\partial n} = -\frac{i}{kZ} \left( M \frac{\partial}{\partial x} - ik \right)^2 p \quad \text{on } \Gamma_{Z\infty}. \tag{6}
\]

Note that tangential derivatives of the pressure appear in the boundary condition (6), which lead to additional difficulties clarified later on.

In order to work in a bounded region \( \Omega \) (see Figure 2(b)), transparent boundary conditions must be specified on the artificial boundaries \( \Sigma_- \) and \( \Sigma_+ \). Such conditions are expressed, thanks to DtN operators:

\[
\nabla p_\cdot n = \frac{\partial p}{\partial n} = -T^\pm (p) \quad \text{on } \Sigma_\pm. \tag{7}
\]

A classical approach is to determine a modal expression of the DtN operators (7). Such expression can be deduced from an expansion of the pressure on the guide modes. Suppose for instance that

\[
p(x, y) = \sum_{n=0}^{\infty} A_n(p) \theta_n(y) e^{i\beta_n(x-L)} \quad \text{for } x > L, \tag{8}
\]

then a simple derivation leads to:

\[
T^+(p) = -\sum_{n=0}^{\infty} i\beta_n A_n(p) \theta_n. \tag{9}
\]

A main difficulty to justify such an expansion is to prove the completeness of the modes. Then, an orthogonality relation is needed to provide the modal amplitudes \( A_n(p) \).

### 3. Derivation of the DtN Operator Without Mean Flow

First let us consider the fluid at rest. The problem is reduced to the Helmholtz equation and the following boundary conditions:

\[
\Delta p + k^2 p = f \quad \text{in } \Omega, \tag{10}
\]

\[
\frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma. \tag{11}
\]
\[
\frac{\partial p}{\partial n} = \frac{ikp}{Z} \quad \text{on } \Gamma_Z, \tag{12}
\]

\[
\frac{\partial p}{\partial n} = -T^\pm_Z(p) \quad \text{on } \Sigma_\pm. \tag{13}
\]

3.1. Modes calculation

The modes are obtained by the classical separation of variables method: \( p(x, y) = \phi(y) e^{i\beta x} \). This leads to the so-called transverse eigenvalue problem (\( k \) is fixed, \( \beta^2 \) are the eigenvalues):

\[
\frac{d^2 \phi}{dy^2} = -(k^2 - \beta^2) \phi \quad \text{for } y \in ]0, h[, \tag{14}
\]

\[
\frac{d\phi}{dy} = 0 \quad \text{at } y = 0, \tag{15}
\]

\[
\frac{d\phi}{dy} = \frac{ik\phi}{Z} \quad \text{at } y = h. \tag{16}
\]

The solutions are \( \phi(y) = a \cos(zy) \), where the transversal wave numbers \( z \) are the solutions of the complex transcendent equation:

\[
-z \tan(zy) = \frac{ik}{Z}. \tag{17}
\]

\( a \) is a normalization coefficient determined in the next section. If \( z \) is a solution, then \(-z\) is also a solution. Owing to the symmetry of the cos function, we only look for roots such that \( \Re(e(z)) \geq 0 \). As in the case of a perfectly rigid duct with a fluid at rest, it is well known that there are two families of modes:

\[
p^\pm_n(x, y) = a_n \cos(z_n y) e^{\pm i\beta_n x},
\]

where the constant of propagation \( \beta_n \) is linked to \( z_n \) by the dispersion relation:

\[
\beta^2_n = k^2 - z^2_n \quad \text{with } \Im(\beta_n) > 0.
\]

All the modes are evanescent and indices \( \pm \) correspond to the direction of mode propagation (\( \pm \) for \( x \to \pm \infty \)).

Solutions for (17) are found numerically by the Newton–Raphson method. This method is very efficient and converges quickly, but does require an initial guess. Such a guess is obtained using an FE scheme to solve the transverse eigenvalue problem. Approximated values of \( \beta_n \) (and thus of \( z_n \)) are found and then refined, thanks to the Newton–Raphson algorithm.

Some numerical values of \( \beta_n \) are listed in Table I.

As expected, all modes are attenuated due to the presence of the absorbent material, but the first three modes are almost propagative. Similar to the rigid wall case (\(|Z| = \infty\)), we still have a finite number of (almost) propagative modes and an infinite number of (strongly) evanescent modes (pseudo cut-off phenomenon).

Once the transverse wavenumbers \( z_n \) are known, it is easy to plot the corresponding modes in the cross-section. As expected, the absorbent material on the wall modifies the shape of the first modes compared with the rigid case (see modes 1, 2, 3, 4 on Figure 4(a)). In particular, the first mode in no longer a plane wave.

3.2. Asymptotic behavior of the modes

In this section, we study more precisely the asymptotic behavior of the modes as \( n \to +\infty \). One can show that the modes behave asymptotically like those of a rigid guide (see the behavior of the
Figure 3. Propagation constants $\beta_n$ for $Z=(1 - i)$ and $k = 7$.

Table I. Roots of $-x_n \tan(x_n h) = ik/Z$ and corresponding $\beta_n$ for $Z=(1 - i)$ and $k = 7$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$\beta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.3517 - 0.1634i</td>
<td>6.8702 + 0.0321i</td>
</tr>
<tr>
<td>2</td>
<td>4.0525 - 0.3690i</td>
<td>5.7254 + 0.2612i</td>
</tr>
<tr>
<td>3</td>
<td>6.8580 - 0.3734i</td>
<td>1.9552 + 1.3099i</td>
</tr>
<tr>
<td>4</td>
<td>9.8138 - 0.3093i</td>
<td>0.4409 + 6.8854i</td>
</tr>
<tr>
<td>5</td>
<td>12.855 - 0.2516i</td>
<td>0.3000 + 10.783i</td>
</tr>
</tbody>
</table>

Figure 4. (a) Modes shape $\theta_n$ in a lined guide for $Z=(1 - i)$ and $k = 7$ and (b) zoom for $n = 60$ and $n = 100$ near the lined wall.

modes at the lined wall on figure 4(b)). Looking at (17) and using $|x_n| \xrightarrow{n \to +\infty} +\infty$, it is clear that:

$$\tan(x_n h) \xrightarrow{n \to +\infty} 0.$$
Furthermore, an asymptotic behavior of $x_n$ is given in [14]:

$$x_n h = n \pi + \frac{k h}{i Z n \pi} + O \left( \frac{1}{n^3} \right).$$

This expression can be easily recovered from (17) and the following ansatz:

$$x_n h = n \pi + \epsilon_n \quad \text{with} \quad |\epsilon_n| \ll 1,$$

but the rigorous proof is not straightforward.

3.3. Bi-orthogonality relation

According to the standard manipulation to establish an orthogonality relation, namely multiplying (14) by $\hat{\theta}_m$ ($\hat{\theta}$ is the complex conjugate of $\theta$), integrating by parts over $[0, h]$, and using the boundary conditions (15) and (16) lead to:

$$\int_0^h d\theta_n \frac{d\hat{\theta}_m}{dy} dy - (k^2 - \beta_n^2) \int_0^h \theta_n \hat{\theta}_m dy = \frac{ik}{Z} \theta_n(h) \hat{\theta}_m(h).$$

(18)

Exchanging the roles of $m$ and $n$ and conjugating, one obtains after substraction:

$$\left( \beta_n^2 - \beta_m^2 \right) \int_0^h \theta_n \hat{\theta}_m dy - i k \left( \frac{1}{Z} + \frac{1}{Z} \right) \theta_n(h) \hat{\theta}_m(h) = 0.$$

As expected, classical results are recovered for a purely imaginary impedance $Z$ (i.e., pure reactance) or for perfectly rigid boundaries ($|Z|=\infty$): in these simple cases, the modes form an orthonormal basis of $L^2([0, h])$ in the sense of the usual inner product on $L^2([0, h])$:

$$(\theta_n, \theta_m) = \int_0^h \theta_n \hat{\theta}_m dy = \delta_{nm}.$$  

(19)

In our case ($Z \notin i \mathbb{R}$), the modal eigenvalue problem is no longer self-adjoint and the modes are no longer orthogonal for the usual inner product (19). However, exchanging the roles of $m$ and $n$ in (18) without conjugating leads to:

$$\left( \theta_n, \theta_m \right)^* = 0 \quad \text{for} \quad n \neq m,$$

where we have set

$$(\theta, \psi)^* = \int_0^h \theta \psi dy.$$  

(20)

According to the definition given by Ochmann and Donner [17], the sequences $(\theta_n)$ and $(\hat{\theta}_m)$ are said to be bi-orthogonal. Let us emphasize that relation (20) does not define an $L^2$-inner product; in particular, it can vanish for $\theta = \psi$, even if $\theta \neq 0$. Since

$$\int_0^h \cos^2(z y) dy = \frac{h}{2} \left[ 1 + \frac{\sin(2zh)}{2zh} \right],$$

it appears indeed that the normalization of the modes with respect to the bi-orthogonality relation (20) is not possible if

$$\sin(2zh) + 2zh = 0.$$  

(21)

To the sequence $z_c^n$ of roots of (21) is associated the sequence of critical values $Z_c^n$:

$$Z_c^n = -\frac{ik}{z_c^n \tan(z_c^n h)},$$  

(22)
Table II. Critical values $\lambda_c^n$ and $Z_c^n$ for $k = 7$.  

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2\lambda_c^n h$</th>
<th>$Z_c^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.2124 − 2.2507i</td>
<td>2.0685 + 1.6583i</td>
</tr>
<tr>
<td>2</td>
<td>10.7125 − 3.1031i</td>
<td>1.1424 + 0.4403i</td>
</tr>
<tr>
<td>3</td>
<td>17.0734 − 3.5511i</td>
<td>0.7665 + 0.2051i</td>
</tr>
<tr>
<td>4</td>
<td>23.3984 − 3.8588i</td>
<td>0.5740 + 0.1197i</td>
</tr>
<tr>
<td>5</td>
<td>29.7081 − 4.0937i</td>
<td>0.4578 + 0.0784i</td>
</tr>
</tbody>
</table>

Numerical values of $\lambda_c^n$ obtained by the Newton-Raphson method and associated critical values $Z_c^n$ are presented in Table II.

It has been proved in [14] that, except for this countable sequence of critical values $Z_c^n$, $(\theta_n)_{n \in \mathbb{N}}$ is a complete family of $L^2(0, h)$.

**Remark**

A complete mathematical study has been made in [14] to solve the transcendent equation (17) in the complex plane. It has been shown that when $Z$ is a critical value, (17) has a double root.

Indeed, introducing $g(z) = z \tan(zh) - ik/Z$, one gets $g'(z) = (\sin(2zh) + 2zh)/2\cos^2(zh)$.

Let us point out that the sequence $Z_c^n$ tends to zero when $n$ tends toward infinity. From now on, we suppose that $Z$ is not equal to a critical impedance and that the eigenfunctions $\theta_n$ are such that:

$$(\theta_n, \theta_m) = \delta_{nm}.$$  \hspace{1cm} (23)

In other words, good properties of self-adjoint eigenvalue problems are recovered in this non-self-adjoint case.

### 3.4. DtN operator and variational formulation

Using previous results, we can now build the DtN operators $T_Z^\pm$ involved in the boundary conditions on $\Sigma_{\pm}$ (13). Since $Z$ is not a critical impedance, the radiated pressure $p$ for $x = L$ can be expanded in a series of the eigenfunctions $\theta_n$:

$$p(x = L, y) = \sum_{n=0}^{\infty} A_n \theta_n(y).$$

The bi-orthogonality property (20) leads to:

$$A_n = (p, \theta_n)_{\Sigma_+}^*,$$  \hspace{1cm} (24)

where in an obvious way and using (20), we define

$$(p, \theta_n)_{\Sigma_+}^* = (p(L, \cdot), \theta_n(\cdot))^* = \int_0^h p(L, y) \theta_n(y) dy.$$  \hspace{1cm} (25)

Thus for $x \geq L$:

$$p(x, y) = \sum_{n=0}^{\infty} (p, \theta_n)_{\Sigma_+}^* \theta_n(y) e^{i\beta_n(x-L)}.$$

The normal derivative of the pressure can be easily calculated and the DtN operator is expressed by:

$$T_Z^\pm(p) = -\nabla p \cdot n = \sum_{n=0}^{\infty} i\beta_n (p, \theta_n)_{\Sigma_+}^* \theta_n.$$  \hspace{1cm} (25)
The variational formulation associated with Equation (10) and boundary conditions (11)–(13) is obtained by multiplying Equation (10) by a test function \( \psi \), integrating over \( \Omega \), and applying Green’s theorem:

\[
\int_{\Omega} \nabla p \cdot \nabla \psi \, d\Omega - k^2 \int_{\Omega} p \psi \, d\Omega - \frac{ik}{Z} \int_{\Gamma_Z} p \psi \, d\Gamma + \int_{\Sigma_\pm} \psi \nabla p \cdot n \, d\Sigma = - \int_{\Omega} f \psi \, d\Omega.
\]

Using the expression of the DtN operator \( T^\pm_Z(p) \) (25), the variational formulation becomes:

\[
\int_{\Omega} \nabla p \cdot \nabla \psi \, d\Omega - k^2 \int_{\Omega} p \psi \, d\Omega - \frac{ik}{Z} \int_{\Gamma_Z} p \psi \, d\Gamma - \sum_{n=0}^{\infty} i\beta_n(p, \theta_n)^\pm (\psi, \theta_n)^\pm_{\Sigma_\pm} = - \int_{\Omega} f \psi \, d\Omega. \tag{26}
\]

In an FE treatment, it is well known that the matrix system obtained by the discretization of the variational formulation is usually sparse. Owing to the non-local nature of the DtN operator, the matrix corresponding to the boundary condition on \( \Sigma_\pm \) is a full matrix. In practice, the DtN operator (25) is calculated by truncating the series to a finite order \( N \) (in Equation (26)).

4. DERIVATION OF THE DtN OPERATOR WITH MEAN FLOW

With a uniform mean flow and a treated boundary, the problem in the two-dimensional guide is described by the convected Helmholtz equation (1) and the boundary conditions (2)–(6). Our purpose is to generalize the method described above to the presence of a flow. As we will see, this generalization is not straightforward.

4.1. Modes calculation

With a mean flow, the modal eigenvalue problem reads:

\[
\frac{d^2 \theta}{dy^2} = [k^2 - (1 - M^2)\beta^2 - 2kM \beta] \theta \quad \text{for } y \in ]0, h[. \tag{27}
\]

\[
\frac{d\theta}{dy} = 0 \quad \text{at } y = 0, \tag{28}
\]

\[
\frac{d\theta}{dy} = \frac{i(M\beta - k)^2 \theta}{kZ} \quad \text{at } y = h. \tag{29}
\]

As in the no-flow case, the solutions are of the form \( \theta(y) = a \cos(xy) \) where the transverse wave number now satisfies:

\[
-\alpha \tan(\alpha h) = \frac{i(\beta^2 + \alpha^2)}{kZ}. \tag{30}
\]

Contrary to the no-flow case, this relation depends on \( \beta \). This relation is coupled to the dispersion relation:

\[
\alpha^2 = k^2 - (1 - M^2)\beta^2 - 2kM \beta. \tag{31}
\]

Actually, searching for solutions to (30)–(31) is equivalent to finding \( \beta \) solution of Equation (30), where \( \alpha = \alpha(\beta) \) is a (complex) square root of \( k^2 - (1 - M^2)\beta^2 - 2kM \beta \). Indeed, since the tangent function is an odd function, the function \( \beta \in \mathbb{C} \rightarrow \alpha(\beta)\tan(\alpha(\beta)h) \) is well defined and independent of the choice of the square root.

One can divide the solutions into two families \( \pm \) according to the sign of the imaginary part of \( \beta \). More precisely \( \beta_\pm^\alpha \) is such that \( \pm \Im(\beta_\pm^\alpha) > 0 \) and \( \alpha_\pm = \alpha(\beta_\pm^\alpha) \) (with \( \Re(\alpha_\pm) \geq 0 \)) is the related transverse wave number. Except for the unstable modes, \((+)\) (resp. \((-))\) modes propagate downstream (resp. upstream).
behave asymptotically like those of a guide with a soft upper wall. Indeed, due to the presence of both impedance and flow, some values of $\alpha_n^\pm$ and $\beta_n^\pm$ are listed in Table III for a uniform mean flow of Mach number $M = 0.3$ and the propagation constants $\beta_n^\pm$ are represented in Figure 5.

Transverse modes for a lined guide with mean flow are represented in Figure 6. The influence of the flow is represented in Figure 7: we notice that low-order modes do not depend on the flow velocity $M$ (as in the case of a rigid boundary), but the effect of the flow becomes significant for high-order modes, particularly concerning their behavior at the treated boundary. This will be discussed in the next paragraph.

### 4.2. Asymptotic behavior of the modes

We have seen in Section 3.2 that in the no-flow case, the modes behave asymptotically ($n \to +\infty$) like those of a rigid guide. It appears that the results completely differ in the presence of a flow. Indeed, due to the presence of both impedance and flow ($|Z| < +\infty$ and $M > 0$), the modes behave asymptotically like those of a guide with a soft upper wall ($p = 0$ on the soft boundary, $\alpha_n^{\text{soft}} = (n + \frac{1}{2})\pi/h$) as it can be seen in Figure (6(b)).

More precisely, from (30) and (31), we deduce that:

$$\tan(\alpha_n^\pm h) \sim n x_n^\pm h,$$

(32)
Figure 6. (a) Downstream modes shape in a lined guide with a flow $M=0.3$, $k=7$, $Z=(1-i)$ and (b) zoom of modes 60 and 100 near the upper wall.

Figure 7. Difference between the modes shape in a lined guide ($Z=(1-i)$) without and with flow ($M=0.3$) for $k=7$.

where we have set:

$$\tau = \frac{i M^2}{k h Z(1-M^2)},$$

and therefore $|\tan(z_n^\pm h)| \xrightarrow{n \to +\infty} +\infty$. This leads us to the following ansatz:

$$z_n^\pm h = n\pi + \frac{\pi}{2} + \varepsilon_n \text{ with } |\varepsilon_n| \ll 1.$$

Injecting this ansatz into (32) provides finally the following asymptotic behavior:

$$z_n^\pm h = n\pi + \frac{\pi}{2} - \frac{1}{n\pi r} + O\left(\frac{1}{n^2}\right).$$
Let us point out that, contrary to the no-flow case, this asymptotic behavior has not been rigorously established. It could seem paradoxical to have completely different behaviors of the modes with and without flow, especially if the flow is slow ($M \ll 1$). In fact, this paradox concerns only high-order modes, and numerical experiments show that low-order modes are only slightly influenced by the presence of a slow flow (see Figure 7). More precisely, one can guess that the asymptotic behavior (34) will be reached for large values of $\tau$. In other words, for small values of $\tau$ ($M$ small and/or $|Z|$ large, which is the most usual case), the asymptotic behavior (34) applies only to very high-order modes and is therefore useless for physical or numerical studies. On the contrary, for large values of $\tau$ ($M \sim 1$ and/or $|Z|$ small), the 'strange' asymptotic behavior (34) will be effective for modes of moderate order, and could be probably observed in practical applications.

4.3. Definition of a new pseudo-inner product

Proceeding as in the no-flow case, we multiply (27) by $\theta_m$ and integrate over $]0, h]$:

$$
- \int_0^h \frac{d^2 \theta_n}{dy^2} \theta_m dy = [k^2 - (1 - M^2) \beta_n^2 - 2kM \beta_n] \int_0^h \theta_n \theta_m dy. \tag{35}
$$

Integrating by parts and using the boundary conditions (28) and (29), Equation (35) becomes:

$$
\int_0^h \frac{d \theta_n}{dy} \frac{d \theta_m}{dy} dy + [2kM \beta_n + (1 - M^2) \beta_n^2 - k^2] \int_0^h \theta_n \theta_m dy
$$

$$
+ \frac{i}{kZ} [2kM \beta_n - M^2 \beta_n^2 - k^2] \theta_n(h) \theta_m(h) = 0.
$$

If $\beta_n \neq \beta_m$, exchanging the roles of $m$ and $n$ and making the subtraction leads to:

$$
\left[ \frac{iM}{kZ} [2k - M(\beta_n + \beta_m)] \theta_n(0) \theta_m(0) + [2kM + (1 - M^2)(\beta_n + \beta_m)] \int_0^h \theta_n \theta_m dy \right] = 0.
$$

This is not an orthogonality relation because it still depends on $\beta_n$ and $\beta_m$. However, one notices that when $(\beta_n + \beta_m)$ is large:

$$
\lim_{\beta_n + \beta_m \to \infty} \left[ \int_0^h \theta_n \theta_m dy - \frac{iM^2}{kZ(1 - M^2)} \theta_n(h) \theta_m(h) \right] = 0.
$$

We introduce then a new bilinear relation (..), for which the modes of the guide become asymptotically bi-orthogonal:

$$
(\phi, \psi) = (\phi, \psi)^* - \tau h \psi(h) \quad \psi(h),
$$

where $\tau$ is defined by (33). The difference with the bi-orthogonality in the no-flow case $(..)^*$ is the presence of an additional term on the absorbing boundary at $y = h$.

4.4. Variational formulation and DtN operator

The variational formulation associated with Equation (1) and boundary conditions (2) and (6) reads:

$$
\int_\Omega \left\{ (1 - M^2) \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial y} - 2ikM \frac{\partial \psi}{\partial x} \psi - k^2 p \psi \right\} d\Omega
$$

$$
- \frac{i}{kZ} \int_{\Gamma_x} \left\{ M^2 \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} + 2ikM \frac{\partial \psi}{\partial x} \psi + k^2 p \psi \right\} d\Gamma
$$

$$
\mp (1 - M^2) \left( \left( \frac{\partial \psi}{\partial x}, \psi \right) \right) \Sigma_+ = \int_\Omega f \psi d\Omega.
$$
Assuming that we can expand the radiation pressure downstream the boundary $\Sigma_+$ (respectively, upstream the boundary $\Sigma_-$):

$$\forall x \geq L, \quad p(x, y) = \sum_{n=0}^{+\infty} A_n^+ \theta_n^+(y) e^{i\beta_n^+(x-L)}, \quad (36)$$

then

$$\left( \left( \frac{\partial p}{\partial x}, \psi \right) \right)_{\Sigma_+} = \sum_{n=0}^{+\infty} iA_n^+ \beta_n^+ ((\theta_n^+), \psi) \Sigma_+.$$

At this stage, we have to face a difficulty: due to the absence of an exact bi-orthogonality relation between the modes, the coefficients $A_n^+$ cannot be obtained by a simple projection of $p$ on $\theta_n^+$, as in (24). In practice, the modal expansion will be truncated at rank $N$ to express the coefficients $A_n^+$ as functions of $p$, replacing (36) by:

$$\forall x \geq L \quad p(x, y) = \sum_{n=0}^{N} A_n^+ \theta_n^+(y) e^{i\beta_n^+(x-L)}.$$

Then, the coefficients $(A_n^+)^{0 \leq n \leq N}$ are solutions of the following linear system:

$$\sum_{n=0}^{N} A_n^+ ((\theta_n^+, \theta_m^+)) = ((p, \theta_m^+))_{\Sigma_+} \quad \text{for} \ m = 0, 1, \ldots, N.$$

Let us call $O_N^+$ the symmetric spectral $N \times N$ matrix defined by:

$$[O_N^+]_{mm} = ((\theta_m^+, \theta_m^+)),$$

then, if $O_N^+$ is invertible,

$$A_n^+ = \sum_{m=1}^{N} [O_N^+]^{-1}_{mn} ((p, \theta_m^+))_{\Sigma_+}.$$

One can remark that for $M = 0$,

$$((\theta_n^+, \theta_m^+)) = ((\theta_n^+, \theta_m^+)) = 0 \quad \text{if} \ n \neq m.$$

Consequently, in the no-flow case, the matrix $O_N^+$ is diagonal and (24) is recovered.

Then, the new DtN operators can finally be defined by:

$$T_{ZM}^{N\pm}(p) = \pm \sum_{n=0}^{N} \sum_{n=0}^{N} i\beta_n^+ [O_N^+]^{-1}_{mn} ((p, \theta_m^+))_{\Sigma_+} \theta_n^+(y).$$

Summing up, we get the following variational formulation: find $p$ such that $\forall \psi$:

$$\int_{\Gamma} \{ (1-M^2) \left( \frac{\partial p}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial p}{\partial y} \frac{\partial \psi}{\partial y} - 2ikM \frac{\partial p}{\partial x} \psi - k^2 p \psi \right) \} \ d\Omega - \frac{i}{kZ} \int_{\Gamma} \left\{ M^2 \left( \frac{\partial p}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial p}{\partial y} \frac{\partial \psi}{\partial y} + k^2 \frac{\partial p}{\partial x} \psi \right) - k^2 p \psi \right\} \ d\Gamma$$

$$\pm(1-M^2) \sum_{n=0}^{N} \sum_{m=0}^{N} i\beta_n^+ [O_N^+]^{-1}_{mn} ((p, \theta_m^+))_{\Sigma_+} ((\psi, \theta_n^+))_{\Sigma_+} = - \int_{\Omega} f \psi \ d\Omega.$$

The formulation remains symmetric as in the no-flow case. The only difference concerns the DtN operators on $\Sigma_\pm$ which require the calculation of the matrices $[O_N^+]^{-1}$ and the use of the new pseudo-inner product.

The invertibility of $[O_N^+]$ is not obvious. One can guess that, as in the no-flow case, a sequence of critical impedance values $Z_n^\alpha$ still exists such that the completeness of the modes $\theta_n^\alpha$ fails. We
conjecture that:

(i) Unlike the no-flow case, $|Z^c_n| \to +\infty$.

(ii) If $Z \neq Z^c_n \forall n$, then the matrices $O^\pm_N$ are invertible for $N$ large enough.

Remark

Note that $O^\pm_N$ can be built using other bilinear relations, for instance the scalar product $(\cdot, \cdot)$. Then the term associated with the DtN operator in the variational formulation takes the non-symmetrical form:

$$\mp(1-M^2) \sum_{n=0}^{N} \sum_{m=0}^{N} i\beta^\pm_n [O^\pm_N]_{nn}^{-1}(p, \theta^\pm_n)(\psi, \theta^\pm_n)\Sigma^\pm,$$

where $[O^\pm_N]_{nn} = (\theta^\pm_n, \theta^\pm_m)$. We consider that the choice we have made to use the orthogonality condition induced by the asymptotic study has at least three advantages.

1. The variational term associated with the DtN operator is symmetric and thus the associated finite element matrix requires less computations.

2. $O^\pm_N$ becomes asymptotically diagonal, which implies that this matrix for moderate values of $N$ has more chances to be invertible; also the truncation error is better controlled due to the weak coupling between modes of low orders and modes of high orders.

3. Our formula to define $O^\pm_N$ is “optimal” since it remains valid for all values of the parameters, more precisely all values of $\tau$ defined in (33) as explained in Section 4.5.

4.5. Properties of the matrices $O^\pm_N$

We have already mentioned that one of the difficulties due to the presence of the mean flow is the fact that the matrices $O^\pm_N$ are not diagonal. We would like to point out now that they still are somewhat close to being diagonal. Following the discussion at the end of Section 4.2 and taking modes of unit amplitude $\theta^\pm_n(y) = \cos(x^\pm_n y)$, we will distinguish two different regimes:

- For large values of $\tau$, we can use the asymptotic behavior (34) to evaluate the coefficients $[O^\pm_N]_{mn}$. Straightforward calculations show that (for large $m$ or $n$, $n \neq m$):

$$\begin{align*}
(\theta^\pm_n, \theta^\pm_m)^* \sim \tau h \quad \theta^\pm_n(h)\theta^\pm_m(h) \sim (-1)^{a+m} \frac{h}{\pi^2 \tau mn},
\end{align*}$$

and that:

$$[O^\pm_N]_{nn} = ((\theta^\pm_n, \theta^\pm_n)) \sim (-1)^{a+m} \frac{h}{2\pi^2 \tau mn(m+n)}.$$

Therefore, extra diagonal terms become neglectful compared with diagonal terms that satisfy:

$$[O^\pm_N]_{mn} = ((\theta^\pm_n, \theta^\pm_m)) \sim \frac{h}{2} \delta_{nm} \quad \text{as } n \to +\infty,$$

- for small values of $\tau$, modes of interest (of low orders) are close to the modes in the no-flow case. Therefore, they satisfy:

$$\begin{align*}
(\theta^\pm_n, \theta^\pm_m)^* \sim \frac{h}{2} \delta_{nm}.
\end{align*}$$

Since

$$[O^\pm_N]_{nn} = ((\theta^\pm_n, \theta^\pm_n)) \sim -\tau h \theta^\pm_n(h)\theta^\pm_m(h),$$

$\tau$ small implies that $[O^\pm_N]_{nn}$ is nearly diagonal.
5. NUMERICAL RESULTS

We now present some numerical results for two different waveguides. In the first two examples, we consider straight-lined guides with and without uniform mean flow in order to test the performance of the DtN operators. The last example is the generalization to a non-uniform lined guide with a potential mean flow. Since we do not have an easy access to an analytical solution, we compare the numerical solution obtained with the DtN operator to the solution obtained with the use of PMLs of thickness equal to 10% of the computational domain. The parameters of the PMLs are well chosen [8], such that the obtained solution is exact up to the finite element approximation. All the following simulations were performed with the FEM library MELINA [18]. Additional examples can be found in [19].

5.1. Acoustic source in a lined guide without mean flow

For the following numerical test case, \( k = 7 \), \( Z = 1 - i \) and the fluid is at rest. The artificial boundaries are located at \( x = 0 \) and \( x = L = 4 \). A monopolar circular source \( f = 1 \) of radius \( R = 0.2 \) is located at the center of the computational domain meshed with 2732 triangles. Lagrangian finite element of order 2 (P2 elements) are used.

Figure 8 shows the real part of the acoustic pressure distribution in the guide obtained with a DtN operator and with PML respectively. 20 modes are used in the DtN operators, the error is less than 1%, and the attenuation due to the presence of the absorbent material is clearly seen.

One great interest of our method is to allow to take the vertical boundaries \( \Sigma_{\pm} \) very close to the source and thus to reduce the computational domain. Evanescent modes have then a non-neglectful contribution on \( \Sigma_{\pm} \), which requires to take enough terms in the DtN operators. In the next test case, the artificial boundaries \( \Sigma_{\pm} \) are located at \( x = \pm 0.3 \). A reference solution is computed using 20 modes in the DtN and the logarithm of the relative error in the \( L^2 \) norm is represented versus the number of terms \( N \) in the DtN (1 \( \leq N \leq 19 \)) in Figure 9. The error decreases quickly and is lower than \( 10^{-3} \) as soon as \( N \geq 5 \). Note that with only the three almost propagative modes, the result is rather good with an error less than 1%.

5.2. Acoustic source in a lined guide with uniform mean flow

The same configuration is now studied with a uniform mean flow of Mach number \( M = 0.3 \). Figure 10, presenting the solutions obtained with a DtN operator and with PML, shows that they
are very close. The error is less than 1% again. In addition to the attenuation, the convective effect of the flow is significant since the acoustic fields radiated downstream and upstream are quite different. We have used 20 modes in the expansion for the DtN operator.

The truncation error in Figure 11 behaves in a similar way than in the no-flow case. With just three modes, the number of almost propagative modes, the error is rather low, less than 1%.

5.3. Acoustic source in a lined guide with potential mean flow

In order to test the method on a non-trivial case, we now consider a non-uniform lined guide with a potential mean flow \( \nabla \phi_0 \). The velocity potential \( \phi_0 = \Re(g) \) and \( \psi_0 = \Im(g) \), where \( g(z) \) is a complex function \( z = x + iy \). In order to use the previously developed DtN operators, the artificial boundaries \( \Sigma_- \) and \( \Sigma_+ \) must be located in regions with a uniform cross-section and with a uniform mean flow. For this purpose, we choose \( g(z) = v_\infty z (1 - a^2/(b^2 + z^2)) \) where \( a \) and \( b \) are constants and \( v_\infty \) the velocity at infinity.
Figure 11. $L^2$ norm error ($\log$) versus the number of modes with DtN boundary conditions near the source ($M=0.3$, $Z=(1-i)$, $k=7$) and real part of the acoustic pressure.

Figure 12. Stream lines for $a=1.36$ and $b=2a$.

$(g(z) \sim v_\infty z$ at infinity) corresponding to a flow whose stream lines are represented in Figure 12. The infinite domain is truncated with artificial boundaries at $x=\pm 4$ and the uniform part of the guide is still of width $h = 1$. We consider a non-flat rigid boundary at the bottom of the guide ($\Gamma$) chosen as a particular streamline of the flow. As previously, an absorbent material of impedance $Z=(1-i)$ is located on the top ($\Gamma_Z$) at $y=h$.

Since the flow is potential, the acoustic perturbations are also potential and the associated acoustic potential $\phi$ satisfies [20, 21]:

$$\text{div}(\rho_0 \textbf{grad} \phi) - \rho_0 \frac{d}{dt} \left( \frac{1}{c_0^2} \frac{d \phi}{dt} \right) = f \quad \text{in } \Omega,$$

where

$$\frac{d \phi}{dt} = M v_0 \cdot \text{grad} \phi - i k \phi.$$

$M = v_\infty / c_\infty$ is the Mach number at infinity where $c_\infty$ is the sound speed at infinity and $k = \omega / c_\infty$. All the quantities $v_0$, $c_0$, and $\rho_0$ are non-dimensional and equal to:

$$v_0 = \frac{\text{grad} \phi_0}{v_\infty}, \quad c_0 = 1 + \frac{\gamma - 1}{2} M^2 (1 - |v_0|^2), \quad \rho_0^{-1} = c_0^2.$$
\( \gamma \) is the ratio of specific heats at constant pressure and volume. Using the Myers condition [22] on the liner, the boundary conditions are given by:

\[
\begin{align*}
\frac{\partial \phi}{\partial n} &= 0 \quad \text{on} \quad \Gamma, \\
\frac{\partial \phi}{\partial n} &= \frac{1}{i k Z} \left[ \left( M \mu_0 \frac{\partial}{\partial x} - i k - M \frac{\partial v_0}{\partial y} \right) \rho_0 \frac{d\phi}{dr} \right] \quad \text{on} \quad \Gamma_Z, \\
\frac{\partial \phi}{\partial n} &= -T_{ZM}^\pm(\phi) \quad \text{on} \quad \Sigma_\pm.
\end{align*}
\]

Figure 13 corresponds to a source located in the center of the computational domain. The real part of the acoustic potential shows the good agreement between results obtained with the DtN operator and with PMLs.

6. CONCLUSIONS

The effects of locally reacting absorbing material at the wall of an infinite waveguide are taken into account in order to express a transparent boundary condition based on a DtN map. This new transparent boundary condition first requires the numerical calculation of the modes in a guide with an absorbing wall with or without a uniform flow. Then, the main difficulty is due to the presence of an absorbing wall because the modes are no longer orthogonal for the usual scalar product. Without flow, a bi-orthogonality condition can be used. With a flow a new bi-orthogonality has been recovered, valid only asymptotically for high-order modes. Then the DtN operator can be defined by calculating a spectral matrix that becomes almost diagonal for high-order modes. The effectiveness of the operator is initially illustrated by the radiation of a circular source located in a straight lined duct with uniform mean flow. The sound attenuation and the convection effects are clearly shown. Finally, a non-uniform lined duct with a potential mean flow is studied in order to show the applicability of our method to more complex problems. The results are in good agreement with those obtained with PMLs.

REFERENCES


