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MATHEMATICAL ANALYSIS OF THE JUNCTION OF TWO ACOUSTIC OPEN WAVEGUIDES*

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Abstract. The present paper concerns the scattering of a time-harmonic acoustic wave by the junction of two open uniform waveguides, where the junction is limited to a bounded region. We consider a two-dimensional problem for which wave propagation is described by the scalar Helmholtz equation. The main difficulty in the modeling of the scattering problem lies in the choice of conditions which characterize the outgoing behavior of a scattered wave. We use here *modal radiation conditions* which extend the classical conditions used for closed waveguides. They are based on the generalized Fourier transforms which diagonalize the transverse contributions of the Helmholtz operator on both sides of the junction. We prove the existence and uniqueness of the solution, which seems to be the first result in this context. The originality lies in the proof of uniqueness, which combines a natural property related to energy fluxes with an argument of analyticity with respect to the generalized Fourier variable.

Key words. open waveguide, Helmholtz equation, radiation condition, generalized Fourier transform

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1. Introduction.

A uniform waveguide may be defined as a propagative medium whose physical features are invariant in one *longitudinal* direction so that waves can propagate in this direction and remain confined to a limited region in the orthogonal *transverse* direction(s). Such a waveguide is said to be open when the cross-section is unbounded; the confinement is then due to a particular arrangement of the inhomogeneities which allows an evanescent behavior of guided waves in the transverse direction(s). In the present paper, we are concerned with the modeling and mathematical analysis of the junction of two different open waveguides, which covers many physical applications in areas such as electromagnetism (junction of optical fibers, or between a fiber and an integrated optical device), acoustics (immersed junction of pipelines), elastodynamics (seismic waves in two layered media separated by a rift), and hydrodynamics (water waves guided by a varying cross-section ocean trench). Figure 1.1 illustrates two examples of junctions which will be considered in the paper, called *abrupt* and *thick*, depending on whether the part which contains the variable cross-section has a zero thickness or not.

The physical problem we are interested in can be expressed in a very simple way: consider an incident guided wave on one side of the junction; what happens to this wave when it meets the junction? It seems clear that the interaction of this incident wave with the junction will produce three kinds of waves: these are a reflected guided wave which propagates in the direction opposite to the incident wave, a transmitted

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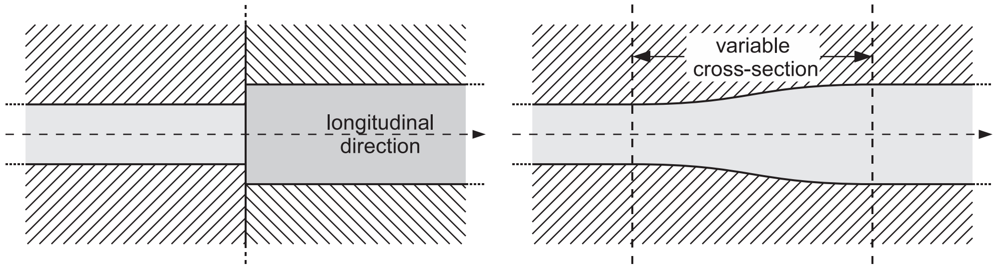

Fig. 1.1. *Examples of abrupt (left) and thick (right) junctions of waveguides (the different levels of gray and hatching represent different homogeneous media).*

guided wave in the other side of the junction, and finally a wave which radiates in the transverse direction(s). Nevertheless the mathematical modeling of this apparently simple problem is far from obvious. What kind of *radiation condition* can describe the outgoing behavior of a scattered wave? The question is well understood in the case of a closed waveguide (that is, when the cross section is bounded, for instance, in Figure 1.1, when the hatching represents a nonpenetrable medium). Indeed, in a uniform closed waveguide, a wave can be described as a discrete superposition of guided and evanescent modes, which leads us to interpret a junction by means of the modal transmission and reflection coefficients. To a certain extent, the *radiation conditions* we shall use generalize such a description to open waveguides. The main issue we shall deal with is proving that, with these conditions, the propagation equations become well-posed. Such a result may seem surprising for those who are familiar with closed waveguides. Indeed, the uniqueness of the solution rules out the existence of *trapped modes* for a junction of open waveguides, whereas such modes are known to occur in perturbed closed waveguides (see, e.g., the review paper [15] as well as Remark 4.10).

Our study falls within the general framework of scattering of time-harmonic waves by unbounded inhomogeneities, among which one can distinguish a category of media gathered under the word "rough" (rough surfaces, rough layers, etc.), which garnered significantly increased interest in recent years as evidenced in the applied mathematical literature (see, e.g., [4, 5, 13] and the references cited therein). Although there is no precise definition of this word, it usually designates a perturbation of a medium invariant in some longitudinal direction(s), where the perturbation is localized in the transverse direction (finite amplitude) but not in the longitudinal one. In this sense, a junction of waveguides could be seen as a rough medium. But all the results of existence and uniqueness of a solution obtained in this context concern cases where guided modes do not exist. In these cases, it is enough to impose a radiation condition in the transverse direction(s), which amounts to saying that the wave can be represented as a superposition of plane waves (propagative and evanescent) which are outgoing in the transverse direction(s). But when guided modes do exist, such a condition cannot distinguish between incoming and outgoing guided waves in the longitudinal direction. As a consequence, it is not adapted to the situation we focus on in the present paper (see section 5 for additional comments on this topic).

Various solutions have been proposed for open waveguides. In the case of a three-dimensional layered medium, Xu [21, 22] uses a decomposition of the scattered field into a finite sum of guided waves and a "free" wave and imposes separately for each of them a usual Sommerfeld radiation condition with the appropriate wavenumber. Ciraolo and Magnanini [6, 7] introduce a similar radiation condition based on the same decomposition of the scattered field using a weaker form of Sommerfeld conditions for the various components. Following a slightly different idea, Jerez-Hanckes and Nédélec [12] propose dividing the propagative medium into two regions (close to and far from the core of the guide) and imposing on each of them Sommerfeld-type conditions. It is likely that all these different conditions are equivalent, and all lead to the well-posedness of the propagation equations. However, it seems that for waveguides with a local (but not small) perturbation, the uniqueness proofs proposed in the above-mentioned papers [6, 21] are incomplete, for they do not deal with the possible evanescent component of a scattered wave.

In the present paper, we use the *modal radiation condition* introduced in [1], which amounts to saying that a scattered wave appears as a superposition of guided and radiation modes which are outgoing in the longitudinal direction. This condition is based on the generalized Fourier transform associated with the transverse part of the propagation equation, which appears as a very efficient theoretical tool for studying scattering problems in a uniform waveguide, especially as regards the proof of uniqueness. We reinforce here this assertion: the use of this transform allows us to prove the solvability of the scattering by a thick junction of uniform waveguides, which is, to the best of our knowledge, the first proof proposed in this context.

As in [1], we consider here a simple two-dimensional acoustic model. For the sake of simplicity, we assume that the problem is symmetric with respect to a longitudinal axis, so that it can be set in a half-plane $\Omega := \{(x,z) \in \mathbb{R}^2;\ z > 0\}$, where x (respectively, z) defines the longitudinal (respectively, transverse) direction. We denote by $\Gamma := \mathbb{R} \times \{0\}$ the symmetry axis. For a given frequency, wave propagation is described by a bounded positive wavenumber function $k = k(x,z)$ which is assumed to be a localized perturbation of a reference function $k_\star = k_\star(x,z)$ in the sense that

$$k(x,z) - k_\star(x,z) \text{ is compactly supported in } [-a,+a] \times [0,b]$$

for some positive numbers a and b, and where $k_\star$ is defined by

$$k_\star(x,z) := k^\pm(z) \text{ if } x \in \mathbb{R}^\pm, \quad \text{where} \quad k^\pm(z) := \begin{cases} k_0 & \text{if } 0 < z < h^\pm, \\ k_\infty & \text{if } z > h^\pm, \end{cases} \tag{1.1}$$

and both k_0 and k_∞ are positive real numbers. As shown in Figure 1.2, function $k_\star$ corresponds to an abrupt junction of two semi-infinite waveguides made with the same materials but whose cores have different heights (respectively, h^- and h^+), whereas k can represent a smooth (thick) junction of the same waveguides, or a penetrable defect in the abrupt junction. We use here the word "waveguide," which may be somewhat improper since the existence of guided waves is subject to some condition on k_0 and k_∞, namely, that $k_0^2 - k_\infty^2 > \pi^2/(2h^\pm)^2$ (see the appendix). However, this assumption is not crucial. All of the results of the paper hold for nonguiding devices.

The problem we are interested in is then defined as follows: considering a given excitation f assumed compactly supported, find the *outgoing* solution u to

$$-\Delta u - k^2\, u = f \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \Gamma. \tag{1.3}$$

We shall give a complete definition of this problem in section 2. First, we make precise in section 2.1 the meaning of the word *outgoing*: on each side of the junction, we use the above-mentioned modal radiation condition expressed by means of the corresponding generalized Fourier transform. We then introduce in section 2.2 the

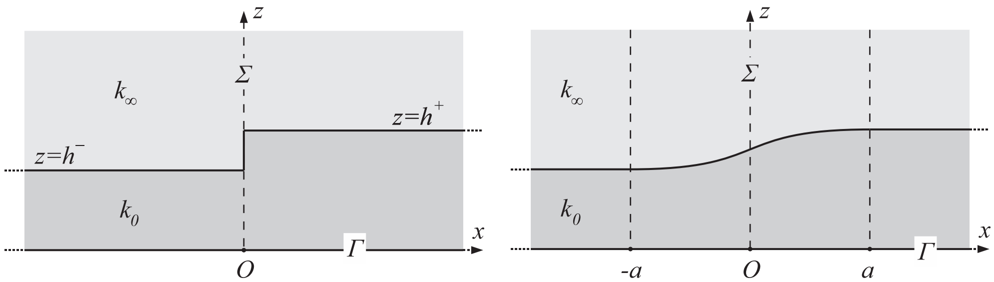

FIG. 1.2. *Our models of abrupt (left) and thick (right) junctions.*

functional space in which u will be sought. In section 2.3, we complete the definition of our scattering problem, which may also model the case of an incident wave, and state the main result of the paper (Theorem 2.3) about the well-posedness of the problem. The idea of the proof is to consider the thick junction as a perturbation of the abrupt one, which leads us to rewrite our scattering problem as a Lippmann–Schwinger equation. The fact that Fredholm's alternative applies follows from the solvability of the scattering problem for the abrupt junction, which is the object of section 3. And uniqueness is proved in section 4.

The analysis of the abrupt junction presented in section 3 combines and extends the ideas developed on one hand in [1] for a uniform waveguide, and on the other hand in [2] for an abrupt junction. The idea is to split the acoustic field into two parts. The first part represents the solution of a radiation problem for two *uncoupled* semi-infinite waveguides, whose properties, collected in section 3.1, essentially follow from [1]. The second part is a correction which takes into account the coupling between both waveguides, which leads to a *coupling equation* set on the junction line Σ (see Figure 1.2) that was partly studied in [2]. Section 3.2 completes this study.

The originality of the paper is mainly contained in section 4, which explains the proof of uniqueness for the thick junction. The general idea of the proof is similar to [1]. The first step is based on an energy argument. We show in section 4.1 that if there is no excitation, then the energy flux across any infinite transverse section situated outside the junction vanishes, which implies that in the modal decomposition of the acoustic field, the components associated with propagative modes vanish. Following the method proposed in [20] and reformulated for the generalized Fourier transform in [1], the trick then consists in using an analyticity argument to deduce that the other components of the field, associated with evanescent modes, also vanish. But the implementation of this second step is far more intricate for the junction than for a perturbed uniform waveguide because of the use of both generalized Fourier transforms associated with both semi-infinite waveguides. We need a preliminary study of the decay properties of the solution in the transverse direction (section 4.2). The analyticity is then deduced from the above-mentioned *coupling equation* (section 4.3).

We conclude the paper with some comments about our method: these include criticisms, conjectures, and possible generalizations of the method.

2. Definition of the scattering problem.

2.1. Modal radiation conditions. In many physical textbooks (see, e.g., [19]), it is generally admitted that in an open uniform waveguide, any time-harmonic wave can be represented as the sum of a finite superposition of *guided modes* and a continuous superposition of *radiation modes*, where both superpositions involve right-going and left-going modes. The radiation conditions we shall use here are based on such decompositions in both semi-infinite waveguides located on both sides of the junction (i.e., for $|x| \geq a$): the idea is simply to keep the *outgoing* components, that is, the right-going modes for $x > a$ and the left-going ones for $x < -a$. The mathematical tool that allows us to justify these decompositions is the generalized Fourier transform associated with the transverse part of the propagation equations. We summarize here the main results of this transform. More details can be found in [1, 11].

The modes are obtained by the method of separation of variables applied to the propagation equations (1.2)–(1.3) restricted to the right or left semi-infinite waveguide located outside the junction, i.e.,

$$-\Delta u - k^{\pm}(z)^2\, u = 0 \quad \text{in } \{(x,z) \in \Omega;\ \pm x > a\}, \tag{2.1}$$

$$u(x,0) = 0 \quad \text{for } \pm x > a, \tag{2.2}$$

where $k^{\pm}(z)$ is defined in (1.1). Setting $u(x,z) = \varphi(z)\exp(\gamma x)$, we are led to find $\lambda = \gamma^2 \in \mathbb{C}$ and $\varphi \neq 0$ such that

$$-\varphi'' - k^{\pm}(z)^2\, \varphi = \lambda\, \varphi \quad \text{in } \mathbb{R}^+, \tag{2.3}$$

$$\varphi(0) = 0. \tag{2.4}$$

In other words, we search for the spectral elements of the unbounded self-adjoint operator $A^{\pm}$ defined in $L^2(\mathbb{R}^+)$ by

$$A^{\pm}\varphi := -\varphi'' - k^{\pm}(z)^2\, \varphi \quad \forall \varphi \in \mathrm{D}(A^{\pm}) := H^2(\mathbb{R}^+) \cap H^1_0(\mathbb{R}^+), \tag{2.5}$$

where we use standard notation for the Sobolev spaces H^s and H^s_0. Its spectrum $\Lambda^{\pm}$ is composed of two parts: a continuous spectrum $\Lambda^{\pm}_{\mathrm{c}} = \Lambda_{\mathrm{c}} := [-k^2_{\infty}, +\infty)$ and a finite point spectrum $\Lambda^{\pm}_{\mathrm{p}}$ which is nonempty if and only if $k_0^2 - k_\infty^2 > \pi^2/(2h^{\pm})^2$ (in this case, $\Lambda^{\pm}_{\mathrm{p}} \subset (-k_0^2 + \pi^2/(2h^{\pm})^2, -k_\infty^2)$; see (A.1)). For all $\lambda \in \Lambda^{\pm}$, the solutions to (2.3)–(2.4) form a one-dimensional space spanned by some function $\Phi^{\pm}_{\lambda}(z)$. We show in the appendix the expression of a family $\{\Phi^{\pm}_{\lambda}(z);\ \lambda \in \Lambda^{\pm}\}$ which has the remarkable property that for each z, $\Phi^{\pm}_{\lambda}(z)$ extends to an entire function of $\lambda \in \mathbb{C}$. Notice that if $\lambda \in \Lambda^{\pm}_{\mathrm{p}}$, the function $\Phi^{\pm}_{\lambda}$ belongs to $\mathrm{D}(A)$: it is an eigenfunction associated with the eigenvalue λ. On the other hand, if $\lambda \in \Lambda_{\mathrm{c}}$, then $\Phi^{\pm}_{\lambda}$ no longer belongs to $L^2(\mathbb{R}^+)$: it is often called a generalized eigenfunction.

The family $\{\Phi^{\pm}_{\lambda};\ \lambda \in \Lambda^{\pm}\}$ satisfies some orthogonality and completeness properties which can be stated precisely by introducing the associated generalized Fourier transform, that is, the operator of "decomposition" on this family, given by

$$(\mathcal{F}_{\pm}\varphi)(\lambda) := \int_{\mathbb{R}^+} \varphi(z)\, \Phi^{\pm}_{\lambda}(z)\, \mathrm{d}z \quad \forall \lambda \in \Lambda^{\pm}, \tag{2.6}$$

for all $\varphi \in L^2(\mathbb{R}^+)$ with compact support. Using a density argument, $\mathcal{F}_{\pm}$ extends to a unitary transformation from $L^2(\mathbb{R}^+)$ to a spectral space of the form $L^2(\Lambda^{\pm}; \mathrm{d}\mu^{\pm})$, which denotes the space of square integrable functions on $\Lambda^{\pm}$ for the measure $\mathrm{d}\mu^{\pm} := \sum_{\lambda \in \Lambda^{\pm}_{\mathrm{p}}} \rho^{\pm}_{\lambda}\, \delta_{\lambda} + \rho^{\pm}_{\lambda}\, \mathrm{d}\lambda|_{\Lambda_{\mathrm{c}}}$, where δ_λ is the Dirac measure at $\lambda \in \Lambda^{\pm}_{\mathrm{p}}$, $\mathrm{d}\lambda|_{\Lambda_{\mathrm{c}}}$ is the Lebesgue measure restricted to Λ_{c}, and $\rho^{\pm}_{\lambda}$ is a weight function (see (A.2)). In other words, a function $\widehat{\varphi} : \Lambda^{\pm} \mapsto \mathbb{C}$ belongs to $L^2(\Lambda^{\pm}; \mathrm{d}\mu^{\pm})$ if

$$\|\widehat{\varphi}\|^2_{L^2(\Lambda^{\pm};\mathrm{d}\mu^{\pm})} := \int_{\Lambda^{\pm}} |\widehat{\varphi}(\lambda)|^2\, \mathrm{d}\mu(\lambda) = \sum_{\lambda \in \Lambda^{\pm}_{\mathrm{p}}} \rho^{\pm}_{\lambda}\, |\widehat{\varphi}(\lambda)|^2 \ + \int_{\Lambda_{\mathrm{c}}} |\widehat{\varphi}(\lambda)|^2\, \rho^{\pm}_{\lambda}\, \mathrm{d}\lambda < \infty.$$

The inverse transform $\mathcal{F}_\pm^{-1}$ appears as the operator of "recomposition" on the family $\{\Phi_\lambda^\pm;\ \lambda\in\Lambda^\pm\}$: for all $\widehat{\varphi}\in L^2(\Lambda^\pm;\mathrm{d}\mu^\pm)$,

$$\mathcal{F}_\pm^{-1}\widehat{\varphi} \;=\; \int_{\Lambda^\pm}\widehat{\varphi}(\lambda)\,\Phi_\lambda^\pm\,\mathrm{d}\mu^\pm(\lambda) = \sum_{\lambda\in\Lambda_\mathrm{p}^\pm}\rho_\lambda^\pm\,\widehat{\varphi}(\lambda)\,\Phi_\lambda^\pm \;+\; \int_{\Lambda_\mathrm{c}}\widehat{\varphi}(\lambda)\,\Phi_\lambda^\pm\,\rho_\lambda^\pm\,\mathrm{d}\lambda. \tag{2.7}$$

If $\widehat{\varphi}=\mathcal{F}_\pm\varphi$ for $\varphi\in L^2(\mathbb{R}^+)$, this formula yields the decomposition of φ on the family $\{\Phi_\lambda^\pm;\ \lambda\in\Lambda^\pm\}$ which may be seen as a generalized orthonormal basis.

An essential property of $\mathcal{F}_\pm$ is that it diagonalizes $A^\pm$ in the sense that $A^\pm\varphi=\mathcal{F}_\pm^{-1}\lambda\,\mathcal{F}_\pm\,\varphi$ for all $\varphi\in\mathrm{D}(A^\pm)$. Hence, if we apply formally $\mathcal{F}_\pm$ to (2.1)–(2.2) (which has to be justified since $u(x,\cdot)\notin\mathrm{D}(A^\pm)$ in general; see section 2.2), we obtain

$$-\frac{\partial^2}{\partial x^2}\mathcal{F}_\pm u(x,\lambda)+\lambda\,\mathcal{F}_\pm u(x,\lambda)=0\quad\text{for}\pm x>a\text{ and }\lambda\in\Lambda^\pm.$$

For all $\lambda\in\Lambda^\pm$, the solutions of this differential equation are linear combinations of exponential functions: $\mathcal{F}_\pm u(x,\lambda)=\widehat{\alpha}^\pm(\lambda)\ \mathrm{e}^{-\sqrt{\lambda}\,(|x|-a)}+\widehat{\beta}^\pm(\lambda)\ \mathrm{e}^{+\sqrt{\lambda}\,(|x|-a)}$ for $\pm x>a$. Using (2.7), we deduce that

$$u(x,z)=\int_{\Lambda^\pm}\Big\{\widehat{\alpha}^\pm(\lambda)\ \mathrm{e}^{-\sqrt{\lambda}\,(|x|-a)}\,\Phi_\lambda^\pm(z)+\widehat{\beta}^\pm(\lambda)\ \mathrm{e}^{+\sqrt{\lambda}\,(|x|-a)}\,\Phi_\lambda^\pm(z)\Big\}\,\mathrm{d}\mu^\pm(\lambda) \tag{2.8}$$

for $\pm x>a$. This formula is nothing but the above-mentioned decomposition of u into a finite superposition of *guided modes*, associated with the point spectrum $\Lambda_\mathrm{p}^\pm$, and a continuous superposition of *radiation modes*, associated with the continuous spectrum Λ_c. In order to distinguish outgoing and incoming modes, we have to make precise the definition of $\sqrt{\lambda}$. Throughout the paper, we shall use the following definition of the complex square root (where the branch cut is chosen on $\mathrm{i}\mathbb{R}^+$):

$$\sqrt{\zeta}:=|\zeta|^{1/2}\ \mathrm{e}^{\mathrm{i}(\arg\zeta)/2}\quad\text{for }\zeta\in\mathbb{C}\text{ with }-3\pi/2<\arg\zeta\le\pi/2. \tag{2.9}$$

In particular, for a negative $\lambda\in\Lambda^\pm$, we have $\sqrt{\lambda}=-\mathrm{i}|\lambda|^{1/2}$. As a consequence, if we assume a time-dependence in the form $\mathrm{e}^{-\mathrm{i}\omega t}$, we see that for $\lambda<0$, function $\mathrm{e}^{-\sqrt{\lambda}\,(|x|-a)}\,\Phi_\lambda^\pm(z)$ represents a propagative outgoing mode, and $\mathrm{e}^{+\sqrt{\lambda}\,(|x|-a)}\,\Phi_\lambda^\pm(z)$ is incoming. And for $\lambda>0$, the former is evanescent as $|x|\to\infty$, whereas the latter is exponentially increasing. This justifies the following definition of our outgoing radiation conditions, which simply consists in keeping outgoing or evanescent modes in the decomposition of u.

Definition 2.1. *We say that a solution u to* (2.1)–(2.2) *satisfies the modal radiation conditions* ($\mathcal{R}^\pm$) *if there exist two functions $\widehat{\alpha}^\pm:\Lambda^\pm\mapsto\mathbb{C}$ such that*

$$\mathcal{F}_\pm u(x,\lambda)=\widehat{\alpha}^\pm(\lambda)\ \mathrm{e}^{-\sqrt{\lambda}\,(|x|-a)}\quad\textit{for }\pm x\ge a\textit{ and }\lambda\in\Lambda^\pm.$$

2.2. Functional framework. In this paragraph, we focus on the functional space in which we shall search for the solution to (1.2)–(1.3). The first difficulty concerns Definition 2.1, which makes sense only if $\mathcal{F}_\pm$ can actually be applied to $u(x,\cdot)$ for all x such that $\pm x\ge a$. But this function does not belong to $L^2(\mathbb{R}^+)$ in general (think of the case of a homogeneous medium, i.e., $k_0=k_\infty$, for which we have $u(x,z)=O(z^{-1/2})$ for fixed x). It is shown in [11] that, as the usual Fourier transform, each generalized Fourier transform $\mathcal{F}_\pm$ extends to an isomorphism from a space of "physical" distributions to a space of "spectral" distributions, both similar to the Schwartz space $\mathcal{S}'(\mathbb{R})$. More precisely, these extensions hold if

$$k_0^2 - k_\infty^2 \neq \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(h^\pm)^2} \quad \forall n \in \mathbb{N}, \tag{2.10}$$

which rules out the *cutoff frequencies* of both waveguides. With this condition, the application of $\mathcal{F}_\pm$ to (2.1)–(2.2) (which yields (2.8)) is easily justified by assuming that $u(x,\cdot)$ is in the proper distribution space for each x.

However, for technical reasons which will appear in what follows, such an assumption is not sufficient for our purposes. Indeed, we shall assume that for each x, $u(x,\cdot)$ belongs to some space (see Definition 2.2 below) which is closely related to the notion of *energy flux.* To see this, let $\Sigma_{\pm a} := \{\pm a\} \times \mathbb{R}^+$ denote the transverse sections located at $x = \pm a$, and consider the integral $\int_{\Sigma_{\pm a}} (\partial u / \partial |x|)\, \overline{u}\, \mathrm{d}z$, whose imaginary part is known to represent the longitudinal energy flux across $\Sigma_{\pm a}$. Using formally a Parseval-like equality (recall that $\mathcal{F}_\pm$ is unitary) and the modal radiation conditions $(\mathcal{R}^\pm)$, which show that $\mathcal{F}_\pm u(\pm a, \lambda) = \widehat{\alpha}^\pm(\lambda)$ and $(\partial/\partial|x|)\mathcal{F}_\pm u(\pm a,\lambda) = -\sqrt{\lambda}\,\widehat{\alpha}^\pm(\lambda)$, we obtain

$$\int_{\Sigma_{\pm a}} \frac{\partial u}{\partial |x|}\,\overline{u}\,\mathrm{d}z = \int_{\Lambda^\pm} \frac{\partial \mathcal{F}_\pm u}{\partial |x|}(\pm a,\lambda)\,\overline{\mathcal{F}_\pm u(\pm a,\lambda)}\,\mathrm{d}\mu^\pm(\lambda) \tag{2.11}$$

$$= \int_{\Lambda^\pm} -\sqrt{\lambda}\,|\widehat{\alpha}^\pm(\lambda)|^2\,\mathrm{d}\mu^\pm(\lambda). \tag{2.12}$$

Hence, assuming that the integral of the left-hand side is bounded amounts to assuming that $\mathcal{F}_\pm u(\pm a,\cdot) = \widehat{\alpha}^\pm$ belongs to the space

$$\widehat{V}_\pm := \left\{ \widehat{\varphi} : \Lambda^\pm \mapsto \mathbb{C};\ |\lambda|^{1/4}\widehat{\varphi}(\lambda) \in L^2(\Lambda^\pm; \mathrm{d}\mu^\pm) \right\}. \tag{2.13}$$

In this case, $(\partial/\partial|x|)\mathcal{F}_\pm u(\pm a,\lambda) = -\sqrt{\lambda}\,\widehat{\alpha}^\pm(\lambda)$ belongs to

$$\widehat{V}'_\pm := \left\{ \widehat{\varphi} : \Lambda^\pm \mapsto \mathbb{C};\ |\lambda|^{-1/4}\widehat{\varphi}(\lambda) \in L^2(\Lambda^\pm; \mathrm{d}\mu^\pm) \right\}.$$

These spaces can be equipped, respectively, with the norms

$$\|\widehat{\varphi}\|_{\widehat{V}_\pm} := \||\lambda|^{1/4}\widehat{\varphi}\|_{L^2(\Lambda^\pm;\mathrm{d}\mu^\pm)} \quad \text{and} \quad \|\widehat{\varphi}\|_{\widehat{V}'_\pm} := \||\lambda|^{-1/4}\widehat{\varphi}\|_{L^2(\Lambda^\pm;\mathrm{d}\mu^\pm)}. \tag{2.14}$$

Moreover, $\widehat{V}'_\pm$ appears as the dual space of $\widehat{V}_\pm$ by considering the duality product

$$\langle \widehat{\varphi}, \widehat{\psi} \rangle_{\widehat{V}'_\pm, \widehat{V}_\pm} := (|\lambda|^{-1/4}\widehat{\varphi}\,,\, |\lambda|^{+1/4}\widehat{\psi})_{L^2(\Lambda^\pm;\mathrm{d}\mu^\pm)} \quad \forall \widehat{\varphi} \in \widehat{V}'_\pm,\ \forall \widehat{\psi} \in \widehat{V}_\pm.$$

None of these spaces are contained in $L^2(\Lambda^\pm; \mathrm{d}\mu^\pm)$, but it is shown in [1] that they are both embedded in the above-mentioned space of spectral distributions. This allows us to define

$$V_\pm := \mathcal{F}_\pm^{-1}(\widehat{V}_\pm) \quad \text{and} \quad V'_\pm := \mathcal{F}_\pm^{-1}(\widehat{V}'_\pm),$$

which can be equipped with the corresponding norms

$$\|\psi\|_{V_\pm} := \|\mathcal{F}_\pm\psi\|_{\widehat{V}_\pm} = \left\| |\lambda|^{+1/4}\mathcal{F}_\pm\psi \right\|_{L^2(\Lambda^\pm;\mathrm{d}\mu^\pm)} \quad \forall \psi \in V_\pm,$$

$$\|\varphi\|_{V'_\pm} := \|\mathcal{F}_\pm\varphi\|_{\widehat{V}'_\pm} = \left\| |\lambda|^{-1/4}\mathcal{F}_\pm\varphi \right\|_{L^2(\Lambda^\pm;\mathrm{d}\mu^\pm)} \quad \forall \varphi \in V'_\pm.$$

These spaces are clearly dual one to each other, using the duality product defined by

$$\langle \varphi, \psi \rangle_{V'_\pm, V_\pm} = \langle \mathcal{F}_\pm \varphi, \mathcal{F}_\pm \psi \rangle_{\widehat{V}'_\pm, \widehat{V}_\pm} \quad \forall \varphi \in V'_\pm,\ \forall \psi \in V_\pm,$$

which is nothing but the proper formulation of the Parseval-like equality (2.11). Let us finally mention the following continuous embeddings [17]:

$$H^{1/2}(\mathbb{R}^+) \subset V_\pm \subset H^{1/2}_{\mathrm{loc}}(\mathbb{R}^+), \tag{2.15}$$

where, for an unbounded domain $X \in \mathbb{R}^n$, we denote by $H^s_{\mathrm{loc}}(X)$ the set of functions whose restrictions to all bounded $Y \subset X$ belong to $H^s(Y)$.

We shall search for a solution to (1.2)–(1.3) which satisfies $u(x,\cdot) \in V_\pm$ for all $x \in \mathbb{R}^\pm$, which leads us to the following definition (see some comments about this functional framework in section 5).

Definition 2.2. *Let $\Omega^\pm := \mathbb{R}^\pm \times \mathbb{R}^+$ denote the domains corresponding to the right and left semiwaveguides of the abrupt junction. Define*

$$\begin{aligned} \mathcal{H}^\pm &:= \left\{ u \in H^1_{\mathrm{loc}}(\Omega^\pm);\ u(x,\cdot) \in V_\pm\ \forall x \in \mathbb{R}^\pm \right\}, \\ \mathcal{H} &:= \left\{ u \in H^1_{\mathrm{loc}}(\Omega);\ u|_{\Omega^-} \in \mathcal{H}^- \textit{ and } u|_{\Omega^+} \in \mathcal{H}^+ \right\}. \end{aligned}$$

2.3. Main results. We can now give a precise definition of our scattering problem, which may model not only the response of the junction to a localized excitation $f \in L^2(\Omega)$ but also its response to a given incident wave. For the sake of simplicity, we consider only the case of an incident wave coming from the left, for instance, an incoming guided mode $\mathrm{e}^{-\sqrt{\lambda}\,x}\,\Phi^-_\lambda(z)$ for $\lambda \in \Lambda^-_{\mathrm{p}}$, or more generally an incoming superposition of propagative guided and radiation modes, as described by (2.8), where we choose $\widehat{\alpha}^- = 0$. Such a superposition can be written equivalently as

$$u_0(x,z) = \mathcal{F}_-^{-1}\left(\mathrm{e}^{-\sqrt{\lambda}\,x}\widehat{\beta}_0\right)(z), \tag{2.16}$$

where we assume that $\widehat{\beta}_0 \in \widehat{V}_-$ has a compact support contained in $\Lambda^- \cap \mathbb{R}^-$ (without this assumption, the above expression does not necessarily make sense since $\mathrm{e}^{-\sqrt{\lambda}\,x}$ is exponentially increasing as $\lambda \to +\infty$ or $x \to -\infty$). Such a u_0 can represent a (non-Gaussian) beam coming from some oblique direction (note that the generalized Fourier transform allows us to extend the notion of a beam to stratified media). Then our scattering problem is

$$(P) \qquad \begin{cases} \text{Find } u \in \mathcal{H} \text{ such that} \\ -\Delta u - k^2\, u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma, \\ u - u_0 \text{ satisfies } (\mathcal{R}^-) \text{ and } u \text{ satisfies } (\mathcal{R}^+), \end{cases}$$

where we recall that the radiation conditions $(\mathcal{R}^\pm)$ are defined in Definition 2.1. Apparently this formulation of the scattering problem does not contain any condition on the behavior of u in the transverse direction. Actually such a condition is hidden in the definition of $\mathcal{H}$. But this condition is neither a radiation condition in the transverse direction (it does not distinguish between outgoing and incoming waves in the transverse direction) nor a decay condition (in particular, it allows for some slowly decaying oscillating behavior due to the possible singularity of $\mathcal{F}_\pm u(x,\cdot)$ at $\lambda = 0$; see (2.13)).

The aim of this paper is to prove the following result.

THEOREM 2.3. *On the assumption* (2.10)*, for all* $f \in L^2(\Omega)$ *with support contained in* $[-a,+a] \times [0,b]$*, and all incident waves* u_0 *given by* (2.16)*, where* $\widehat{\beta}_0 \in \widehat{V}_-$ *has a compact support contained in* $\Lambda^- \cap \mathbb{R}^-$*, problem* (P) *has a unique solution which depends continuously on* f *and* u_0 *in the sense that there exists* $C > 0$ *such that*

$$\|u(x,\cdot)\|_{V_\pm} \leq C \, \left(\|f\|_{L^2(\Omega)} + \|\widehat{\beta}_0\|_{\widehat{V}_-} \right) \quad \forall x \in \mathbb{R}^\pm \tag{2.17}$$

(where $\|\cdot\|_{\widehat{V}_-}$ *is defined in* (2.14)*), and for all bounded domains* $\mathcal{O} \subset \Omega$*, there exists* $C(\mathcal{O}) > 0$ *such that*

$$\|u\|_{H^1(\mathcal{O})} \leq C(\mathcal{O}) \, \left(\|f\|_{L^2(\Omega)} + \|\widehat{\beta}_0\|_{\widehat{V}_-} \right). \tag{2.18}$$

The proof is based on a perturbation approach which consists in considering our *thick* junction of waveguides as a perturbation of an *abrupt* junction of the same waveguides, just as we defined k as a perturbation of $k_\star$ (see (1.1)). We thus introduce the following scattering problem for the abrupt junction:

$$(P_\star) \qquad \begin{cases} \text{Find } u_\star \in \mathcal{H} \text{ such that} \\ -\Delta u_\star - k_\star^2 \, u_\star = f_\star \quad \text{in } \Omega, \\ u_\star = 0 \quad \text{on } \Gamma, \\ u_\star - u_0 \text{ satisfies } (\mathcal{R}^-) \text{ and } u_\star \text{ satisfies } (\mathcal{R}^+). \end{cases}$$

Section 3 is devoted to the proof of the following theorem.

THEOREM 2.4. *The statement of Theorem* 2.3 *is valid for problem* $(P_\star)$*.*

This result ensures the existence of a continuous operator T which maps the pair of data $(f_\star, \widehat{\beta}_0)$ to the unique solution $u_\star = T\,(f_\star, \widehat{\beta}_0)$ to $(P_\star)$. Going back to our initial problem (P), we can rewrite the Helmholtz equation as $-\Delta u - k_\star^2\, u = f + (k^2 - k_\star^2) u$ in Ω, which shows that $u = T\,(f + (k^2 - k_\star^2)u, \widehat{\beta}_0)$. As a consequence, if $\Omega_0 \subset \Omega$ denotes a bounded domain which contains the support of the perturbation (i.e., the support of $k^2 - k_\star^2$) and the support of f, and K is the operator defined on $L^2(\Omega_0)$ by

$$Kv := \{T\,((k^2 - k_\star^2)v, 0)\}|_{\Omega_0} \quad \forall v \in L^2(\Omega_0),$$

then the restriction of u to Ω_0 (still denoted by u for simplicity) is a solution to

$$(I - K)\, u = \{T\,(f, \widehat{\beta}_0)\}|_{\Omega_0} \quad \text{ in } L^2(\Omega_0). \tag{2.19}$$

And conversely, if u is a solution to (2.19), it is readily seen that $T\,(f + (k^2 - k_\star^2)u, \widehat{\beta}_0)$ is an extension of u to the whole domain Ω, which is a solution to (P). In other words, (P) is equivalent to the Lippmann–Schwinger equation (2.19).

Notice now that Theorem 2.4 implies that K can be considered as a continuous operator from $L^2(\Omega_0)$ to $H^1(\Omega_0)$. Therefore, as Ω_0 is bounded, the compact embedding of $H^1(\Omega_0)$ into $L^2(\Omega_0)$ shows that K is a compact operator in $L^2(\Omega_0)$. Thus (2.19) comes within Fredholm's alternative: the existence of a solution follows from its uniqueness. Thanks to the equivalence between both problems, this latter property results from the uniqueness of the solution to (P) proved in section 4. Finally, the stability properties (2.17) and (2.18) are just consequences of the same stability properties for $(P_\star)$ and the continuity of $(I - K)^{-1}$.

3. Analysis of the abrupt junction. The proof of Theorem 2.4 is based on a decomposition of the acoustic field into two parts: $u_\star = u_1 + u_2$. The first part, u_1, represents the field generated by the same source $f_\star$ in both semiwaveguides $\Omega^\pm := \mathbb{R}^\pm \times \mathbb{R}^+$, which are uncoupled by imposing a Dirichlet condition on the junction line $\Sigma := \{0\} \times \mathbb{R}^+$. The second part, u_2, is a corrective term which takes into account the coupling between both semiwaveguides. For the latter, the support of the source is reduced to Σ, which leads us to a *coupling equation* formulated on Σ. This decomposition allows us to use existing results associated with each part [1, 2, 17].

For a function v defined in Ω, we denote by $v^\pm := v|_{\Omega^\pm}$ its restrictions to $\Omega^\pm$ and by $[v]_\Sigma := v^+|_\Sigma - v^-|_\Sigma$ its jump across Σ. Problem $(P_\star)$ can be formulated equivalently by writing the equations satisfied by $u_\star^\pm$ separately, together with the transmission conditions

$$[u_\star]_\Sigma = 0 \quad \text{and} \quad \left[\frac{\partial u_\star}{\partial x}\right]_\Sigma = 0,$$

which express the coupling between both waveguides. Replacing them by a Dirichlet condition leads to two uncoupled problems $(P_1^\pm)$ defined by

$$(P_1^\pm) \quad \begin{cases} \text{Find } u_1^\pm \in \mathcal{H}^\pm \text{ such that} \\ -\Delta u_1^\pm - k^\pm(z)^2\, u_1^\pm = f_\star^\pm \quad \text{in } \Omega^\pm, \\ u_1^\pm = 0 \quad \text{on } \Gamma^\pm := \mathbb{R}^\pm \times \{0\}, \\ u_1^\pm = 0 \quad \text{on } \Sigma, \\ u_1^\pm \text{ satisfies } (\mathcal{R}^\pm), \end{cases}$$

where we have chosen to remove the incident wave. Hence, $u_\star = u_1 + u_2$ will be a solution to $(P_\star)$ if and only if u_2 is a solution to

$$(P_2) \quad \begin{cases} \text{Find } u_2 \text{ such that } u_2^\pm \in \mathcal{H}^\pm \text{ and} \\ -\Delta u_2^\pm - k^\pm(z)^2\, u_2^\pm = 0 \quad \text{in } \Omega^\pm, \\ u_2^\pm = 0 \quad \text{on } \Gamma^\pm, \\ [u_2]_\Sigma = 0 \text{ and } [\partial u_2/\partial x]_\Sigma = -\left[\partial u_1/\partial x\right]_\Sigma, \\ u_2^- - u_0 \text{ satisfies } (\mathcal{R}^-) \text{ and } u_2^+ \text{ satisfies } (\mathcal{R}^+). \end{cases}$$

In sections 3.1 and 3.2, we deal successively with $(P_1^\pm)$ and (P_2): we show that both problems are well-posed (which proves Theorem 2.4) and collect some properties of their respective solutions which will be used in section 4.

3.1. The uncoupled problems. Most results stated here are deduced from the case of an infinite uniform waveguide by a symmetry argument (with respect to Σ).

Proposition 3.1. *On the assumption* (2.10), *for all* $f_\star^\pm \in L^2(\Omega^\pm)$ *with support contained in* $\{0 \le \pm x \le a\} \times [0, b]$, *problem* $(P_1^\pm)$ *has a unique solution which depends continuously on* $f_\star^\pm$ *in the sense that there exists* $C > 0$ *such that*

$$\|u_1^\pm(x,\cdot)\|_{V_\pm} \le C\ \|f_\star^\pm\|_{L^2(\Omega^\pm)} \quad \forall x \in \mathbb{R}^\pm, \tag{3.1}$$

and for all bounded domain $\mathcal{O} \subset \Omega^\pm$, *there exists* $C(\mathcal{O}) > 0$ *such that*

$$\|u_1^\pm\|_{H^1(\mathcal{O})} \le C(\mathcal{O})\ \|f_\star^\pm\|_{L^2(\Omega^\pm)}. \tag{3.2}$$

Proof. Consider, for instance, the right-hand semi-infinite waveguide Ω^+, and suppose that instead of the Dirichlet boundary condition on Σ, this waveguide is continued on the left-hand side so that it becomes a uniform infinite waveguide. Consider then the corresponding radiation problem, i.e., the same problem as $(P_\star)$ where $k_\star$ is replaced by k^+, $f_\star$ is replaced by $f_\star^+$, and $u_0 = 0$. We know (see Propositions 3.1 and 3.3 of [1]) that its solution, denoted by u_{ref}, is unique and has the following integral representation:

$$u_{\mathrm{ref}}(x,z) = \int_{\Omega^+} G^+(x,z\,;x',z')\, f_\star^+(x',z')\,\mathrm{d}x'\,\mathrm{d}z', \tag{3.3}$$

where G^+ is the Green's function of the uniform waveguide, given by

$$G^+(x,z\,;x',z') := \int_{\Lambda^+} \frac{1}{2\sqrt{\lambda}}\,\mathrm{e}^{-\sqrt{\lambda}|x-x'|}\,\Phi_\lambda^+(z)\,\Phi_\lambda^+(z')\,\mathrm{d}\mu^+(\lambda).$$

Moreover, u_{ref} belongs to $H^1_{\mathrm{loc}}(\Omega)$, is such that $u_{\mathrm{ref}}(x,\cdot) \in V_+$ for all $x \in \mathbb{R}$, and satisfies stability estimates similar to (3.1)–(3.2). To conclude, we simply have to notice that $u_1^+(x,z) := u_{\mathrm{ref}}(x,z) - u_{\mathrm{ref}}(-x,z)$ is the only solution to (P_1^+). Of course, the same idea applies to (P_1^-). ∎

We will need in what follows some properties of the solution to $(P_1^\pm)$, which are collected in the following proposition.

PROPOSITION 3.2. *If $u_1^\pm$ is the solution to $(P_1^\pm)$, then the following hold:*

(i) *It belongs to $L^\infty(\Omega^\pm)$.*

(ii) *When $z \to +\infty$, we have*

$$u_1^\pm(x,z) = O\left(\frac{|x|}{z^{3/2}}\right),\ \frac{\partial u_1^\pm}{\partial x}(x,z) = O\left(\frac{1}{z^{3/2}}\right),\ \text{and}\ \frac{\partial u_1^\pm}{\partial z}(x,z) = O\left(\frac{1}{z^{1/2}}\right), \tag{3.4}$$

where the last two asymptotic behaviors hold uniformly with respect to x in a bounded domain.

(iii) *$\partial u_1^\pm/\partial x(x,\cdot) \in V'_\pm$ for all $x \in \mathbb{R}^\pm$, and there exists $C > 0$ such that*

$$\left\| \frac{\partial u_1^\pm}{\partial x}(x,\cdot) \right\|_{V'_\pm} \le C\, \|f_\star^\pm\|_{L^2(\Omega^\pm)} \quad \forall x \in \mathbb{R}^\pm. \tag{3.5}$$

(iv) *For all fixed $x \in \mathbb{R}^\pm$, functions $\mathcal{F}_\pm u_1^\pm(x,\lambda)$ and $(\partial/\partial x)\mathcal{F}_\pm u_1^\pm(x,\lambda)$ extend to analytic functions of $\lambda \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}^+$.*

Proof. Again, for u_1^+ (and similarly for u_1^-), these properties can be deduced from those of u_{ref} defined in (3.3). Items (i) and (ii) are stated in [1] except for the behavior of $u_1^\pm$ in (3.4): we actually have $u_{\mathrm{ref}}(x,z) = O(z^{-1/2})$. To obtain the announced result, we use the fact that $u_1^+(0,z) = 0$ and the behavior of $\partial u_1^+/\partial x$ which yield

$$\left|u_1^+(x,z)\right| = \left| \int_0^x \frac{\partial u_1^+}{\partial x}(t,z)\,\mathrm{d}x \right| \le |x| \sup_{t\in[0,x]} \left| \frac{\partial u_1^+}{\partial x}(t,z) \right| = O\left(\frac{|x|}{z^{3/2}}\right).$$

Item (iii) is proved by a straightforward adaptation of the proof of [1, Proposition 3.3]. Finally, (iv) is also proved in [1], but we recall the idea here, since this property plays a crucial role in section 4. We simply have to note that (3.3) can be written equivalently as

$$\mathcal{F}_+ u_{\mathrm{ref}}(x,\lambda) = \int_{\Omega^+} \frac{\mathrm{e}^{-\sqrt{\lambda}|x-x'|}}{2\sqrt{\lambda}}\,\Phi_\lambda^+(z')\, f_\star^+(x',z')\,\mathrm{d}x'\,\mathrm{d}z'.$$

On one hand, $\Phi_\lambda^+(z)$ extends to an entire function of $\lambda \in \mathbb{C}$ (see the appendix), and on the other hand, $\sqrt{\lambda}$ is analytic in $\mathbb{C} \setminus \mathrm{i}\mathbb{R}^+$ (by virtue of our choice (2.9) of a determination of the complex square root). The conclusion follows, since $f_\star^+$ is compactly supported. The same argument applies for $(\partial/\partial x)\mathcal{F}_\pm u_1^\pm(x,\lambda)$. ∎

3.2. The coupling problem. In order to construct a solution to (P_2), first suppose that we know $\phi := u_2|_\Sigma$ (which is well defined since $[u_2]_\Sigma = 0$). Then, following the reasoning of section 2.1, we can solve the equations satisfied by $u_2^\pm$ in both semi-infinite waveguides $\Omega^\pm$. We obtain formally

$$u_2^- - u_0 = \mathcal{F}_-^{-1}\left\{\mathrm{e}^{+\sqrt{\lambda}\,x}\left(\mathcal{F}_-\phi - \widehat{\beta}_0\right)\right\} \quad \text{and} \quad u_2^+ = \mathcal{F}_+^{-1}\left\{\mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi\right\}, \tag{3.6}$$

where we recall that u_0 is given by (2.16). In particular, the normal derivatives $\partial u_2^\pm/\partial x$ on Σ are readily derived from these expressions. Hence the transmission condition $[\partial u_2/\partial x]_\Sigma = -\left[\partial u_1/\partial x\right]_\Sigma$ will be satisfied if and only if ϕ is a solution to the following *coupling equation*:

$$\mathcal{F}_-^{-1}\sqrt{\lambda}\,\mathcal{F}_-\phi + \mathcal{F}_+^{-1}\sqrt{\lambda}\,\mathcal{F}_+\phi = \left[\frac{\partial u_1}{\partial x}\right]_\Sigma + 2\,\mathcal{F}_-^{-1}\sqrt{\lambda}\,\widehat{\beta}_0. \tag{3.7}$$

PROPOSITION 3.3. *On the assumptions of Proposition* 3.1*, for all incident waves u_0 given by* (2.16) *where $\widehat{\beta}_0 \in \widehat{V}_-$ has a compact support contained in $\Lambda^- \cap \mathbb{R}^-$, problem (P_2) has a unique solution which depends continuously on $f_\star$ and u_0 in the sense that it satisfies stability estimates similar to* (2.17)–(2.18)*. This solution is given by* (3.6)*, where $\phi \in V_- \cap V_+$ is the unique solution to the coupling equation* (3.7)*.*

Proof. Equation (3.7) is introduced and studied in [17]. We recall here briefly the proof for well-posedness which follows surprisingly from the Lax–Milgram theorem. Indeed, its variational formulation writes as follows: Find $\phi \in V_- \cap V_+$ such that

$$a^-(\phi,\psi) + a^+(\phi,\psi) = \ell(\psi) \quad \forall \psi \in V_- \cap V_+, \quad \text{where}$$

$$\begin{aligned} a^\pm(\phi,\psi) &:= \int_{\Lambda^\pm} \sqrt{\lambda}\,\mathcal{F}_\pm\phi(\lambda)\,\overline{\mathcal{F}_\pm\psi(\lambda)}\,\mathrm{d}\mu^\pm(\lambda), \\ \ell(\psi) &:= \int_\Sigma \left(\frac{\partial u_1^+}{\partial x} - \frac{\partial u_1^-}{\partial x}\right)\overline{\psi}\,\mathrm{d}z + 2\int_{\Lambda^-} \sqrt{\lambda}\,\widehat{\beta}_0(\lambda)\,\overline{\mathcal{F}_-\psi(\lambda)}\,\mathrm{d}\mu^-(\lambda). \end{aligned}$$

Strictly speaking, the integrals on $\Lambda^\pm$ (respectively, on Σ) should be written as duality products between $\widehat{V}'_\pm$ and $\widehat{V}_\pm$ (respectively, $V'_\pm$ and $V_\pm$). The sesquilinear form $a^- + a^+$ is clearly continuous and coercive in $V_- \cap V_+$. Moreover, by virtue of (3.5), we have

$$|\ell(\psi)| \leq C\ \left\{\|f_\star^+\|_{L^2(\Omega^+)}\,\|\psi\|_{V_+} + \left(\|f_\star^-\|_{L^2(\Omega^-)} + \|\widehat{\beta}_0\|_{\widehat{V}_-}\right)\,\|\psi\|_{V_-}\right\}$$

for all $\psi \in V_- \cap V_+$, which shows that ℓ is continuous. Hence the solution ϕ to (3.7) is uniquely defined, and there exists $C' > 0$ such that

$$\|\phi\|_{V_\pm} \leq C'\ \left(\|f_\star\|_{L^2(\Omega)} + \|\widehat{\beta}_0\|_{\widehat{V}_-}\right). \tag{3.8}$$

Let us now verify that the functions $u_2^\pm$ defined in (3.6) actually provide a solution to (P_2). We deal here with u_2^+, but the same method applies for u_2^- with minor changes. First notice that for all $x \in \mathbb{R}^+$,

$$\|u_2^+(x,\cdot)\|_{V_+} = \|\mathcal{F}_+ u_2^+(x,\cdot)\|_{\widehat{V}_+} = \|\mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi\|_{\widehat{V}_+} \leq \|\mathcal{F}_+\phi\|_{\widehat{V}_+} = \|\phi\|_{V_+},$$

since $\mathrm{e}^{-\sqrt{\lambda}x} \leq 1$. Combined with (3.8), this yields the stability estimate of type (2.17). Moreover, from (2.15), this shows that $u_2^+ \in L^2_{\mathrm{loc}}(\Omega^+)$. In order to see that u_2^+ satisfies the equations of (P_2) in Ω^+, rewrite (3.6) more explicitly as

$$u_2^+(x,z) = \int_{\Lambda^+} \mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi(\lambda)\,\Phi_\lambda^+(z)\,\mathrm{d}\mu^+(\lambda) \quad \forall x > 0. \tag{3.9}$$

Thanks to the exponential decay of $\exp(-\sqrt{\lambda}\,x)$ as $\lambda \to +\infty$, this expression can be derived with respect to x and z by permuting the derivative and the integral sign (by virtue of Lebesgue's dominated convergence theorem). This shows that u_2^+ is $\mathcal{C}^1$ in $(0,+\infty)\times[0,+\infty)$, and $\mathcal{C}^\infty$ in both $(0,+\infty)\times[0,h]$ and $(0,+\infty)\times[h,+\infty)$, and satisfies

$$\begin{cases} -\Delta u_2^+ - k^+(z)^2\, u_2^+ = 0 \quad \text{in } \Omega^+, \\ u_2^+ = 0 \quad \text{on } \Gamma^+. \end{cases} \tag{3.10}$$

The problem is to understand in what sense $u_2^+ = \phi$ on Σ since we do not yet know that $u_2^+ \in H^1_{\mathrm{loc}}(\Omega^+)$. For a given $x > 0$, define $\Omega_x := (x,+\infty)\times\mathbb{R}^+$ and $\Sigma_x := \{x\}\times\mathbb{R}^+$. Using (3.10) and Green's formula, we see that for $v \in \mathcal{C}^\infty(\overline{\Omega^+})$ with compact support and such that $v|_{\Sigma\cup\Gamma} = 0$, we have

$$\int_{\Omega_x} u_2^+\,\overline{(-\Delta v - k^+(z)^2\, v)}\,\mathrm{d}x\,\mathrm{d}z = \int_{\Sigma_x} u_2^+\,\overline{\frac{\partial v}{\partial x}}\,\mathrm{d}z - \int_{\Sigma_x} \frac{\partial u_2^+}{\partial x}\,\overline{v}\,\mathrm{d}z.$$

What is the limit of this expression as $x \to 0^+$? First, the integral on Ω_x simply converges to the same integral on Ω^+ since $u_2^+ \in L^2_{\mathrm{loc}}(\Omega^+)$. Then we can interpret the integrals on Σ_x as duality products between V_+ and V_+'. Notice, on one hand, that

$$\lim_{x\to 0^+} \|u_2^+(x,\cdot) - \phi\|_{V_+} = \lim_{x\to 0^+} \|(\mathrm{e}^{-\sqrt{\lambda}\,x} - 1)\,\mathcal{F}_+\phi\|_{\widehat{V}_+} = 0$$

(by Lebesgue's dominated convergence theorem) and, on the other hand, that

$$\left\|\frac{\partial u_2^+}{\partial x}(x,\cdot)\right\|_{V_+'} = \left\|-\sqrt{\lambda}\,\mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi\right\|_{\widehat{V}_+'} \le \left\||\lambda|^{1/2}\,\mathcal{F}_+\phi\right\|_{\widehat{V}_+'} = \|\mathcal{F}_+\phi\|_{\widehat{V}_+} = \|\phi\|_{V_+}.$$

Moreover, from (2.15), we know that $\|v(x,\cdot)\|_{V_+} \le C\,\|v(x,\cdot)\|_{H^{1/2}(\mathbb{R}^+)} \to 0$ as $x \to 0^+$. This shows finally that for $v \in \mathcal{C}^\infty(\overline{\Omega^+})$ with compact support and such that $v|_{\Sigma\cup\Gamma} = 0$,

$$\int_{\Omega^+} u_2^+\,\overline{(-\Delta v - k^+(z)^2\, v)}\,\mathrm{d}x\,\mathrm{d}z = \left\langle \phi\,,\, \frac{\partial v}{\partial x}\bigg|_{\Sigma} \right\rangle_{V_+,V_+'},$$

which means that u_2^+ is a very weak solution to (3.10) together with the boundary condition $u_2^+ = \phi$ on Σ. As $\phi \in H^{1/2}_{\mathrm{loc}}(\Sigma)$ (see (2.15)), standard arguments of regularity for elliptic equations (see, e.g., [16]) show that u_2^+ belongs to $H^1_{\mathrm{loc}}(\Omega^+)$ and satisfies a stability estimate of type (2.18).

And this is the only solution to (3.10) which satisfies $u_2^+ = \phi$ on Σ: using a symmetry argument, this is readily deduced from the uniqueness property in a uniform waveguide [1, Proposition 3.1]. Consequently, as the coupling equation (3.7) is well-posed, the uniqueness of the solution to (P_2) follows. ◻

PROPOSITION 3.4. *If u_2 is a solution to (P_2), then*
(i) *it belongs to $L^\infty(\Omega) + L^2(\Omega)$; and*
(ii) *there exists $C > 0$ such that for all $x \ne 0$ and $z \in \mathbb{R}^+$,*

$$|u_2(x,z)| \le C + \frac{C}{|x|^{1/2}}, \quad \left|\frac{\partial u_2}{\partial x}(x,z)\right| \le C + \frac{C}{|x|^{3/2}}, \quad \textit{and} \quad \left|\frac{\partial u_2}{\partial z}(x,z)\right| \le C + \frac{C}{|x|^{3/2}}.$$

Proof. As above, we consider only the case $x > 0$ (similar arguments can be used for $x < 0$). For (ii), we use the Cauchy–Schwarz inequality in (3.9), which gives

$$|u_2^+(x,z)|^2 \le J(x,z)\,\|\phi\|_{V_+}^2, \quad \text{where} \quad J(x,z) := \int_{\Lambda^+} |\lambda|^{-1/2} \left| \mathrm{e}^{-\sqrt{\lambda}\,x}\,\Phi_\lambda^+(z)\right|^2 \mathrm{d}\mu^+(\lambda).$$

We decompose the latter integral into three parts: these are a discrete part on Λ_{p}^+, which is clearly bounded uniformly with respect to x and z, and two continuous contributions, respectively, on $(-k_\infty^2, 0)$ and $(0, +\infty)$. Then, using Lemma A.1, we have

$$J(x,z) \le C \left(1 + \int_{-k_\infty^2}^0 \frac{1}{\sqrt{|\lambda|\,(\lambda + k_\infty^2)}}\,\mathrm{d}\lambda + \int_0^{+\infty} \frac{\mathrm{e}^{-2\sqrt{\lambda}\,x}}{\sqrt{\lambda}}\,\mathrm{d}\lambda\right).$$

The first integral is bounded, and the second one is equal to x^{-1}, which yields the announced estimate for $|u_2(x,z)|$. The same idea applies for $\partial u_2^+/\partial x$ and $\partial u_2^+/\partial z$ (recall that we can permute a derivative and the integral sign if $x \ne 0$). In the expression of J, the quantity $|\lambda|^{-1/2}$ has to be replaced by $|\lambda|^{1/2}$ for the former, and Φ_λ^+ by $\partial\Phi_\lambda^+/\partial z$ for the latter. The conclusion again follows from Lemma A.1.

To prove (i), we choose a smooth cutoff function $\widehat{\chi} : \Lambda^+ \mapsto [0,1]$ equal to 1 if $\lambda < \lambda_0$ and to 0 for $\lambda > \lambda_1$, where $0 < \lambda_0 < \lambda_1$. Hence we have $u_2^+ = u_2^{\mathrm{a}} + u_2^{\mathrm{b}}$, where

$$u_2^{\mathrm{a}}(x,\cdot) := \mathcal{F}_+^{-1}\left(\widehat{\chi}\,\mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi\right) \quad \text{and} \quad u_2^{\mathrm{b}}(x,\cdot) := \mathcal{F}_+^{-1}\left((1-\widehat{\chi})\,\mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi\right).$$

We can proceed as above for u_2^{a}: the new expression of J is given by an integral on a bounded part of Λ^+, which shows that $u_2^{\mathrm{a}} \in L^\infty(\Omega^+)$. On the other hand,

$$\begin{aligned}
\int_{\Omega^+} |u_2^{\mathrm{b}}(x,z)|^2\,\mathrm{d}x\,\mathrm{d}z &= \int_{\Lambda^+}\int_{\mathbb{R}^+} \left|(1-\widehat{\chi}(\lambda))\,\mathrm{e}^{-\sqrt{\lambda}\,x}\,\mathcal{F}_+\phi(\lambda)\right|^2 \mathrm{d}x\,\mathrm{d}\mu^+(\lambda)\\
&\le \int_{\lambda_0}^{+\infty} |\mathcal{F}_+\phi(\lambda)|^2 \int_{\mathbb{R}^+} \mathrm{e}^{-2\sqrt{\lambda}\,x}\,\mathrm{d}x\,\rho_\lambda^+\,\mathrm{d}\lambda\\
&\le C\int_{\lambda_0}^{+\infty} \frac{1}{\sqrt{\lambda}}\,|\mathcal{F}_+\phi(\lambda)|^2\,\rho_\lambda^+\,\mathrm{d}\lambda \le C\|\phi\|_{V_+}^2,
\end{aligned}$$

which shows that $u_2^{\mathrm{b}} \in L^2(\Omega^+)$. ◻

4. Uniqueness. As mentioned in the introduction, the originality of the paper consists in the proof of the following theorem, which is decomposed in three stages detailed in the three following subsections.

THEOREM 4.1. *Problem* (P) *has at most one solution.*

From now on, u denotes a solution to (P) with homogeneous data, that is, with $f = 0$ and $u_0 = 0$. As in section 2.3, it can be interpreted as a solution to $(P_\star)$ with a localized excitation $f_\star = (k^2 - k_\star^2)u$ (and no incident wave), which yields the decomposition $u = u_1 + u_2$ introduced in section 3, which will be used hereafter.

4.1. Longitudinal energy fluxes. In order to explain the idea of this first stage, let us assume for one instant that $u \in H^1(S)$, where S denotes the strip $(-a, +a) \times \mathbb{R}^+$. By multiplying the Helmholtz equation by $\overline{u}$, integrating in S, and using Green's formula (which is allowed thanks to our simplifying assumption), we obtain

$$\int_S \left(|\nabla u|^2 - k^2\,|u|^2\right)\,\mathrm{d}x\,\mathrm{d}z = \int_{\Sigma_{-a}\cup\Sigma_{+a}} \frac{\partial u}{\partial |x|}\,\overline{u}\,\mathrm{d}z,$$

where we recall that $\Sigma_{\pm a} := \{\pm a\} \times \mathbb{R}^+$. Taking the imaginary part then yields

$$(4.1) \qquad E^+(u) + E^-(u) = 0, \quad \text{where} \quad E^\pm(u) := \operatorname{Im} \int_{\Sigma_{\pm a}} \frac{\partial u}{\partial |x|}\, \overline{u}\, \mathrm{d}z,$$

which means that the sum of the *longitudinal energy fluxes* across $\Sigma_{\pm a}$ vanishes. These quantities can be expressed by means of the modal components $\widehat{\alpha}^\pm$ involved in the modal radiation condition $(\mathcal{R}^\pm)$. Indeed, from (2.11)–(2.12), we have

$$E^\pm(u) = \operatorname{Im} \int_{\Lambda^\pm} \frac{\partial \mathcal{F}_\pm u}{\partial |x|}(\pm a, \lambda)\, \overline{\mathcal{F}_\pm u(\pm a, \lambda)}\, \mathrm{d}\mu^\pm(\lambda) = \int_{\Lambda^\pm \cap \mathbb{R}^-} \sqrt{|\lambda|}\, |\widehat{\alpha}^\pm(\lambda)|^2\, \mathrm{d}\mu^\pm(\lambda),$$

since $\sqrt{\lambda} = -\mathrm{i}\sqrt{|\lambda|}$ if $\lambda < 0$. Hence, as both quantities $E^\pm(u)$ are nonnegative, we deduce that $\widehat{\alpha}^\pm(\lambda) = 0$ for all $\lambda \in \Lambda^\pm \cap \mathbb{R}^-$. To sum up, if u belonged to $H^1(S)$, we would have proved the following result.

PROPOSITION 4.2. *If u is a solution to (P) with $f = 0$ and $u_0 = 0$, then the modal components of u associated with propagative modes vanish, that is, $\mathcal{F}_\pm u(x, \lambda) = 0$ for $\pm x \geq a$ and $\lambda \in \Lambda^\pm \cap \mathbb{R}^-$.*

Proof. In the above lines, our simplifying assumption ($u \in H^1(S)$) was used only to obtain (4.1): the remainder of the proof still applies without this assumption. Thus we have to prove (4.1) without assuming that $u \in H^1(S)$. The idea is to use Green's formula in the rectangle $(-a, +a) \times (0, R)$ instead of the strip S, which yields

$$(4.2) \quad \operatorname{Im} \int_{\Sigma^R_{-a} \cup \Sigma^R_{+a}} \frac{\partial u}{\partial |x|}\, \overline{u}\, \mathrm{d}z + I^R = 0, \quad \text{where} \quad I^R := \operatorname{Im} \int_{-a}^{+a} \frac{\partial u}{\partial z}(x, R)\, \overline{u(x, R)}\, \mathrm{d}x$$

and $\Sigma^R_{\pm a} := \{\pm a\} \times (0, R)$. Noticing that $u(\pm a, \cdot) \in V_\pm$ and $(\partial u/\partial x)(\pm a, \cdot) \in V'_\pm$ (because of the radiation conditions $(\mathcal{R}^\pm)$), we have

$$E^\pm(u) = \lim_{R \to +\infty} \operatorname{Im} \int_{\Sigma^R_{\pm a}} \frac{\partial u}{\partial |x|}\, \overline{u}\, \mathrm{d}z < +\infty,$$

where $E_\pm(u)$ is given in (4.1). It remains to verify that $\lim_{R \to +\infty} I^R = 0$. In order to prove this, we use the decomposition $u = u_1 + u_2$, which gives

$$I^R = \sum_{i,j=1,2} I^R_{i,j}, \quad \text{where} \quad I^R_{i,j} := \operatorname{Im} \int_{-a}^{+a} \frac{\partial u_i}{\partial z}(x, R)\, \overline{u_j(x, R)}\, \mathrm{d}x.$$

Each of these four integrals vanish as $R \to +\infty$. For $I^R_{1,1}$, it is a straightforward consequence of (3.4). For $I^R_{1,2}$ and $I^R_{2,1}$, we use again (3.4) as well as Proposition 3.4(ii) which show that for large enough R, the integrands $(\partial u_i/\partial z)(x, R)\, \overline{u_j(x, R)}$ are bounded, respectively, by $C\,(1 + |x|^{-1/2})\, R^{-1/2}$ and $C\,(|x| + |x|^{-1/2})\, R^{-3/2}$. Finally, for $I^R_{2,2}$, Proposition 3.4 is not sufficient to obtain the expected result. Instead we reverse the argument we have used for u. Indeed, we know from (3.6) that for $x \neq 0$,

$$\operatorname{Im} \int_{\mathbb{R}^+} \frac{\partial u_2^\pm}{\partial |x|}(x, z)\, \overline{u_2^\pm(x, z)}\, \mathrm{d}z = \int_{\Lambda^\pm \cap \mathbb{R}^-} \sqrt{|\lambda|}\, |\mathcal{F}_\pm \phi(\lambda)|^2\, \mathrm{d}\mu^\pm(\lambda),$$

which depends only on the sign of x. Moreover, using the equations satisfied by u_2^+ in the rectangle $(0, a) \times (0, R)$, we obtain an equation similar to (4.2) for u_2^+. Hence when R tends to $+\infty$, the energy flux $\operatorname{Im} \int_0^a (\partial u_2^+/\partial z)(x, R)\, \overline{u_2^+(x, R)}\, \mathrm{d}x$ must tend to 0. And the same result for u_2^- finally shows that $I^R_{2,2} \to 0$. $\square$

4.2. Transverse behavior of the solution. In section 4.3, we will prove that the other modal components of u, that is, those associated with evanescent modes, also vanish, using an argument of analyticity. But we need a preliminary result summarized in the following proposition.

PROPOSITION 4.3. *If u is a solution to (P) with $f = 0$ and $u_0 = 0$, then both $u|_\Sigma$ and $(\partial u/\partial x)|_\Sigma$ belong to $L^1(\mathbb{R}^+)$.*

The proof will be decomposed as a sequence of four lemmas. The first lemma concerns the decay properties of u in the longitudinal direction, which follow from Proposition 4.2 using the generalized Fourier transform in the transverse direction. In the next three lemmas, we reverse this point of view, using the Fourier transform in the longitudinal direction in order to get some information about the behavior of u in the transverse direction.

LEMMA 4.4. *For all $n \in \mathbb{N}$, there exists a constant $C_n > 0$ such that*

$$\left|\frac{\partial^n u}{\partial x^n}(x,z)\right| \leq \frac{C_n}{(|x|-a)^{n+1/2}} \quad \forall |x| \geq a,\ \forall z \in \mathbb{R}^+.$$

Proof. The modal radiation conditions $(\mathcal{R}^\pm)$ and Proposition 4.2 yield

$$\left|\frac{\partial^n u}{\partial x^n}(x,z)\right| = \left|\int_{\mathbb{R}^+} \widehat{\alpha}^\pm(\lambda)\,\lambda^{n/2}\,\mathrm{e}^{-\sqrt{\lambda}(|x|-a)}\,\Phi^\pm_\lambda(z)\,\rho^\pm_\lambda\,\mathrm{d}\lambda\right| \quad \text{for } \pm x \geq a.$$

The lemma follows from the same arguments as in the proof of Proposition 3.4(ii) and the fact that

$$\int_0^{+\infty} \lambda^{n-1/2}\,\mathrm{e}^{-2\sqrt{\lambda}\,(|x|-a)}\,\mathrm{d}\lambda = \frac{2\,(2n)!}{\{2(|x|-a)\}^{2n+1}}.$$

Note that no constant term occurs here since $\widehat{\alpha}^\pm(\lambda) = 0$ for negative λ. ∎

Define now

$$\widetilde{u}(\xi,z) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x,z)\,\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x \quad \forall \xi \in \mathbb{R}.$$

Note that from Propositions 3.2(i) and 3.4(i), we know that $u \in L^2(\Omega) + L^\infty(\Omega)$; hence u is a tempered distribution. As a consequence, this definition of the Fourier transform of u must be interpreted in the sense of distributions.

LEMMA 4.5. *For $z \geq b$, we have*

$$\xi\,\widetilde{u}(\xi,z) = \begin{cases} A(\xi)\,\mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}\,(z-b)} & \text{for } |\xi| > k_\infty, \\ B(\xi)\,\mathrm{e}^{+\mathrm{i}\sqrt{k_\infty^2-\xi^2}\,(z-b)} + D(\xi)\,\mathrm{e}^{-\mathrm{i}\sqrt{k_\infty^2-\xi^2}\,(z-b)} & \text{for } |\xi| < k_\infty, \end{cases} \tag{4.3}$$

where A, B, and D are $\mathcal{C}^\infty$ functions, except possibly at $\xi = 0$ (B and D are continuous at $\xi = 0$). These functions vanish as well as all their derivatives at $\pm k_\infty$. Moreover, as $|\xi| \to \infty$, we have $(\partial^m/\partial\xi^m)A(\xi) = O(|\xi|^{-n})$ for all integers n and m.

Proof. By Lemma 4.4 and the fact that u is $\mathcal{C}^\infty$ above the inhomogeneities, we have $x^m(\partial^n u/\partial x^n)(\cdot,z) \in L^1(\mathbb{R})$ for all integers $m < n$ and all $z \geq b$, so $(\partial^m/\partial\xi^m)(\xi^n\,\widetilde{u}(\xi,z))$ is a continuous function of ξ which tends to 0 as $|\xi| \to \infty$ (by the Riemann–Lebesgue theorem). Thus $\xi\,\widetilde{u}(\xi,z)$ is $\mathcal{C}^\infty$ except possibly at $\xi = 0$. Formula (4.3) is readily obtained by noticing that $\partial u/\partial x$ satisfies the Helmholtz equation with wavenumber k_∞ for $z \geq b$. Note that no exponentially increasing component (as $|\xi| \to \infty$) appears since $\widetilde{u}$ is a tempered distribution.

The properties of A, B, and D follow from those of $\xi\,\widetilde{u}$: these are $\mathcal{C}^\infty$ functions except possibly at $\xi = 0$. In particular, for $|\xi| > k_\infty$,

$$\frac{\partial}{\partial \xi}(\xi\,\widetilde{u}(\xi,z)) = A'(\xi)\,\mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}\,(z-b)} - A(\xi)\,\frac{\xi\,(z-b)}{\sqrt{\xi^2-k_\infty^2}}\,\mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}\,(z-b)}$$

must be bounded as $\xi \to \pm k_\infty$; thus $A(\pm k_\infty) = 0$. Looking at higher order derivatives yields $A^{(n)}(\pm k_\infty) = 0$ for all $n \in \mathbb{N}$. Therefore, $(\partial^n/\partial\xi^n)(\xi\,\widetilde{u})(\pm k_\infty, z) = 0$, and the same argument for $|\xi| < k_\infty$ then shows that $B^{(n)}(\pm k_\infty) = D^{(n)}(\pm k_\infty) = 0$ for all $n \in \mathbb{N}$. Finally, note that for all $n \in \mathbb{N}$, we have $\xi^n\,A(\xi) = \xi^{n+1}\,\widetilde{u}(\xi,b) \to 0$ as $|\xi| \to +\infty$; thus $A(\xi) = O(|\xi|^{-n})$. Similarly, noticing that $(\partial/\partial\xi)(\xi^{n+1}\,\widetilde{u})(\xi,b) \to 0$ as $|\xi| \to +\infty$ gives the same property for $A'(\xi)$, and so on, for higher derivatives. ∎

Using the inverse Fourier transform, the above lemma leads us to the following decomposition for all $n \geq 1$:

$$\frac{\partial^n u}{\partial x^n} = u_{\mathrm{e}}^{(n)} + u_{\mathrm{p}}^{(n)}, \quad \text{where}$$

$$u_{\mathrm{e}}^{(n)}(x,z) := \frac{1}{\sqrt{2\pi}}\int_{|\xi|>k_\infty} \xi^{n-1}\,A(\xi)\,\mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}(z-b)}\,\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi \quad \text{and}$$

$$u_{\mathrm{p}}^{(n)}(x,z) := \frac{1}{\sqrt{2\pi}}\int_{|\xi|<k_\infty} \xi^{n-1}\left(B(\xi)\mathrm{e}^{\mathrm{i}\sqrt{k_\infty^2-\xi^2}\,(z-b)} + D(\xi)\mathrm{e}^{-\mathrm{i}\sqrt{k_\infty^2-\xi^2}\,(z-b)}\right)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi,$$

which represent two continuous superpositions of modes that are, respectively, evanescent and propagative in the transverse direction. Their respective behaviors at large distance are given in the following two lemmas.

LEMMA 4.6. *For all integers n, p, and q, there exists $C > 0$ such that*

$$\left|u_{\mathrm{e}}^{(n)}(x,z)\right| \leq C\ |x|^{-p}\ (z-b)^{-q} \quad \forall x \neq 0,\ \forall z > b.$$

Proof. By Lemma 4.5, for all $n \in \mathbb{N}$, the function f_n defined by

$$f_n(\xi,z) := \xi^{n-1}\,A(\xi)\,\mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}\,(z-b)} \quad \text{for } |\xi| > k_\infty$$

can be continued by 0 for $|\xi| < k_\infty$ so that it becomes $\mathcal{C}^\infty$ on $\mathbb{R}$. It decays faster than any power of $1/|\xi|$ as $|\xi| \to +\infty$, as well as all of its ξ-derivatives. Hence, by successive integrations by parts, we obtain

$$u_{\mathrm{e}}^{(n)}(x,z) = \frac{1}{\sqrt{2\pi}}\left(\frac{\mathrm{i}}{x}\right)^p \int_{|\xi|>k_\infty} \frac{\partial^p f_n}{\partial \xi^p}(\xi,z)\,\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi \quad \forall p \in \mathbb{N}.$$

Noticing that for all $q \in \mathbb{N}$, one can find $C > 0$ such that $|(\partial^p f_n/\partial\xi^p)(\xi,z)| \leq C\ (\xi^2-k_\infty^2)^{q/2}\ \exp(-\sqrt{\xi^2-k_\infty^2}\,(z-b))$, we deduce that

$$\left|u_{\mathrm{e}}^{(n)}(x,z)\right| \leq \frac{C}{|x|^p}\ \int_{|\xi|>k_\infty} (\xi^2-k_\infty^2)^{q/2}\ \mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}\,(z-b)}\,\mathrm{d}\xi.$$

Using the change of variable $t = \sqrt{\xi^2-k_\infty^2}$ and successive integrations by parts, the conclusion follows. ∎

We deal now with the behavior of $u_{\mathrm{p}}^{(n)}(x,z)$ in an oblique direction $z = \tau\,x$.

LEMMA 4.7. *For $\tau \in (0,+\infty)$ and $n \geq 2$, we have the following asymptotic behaviors as $x \to +\infty$:*

$$u_{\mathrm{p}}^{(n)}(x,\tau x) = \frac{P_\tau\,\xi_\tau^{n-1} B(\xi_\tau)\,\mathrm{e}^{\mathrm{i}(1+\tau^2)\xi_\tau x} + \overline{P_\tau}\,(-\xi_\tau)^{n-1} D(-\xi_\tau)\,\mathrm{e}^{-\mathrm{i}(1+\tau^2)\xi_\tau x}}{\sqrt{x}} + O\left(\frac{1}{x}\right),$$

where

$$\xi_\tau := \frac{k_\infty}{\sqrt{1+\tau^2}} \quad \textit{and} \quad P_\tau := \sqrt{2\pi\,\xi_\tau^3}\,\tau\,k_\infty^{-1}\,\mathrm{e}^{-\mathrm{i}(\tau\xi_\tau b+\pi/4)}.$$

Proof. Using the stationary phase theorem (see, e.g., [18]), let us prove that

$$\int_{|\xi|<k_\infty} \xi^{n-1}\,B(\xi)\,\mathrm{e}^{\mathrm{i}(\sqrt{k_\infty^2-\xi^2}(\tau x-b)+\xi x)}\,\mathrm{d}\xi = P_\tau\,\xi_\tau^{n-1}B(\xi_\tau)\,\frac{\mathrm{e}^{\mathrm{i}(1+\tau^2)\xi_\tau x}}{\sqrt{x}} + O\left(\frac{1}{x}\right).$$

Consider the phase function $\theta(\xi) := \xi + \tau\sqrt{k_\infty^2-\xi^2}$ which has only one (nondegenerate) stationary point at $\xi_\tau = k_\infty/\sqrt{1+\tau^2}$. Because of the possible singularity of the amplitude $\mathcal{B}_n(\xi) := \xi^{n-1}\,B(\xi)\,\exp(-\mathrm{i}\sqrt{k_\infty^2-\xi^2 b})$ at $\xi=0$, and because of the behavior of $\theta(\xi)$ at $\xi=\pm k_\infty$, we introduce a $\mathcal{C}^\infty$ function $\chi(\xi)$ equal to 1 near ξ_τ and to 0 near $\pm k_\infty$ and 0. The stationary phase theorem applied to $\int \chi(\xi)\,\mathcal{B}_n(\xi)\,\exp(\mathrm{i}\theta(\xi)x)\,\mathrm{d}\xi$ then yields the dominant term in the above formula. The remaining contribution is $O(x^{-1})$. Indeed, we can use a nonstationary phase argument,

$$\int_{|\xi|<k_\infty}(1-\chi(\xi))\,\mathcal{B}_n(\xi)\,\mathrm{e}^{\mathrm{i}\theta(\xi)x}\,\mathrm{d}\xi = \frac{-1}{x}\int_{|\xi|<k_\infty}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{(1-\chi(\xi))\,\mathcal{B}_n(\xi)}{\mathrm{i}\,\theta'(\xi)}\right)\mathrm{e}^{\mathrm{i}\theta(\xi)x}\,\mathrm{d}\xi,$$

where $(1-\chi)\,\mathcal{B}_n/(\mathrm{i}\,\theta') \in \mathcal{C}^1([-k_\infty,k_\infty])$ thanks to the assumption $n\geq 2$. The same procedure applies for the integral which involves $D(\xi)$. ∎

We can now conclude the proof of Proposition 4.3. Choose $n\geq 2$. Lemma 4.4 tells us that $(\partial^n u/\partial x^n)(x,\tau x) = O(x^{-n-1/2})$ as $x\to+\infty$. By Lemmas 4.6 and 4.7, we infer that the dominant contribution of $u_{\mathrm{p}}^{(n)}$ (which is $O(x^{-1/2})$) must vanish. So, for large enough x, we have

$$P_\tau\,\xi_\tau^{n-1}B(\xi_\tau)\,\mathrm{e}^{\mathrm{i}(1+\tau^2)\xi_\tau x} + \overline{P_\tau}\,(-\xi_\tau)^{n-1}D(-\xi_\tau)\,\mathrm{e}^{-\mathrm{i}(1+\tau^2)\xi_\tau x} = 0.$$

As this holds for all $\tau>0$, we deduce that $B(\xi)=D(-\xi)=0$ for all $\xi\in(0,k_\infty)$. Now, by choosing $\tau\in(-\infty,0)$ and $x\to-\infty$, we would obtain similarly $B(\xi)=D(-\xi)=0$ for all $\xi\in(-k_\infty,0)$. Hence $B=D=0$, which means that the components of $\partial^n u/\partial x^n$ associated with propagative transverse modes vanish. Going back to Lemma 4.5, we deduce that for $z\geq b$, $\widetilde{u}(\xi,z) = \kappa(z)\,\delta_0 + \xi^{-1}\,A(\xi)\,\mathrm{e}^{-\sqrt{\xi^2-k_\infty^2}\,(z-b)}$, where δ_0 is a Dirac measure at $\xi=0$, $\kappa(z)$ is a function which depends only on z, and $A(\xi)$ is understood as a $\mathcal{C}^\infty$ function which vanishes in $[-k_\infty,k_\infty]$. Therefore, $u(x,z) = \kappa(z)/\sqrt{2\pi} + u_{\mathrm{e}}^{(0)}(x,z)$, where Lemma 4.4 imposes that $\kappa(z)=0$. Proposition 4.3 finally follows from Lemma 4.6.

4.3. Spectral analyticity and proof of uniqueness. The third and last stage for the proof of Theorem 4.1 consists in the following proposition.

PROPOSITION 4.8. *If u is a solution to (P) with $f=0$ and $u_0=0$, then for all x such that $\pm x\geq a$, the function $\Lambda_{\mathrm{c}}\ni\lambda\to\mathcal{F}_\pm u(x,\lambda)$ extends continuously to an analytic function in the quadrant $\mathcal{Q} := \{\lambda\in\mathbb{C};\ \mathrm{Re}(\lambda) > -k_\infty^2$ and $\mathrm{Im}(\lambda)<0\}$.*

Proof. We use again the decomposition $u=u_1+u_2$. On one hand, Proposition 3.2 tells us that for all $x\in\mathbb{R}^\pm$, functions $\mathcal{F}_\pm u_1^\pm(x,\lambda)$ and $(\partial/\partial x)\mathcal{F}_\pm u_1^\pm(x,\lambda)$ extend to analytic functions of λ in $\mathbb{C}\setminus\mathrm{i}\mathbb{R}^+$, which clearly contains $\mathcal{Q}$. On the other hand, by (3.6), we have $\mathcal{F}_\pm u_2^\pm = \mathrm{e}^{-\sqrt{\lambda}\,|x|}\,\mathcal{F}_\pm\phi$, where ϕ is the solution to the coupling equation (3.7). Thus it is enough to prove the expected analyticity property for $\mathcal{F}_\pm\phi$, which essentially follows from Lemma 4.9 below. We deal here with $\widehat{\phi}^+ := \mathcal{F}_+\phi$ (but, of course, the same procedure applies for $\mathcal{F}_-\phi$). First rewrite (3.7) as

$$\sqrt{\lambda}\,\widehat{\phi}^+ + \mathcal{F}_+\mathcal{F}_-^{-1}\sqrt{\lambda}\,\mathcal{F}_-\mathcal{F}_+^{-1}\widehat{\phi}^+ = \mathcal{F}_+\left[\frac{\partial u_1}{\partial x}\right]_\Sigma . \tag{4.4}$$

By definition (4.5), we have

$$\mathcal{F}_+\mathcal{F}_-^{-1}\sqrt{\lambda}\,\mathcal{F}_-\mathcal{F}_+^{-1} = \sqrt{\lambda}\,\mathrm{I} - \sqrt{\lambda}\,\widetilde{\mathcal{F}}\mathcal{F}_+^{-1} + \widetilde{\mathcal{F}}\mathcal{F}_-^{-1}\sqrt{\lambda}\,\mathcal{F}_-\mathcal{F}_+^{-1}.$$

Noticing that $\mathcal{F}_+^{-1}\widehat{\phi}^+ = \phi = u|_\Sigma$ and $\mathcal{F}_-^{-1}\sqrt{\lambda}\,\mathcal{F}_-\mathcal{F}_+^{-1}\widehat{\phi}^+ = (\partial u_2^-/\partial x)|_\Sigma$ then yields

$$\mathcal{F}_+\mathcal{F}_-^{-1}\sqrt{\lambda}\,\mathcal{F}_-\mathcal{F}_+^{-1}\widehat{\phi}^+ = \sqrt{\lambda}\,\widehat{\phi}^+ - \sqrt{\lambda}\,\widetilde{\mathcal{F}}u\Big|_\Sigma + \widetilde{\mathcal{F}}\frac{\partial u_2^-}{\partial x}\Big|_\Sigma .$$

Moreover, using again (4.5), the right-hand side of (4.4) reads as

$$\mathcal{F}_+\left[\frac{\partial u_1}{\partial x}\right]_\Sigma = \mathcal{F}_+\,\frac{\partial u_1^+}{\partial x}\Big|_\Sigma - \sigma_\lambda\,\mathcal{F}_-\,\frac{\partial u_1^-}{\partial x}\Big|_\Sigma - \widetilde{\mathcal{F}}\,\frac{\partial u_1^-}{\partial x}\Big|_\Sigma .$$

To sum up, as $\partial u/\partial x = \partial u_1^-/\partial x + \partial u_2^-/\partial x$ on Σ, we see that (4.4) implies

$$2\sqrt{\lambda}\,\widehat{\phi}^+ = \sqrt{\lambda}\,\widetilde{\mathcal{F}}u|_\Sigma - \widetilde{\mathcal{F}}\,\frac{\partial u}{\partial x}\Big|_\Sigma + \mathcal{F}_+\,\frac{\partial u_1^+}{\partial x}\Big|_\Sigma - \sigma_\lambda\,\mathcal{F}_-\,\frac{\partial u_1^-}{\partial x}\Big|_\Sigma .$$

By Proposition 4.3 and Lemma 4.9, the two first terms of the right-hand side extend continuously as analytic functions in $\mathcal{Q}$. As mentioned at the beginning of the proof, the same holds for the two others (function σ_λ is also analytic by Lemma A.2) and thus also for the left-hand side, which completes the proof. ∎

LEMMA 4.9. *For all $\varphi \in L^1(\mathbb{R}^+)$, the function defined by*

$$\widetilde{\mathcal{F}}\varphi(\lambda) := \mathcal{F}_+\varphi(\lambda) - \sigma_\lambda\,\mathcal{F}_-\varphi(\lambda) \quad \forall\lambda \in \Lambda_c \tag{4.5}$$

(where σ_λ is given in (A.4)*) extends continuously to an analytic function in $\mathcal{Q}$.*

Proof. Formula (4.5) can be written equivalently as

$$\widetilde{\mathcal{F}}\varphi(\lambda) := \int_{\mathbb{R}^+} \varphi(z)\,\widetilde{\Phi}_\lambda(z)\,\mathrm{d}z \quad \forall\lambda \in \Lambda_c,$$

where $\widetilde{\Phi}_\lambda(z)$ is defined as in (A.4). Lemma A.2 tells us that for all $z \in \mathbb{R}^+$, this function extends continuously to an analytic function in $\mathcal{Q}$, and this extension is exponentially decreasing when z tends to $+\infty$. As $\varphi \in L^1(\mathbb{R}^+)$, the conclusion simply follows from Lebesgue's dominated convergence theorem. ∎

We are now able to prove the uniqueness result of Theorem 4.1. First, we know from Proposition 4.2 that if u is a solution to (P) with $f = 0$ and $u_0 = 0$, then for $x \geq a$, we have $\mathcal{F}_+u(x,\lambda) = 0$ for all $\lambda \in \Lambda^+ \cap \mathbb{R}^-$. Moreover, Proposition 4.8 tells us that $\mathcal{F}_+u(x,\lambda)$ extends continuously to an analytic function in $\mathcal{Q}$. Thus, by the Schwarz reflection principle [8], as $\mathcal{F}_+u(x,\lambda)$ is real (because it is equal to 0) on $\Lambda_c \cap \mathbb{R}^-$, we deduce that $\mathcal{F}_+u(x,\lambda)$ has an analytic continuation in $\{\lambda \in \mathbb{C};\ \mathrm{Re}(\zeta) > -k_\infty^2\} \setminus \mathbb{R}^+$. Therefore, $\mathcal{F}_+u(x,\lambda)$ vanishes in this domain and so also on $\mathbb{R}^+$ (since it is continuous on $\mathcal{Q} \cup \mathbb{R}^+$).

To conclude, we have proved that for all $x \geq a$ and $z \in \mathbb{R}^+$, we have $u(x, z) = 0$. By the unique continuation principle (see, for example, [14]), we finally deduce that $u = 0$.

Remark 4.10. As mentioned in the introduction, this result shows that open and closed waveguides lead to very different phenomena. Indeed, *trapped modes* (corresponding to nonuniqueness cases) are known to occur in closed waveguides [15]. It is therefore worth noticing that the above proof of uniqueness cannot apply for a junction of closed waveguides. Indeed, in this case, the transverse operators $A^{\pm}$ have pure point spectra, since they are defined in bounded cross-sections. Of course, Proposition 4.2 holds thanks to a similar energy argument: with no excitation, the modal components of u associated with propagative modes (that is, with negative values of $\lambda \in \Lambda^{\pm}$) vanish. But since $\Lambda^{\pm} \cap \mathbb{R}^-$ is a finite set, we cannot use our analyticity argument to deduce that the other components also vanish. We see here that the presence of a continuous spectrum actually plays a crucial role in our approach.

5. Conclusion. To our knowledge, this paper presents the first existence and uniqueness result for the solution to the time-harmonic acoustic wave equation for the junction of two uniform open waveguides. Here we criticize ourselves and discuss the possible generalizations of our proof, or the obstacles to such generalizations.

Let us first recall that our results also apply when guided modes do not exist, that is, when $\Lambda_{\mathrm{p}}^{\pm} = \emptyset$, which occurs if $(k_0^2 - k_\infty^2)\ \max(h^-, h^+)^2 < \pi^2/4$ (see the appendix). In this situation, our results intersect other results available in the literature in the context of rough media. In particular, using a radiation condition in the transverse direction, the problem is shown to be well-posed in [5, Theorem 7.5] for suitable variations of k in the transverse direction (for instance, in the situation of Figure 1.2, when $k_0 < k_\infty$) and in [4, Theorem 4] for small enough frequencies (for instance, again in the situation of Figure 1.2, when $\max(k_0, k_\infty)\ \max(h^-, h^+) < \sqrt{2}$). A natural and interesting question is whether the solution which is proved to exist in these papers coincides with ours. We conjecture that these are the same solutions, but unfortunately we did not succeed in proving this, more precisely in verifying that our solution satisfies their radiation condition.

Furthermore, we must admit that our choice of a functional framework (Definition 2.2) is not entirely satisfactory. On one hand, the space $\mathcal{H}$ contains a condition on the transverse behavior of the solution of our scattering problem: $u(x, \cdot) \in V_{\pm}$ for $\pm x \geq 0$. In the case of a local perturbation of a uniform waveguide [1], the function space is simply $H^1_{\mathrm{loc}}(\Omega)$, and such a transverse behavior is deduced from the equations and the fact that the modal components $\widehat{\alpha}^{\pm}$ involved in the radiation conditions are *functions*. For the junction, we did not succeed in doing the same because of the use of two different generalized Fourier transforms. On the other hand, our space $\mathcal{H}$ involves a fictitious section $\Sigma = \{0\} \times \mathbb{R}^+$ which has no physical significance. It simply represents the junction line of the abrupt junction which is needed in our perturbation approach. Of course, if we move this line, the solution must remain the same. But for a rigorous proof of this apparently obvious fact, we have to compare both spaces V_+ and V_-. We conjecture that these spaces coincide, which would show that the solution does not depend on Σ. A similar result was proved in a simpler context [3] using the usual Fourier transform. But in our case, the question is open.

What about the possible generalizations of our method? We have considered here a very simple two-dimensional problem, where both semiwaveguides on both sides of the junction are made with the same homogeneous media characterized by the wavenumber k_0 in the cores ($z < h^{\pm}$) and k_∞ in the claddings ($z > h^{\pm}$). First, if we assume that the wavenumber has distinct values $k_0^+ \neq k_0^-$ in both cores but remains the same in the claddings, all the results and proofs hold. The situation is quite different if the wavenumber has distinct values $k_\infty^+ \neq k_\infty^-$ in both claddings. In this case, the only part which has to be adapted is the study of the decay properties of the solution (section 4.2) for the proof of uniqueness. Indeed, this part is based on the usual Fourier transform of the solution in the longitudinal direction: its use is justified by the fact that the medium is homogeneous for $z > \max\{h^-, h^+\}$. If $k_\infty^+ \neq k_\infty^-$, it is likely that the same results can be obtained using instead the generalized Fourier transform associated with operator $-\mathrm{d}^2/\mathrm{d}x^2 - k_\infty^2(x)$, where $k_\infty(x) = k_\infty^\pm$ if $x \in \mathbb{R}^\pm$. We conjecture that in this case, $V_+ \cap V_- = H^{1/2}(\Sigma)$, which would signify that the solution decays faster in the transverse direction than in a homogeneous cladding, as this is already known for a two-layered medium [9]. Because of this decay, our approach is probably not optimal in this situation.

For more complex acoustic waveguides (stratified, three-dimensional, etc.), the stumbling block of our method lies in a technical but essential result which was only briefly mentioned in the present paper—the possibility to apply the generalized Fourier transform to the solution in any transverse section. Such a generalized Fourier transform can be defined in numerous situations, but it is initially restricted to L^2 functions, which is not sufficient for our purposes. We have to actually extend this transform to a larger space which contains slowly decaying functions. For the particular case dealt with here, this was achieved by introducing a new distribution space similar to the usual Schwartz space of tempered distributions [11]. But as far as we know, no result of this kind is available for a more complex situation! Apart from this technical point, it seems that our method could be applied in a very general context.

On the other hand, we cannot say the same for electromagnetic waveguides in which wave propagation is described by Maxwell's equations. Indeed, the method we have used to study the *coupling equation* (section 3.2) cannot be extended to these vector equations. The same difficulty has been encountered for rough media. An original idea was recently proposed in [10], but it is too early to know if a similar idea could apply in the context of a junction of electromagnetic waveguides.

Appendix. We collect here some basic properties of the generalized eigenfunctions associated with operators $A^\pm$ defined in (2.5). For $\lambda \in \mathbb{C}$, define

$$c_\lambda^\pm(z) := \cos\left(\sqrt{\lambda + k^\pm(z)^2}\,(z - h^\pm)\right) \text{ and } s_\lambda^\pm(z) := \frac{\sin\left(\sqrt{\lambda + k^\pm(z)^2}\,(z - h^\pm)\right)}{\sqrt{\lambda + k^\pm(z)^2}},$$

which is a basis of solutions to (2.3). Note that these definitions do not actually depend on the choice of a determination of the complex square root since they are even functions of $\sqrt{\lambda + k^\pm(z)^2}$. Therefore, for all $z \in \mathbb{R}^+$, both $c_\lambda^\pm(z)$ and $s_\lambda^\pm(z)$ are entire functions of λ. Consider then

$$\Phi_\lambda^\pm(z) := c_\lambda^\pm(0)\, s_\lambda^\pm(z) - s_\lambda^\pm(0)\, c_\lambda^\pm(z),$$

which is an entire family of solutions to (2.3)–(2.4). It is easily seen that $\Phi_\lambda^\pm(z)$ is exponentially increasing as $z \to +\infty$ except in two situations. On one hand, if $k_0^2 - k_\infty^2 > \pi^2/(2h^\pm)^2$, then $\Phi_\lambda^\pm(z)$ becomes exponentially decreasing for the values of $\lambda \in (-k_0^2 + \pi^2/(2h^\pm)^2, -k_\infty^2)$ which satisfy

$$\text{(A.1)} \qquad \tan\left((\lambda + k_0^2)^{1/2}\, h^\pm\right) = -\left(\frac{\lambda + k_0^2}{-\lambda - k_\infty^2}\right)^{1/2}.$$

This dispersion equation has a finite (nonzero) number of roots which constitute the point spectrum $\Lambda_{\mathrm{p}}^{\pm}$ of $A^{\pm}$, and the $\Phi_\lambda^{\pm}(z)$'s are associated eigenfunctions. Note that $\Lambda_{\mathrm{p}}^{\pm} = \emptyset$ if $k_0^2 - k_\infty^2 \leq \pi^2/(2h^{\pm})^2$. On the other hand, if $\lambda \in (-k_\infty^2, +\infty)$, then $\Phi_\lambda^{\pm}(z)$ is a bounded oscillating function. The set $\Lambda_{\mathrm{c}}^{\pm} = \Lambda_{\mathrm{c}} := [-k_\infty^2, +\infty)$ is the continuous spectrum of $A^{\pm}$.

As mentioned in section 2.1, the generalized Fourier transform is the operator of "decomposition" on the family $\{\Phi_\lambda^{\pm}; \lambda \in \Lambda^{\pm} := \Lambda_{\mathrm{p}}^{\pm} \cup \Lambda_{\mathrm{c}}\}$ (see (2.6)). It is unitary from $L^2(\mathbb{R}^+)$ to $L^2(\Lambda^{\pm}; \mathrm{d}\mu^{\pm})$, where $\mathrm{d}\mu^{\pm} := \sum_{\lambda \in \Lambda_{\mathrm{p}}^{\pm}} \rho_\lambda^{\pm}\, \delta_\lambda + \rho_\lambda^{\pm}\, \mathrm{d}\lambda|_{\Lambda_{\mathrm{c}}}$, and

$$\text{(A.2)} \qquad \rho_\lambda^{\pm} := \frac{1}{\|\Phi_\lambda^{\pm}\|_{\mathbb{R}^+}^2} \text{ if } \lambda \in \Lambda_{\mathrm{p}}^{\pm} \quad \text{and} \quad \rho_\lambda^{\pm} := \frac{\beta_\lambda}{\pi\left(c_\lambda^{\pm}(0)^2 + \beta_\lambda^2\, s_\lambda^{\pm}(0)^2\right)} \text{ if } \lambda \in \Lambda_{\mathrm{c}},$$

where $\beta_\lambda := (\lambda + k_\infty^2)^{1/2}$.

In order to prove the two lemmas below, we define, for $\lambda \in \Lambda_{\mathrm{c}}$,

$$\Theta_\lambda^{\pm}(z) := \left(c_\lambda^{\pm}(z) - \mathrm{i}\beta_\lambda\, s_\lambda^{\pm}(z)\right) \mathrm{e}^{-\mathrm{i}\beta_\lambda h^{\pm}} \quad \text{and} \quad \Psi_\lambda^{\pm}(z) := \left(c_\lambda^{\pm}(z) + \mathrm{i}\beta_\lambda\, s_\lambda^{\pm}(z)\right) \mathrm{e}^{\mathrm{i}\beta_\lambda h^{\pm}},$$

and we notice that

$$\text{(A.3)} \qquad \Phi_\lambda^{\pm}(z) = \frac{1}{2\mathrm{i}\beta_\lambda}\left[\Theta_\lambda^{\pm}(0)\, \Psi_\lambda^{\pm}(z) - \Psi_\lambda^{\pm}(0)\, \Theta_\lambda^{\pm}(z)\right] \quad \forall \lambda \in \Lambda_{\mathrm{c}}.$$

Lemma A.1. *For all $\lambda \in \Lambda_{\mathrm{c}}$ and $z \in \mathbb{R}^+$, we have*

$$\left|\Phi_\lambda^{\pm}(z)\right|^2 \rho_\lambda^{\pm} \leq \frac{1}{\pi\beta_\lambda} \quad \text{and} \quad \left|\frac{\partial \Phi_\lambda^{\pm}}{\partial z}(z)\right|^2 \rho_\lambda^{\pm} \leq \frac{\lambda + k^{\pm}(z)^2}{\pi\beta_\lambda}.$$

Proof. For $\lambda \in \Lambda_{\mathrm{c}}$, (A.2) can be written as $\rho_\lambda^{\pm} := \pi^{-1}\beta_\lambda |\Theta_\lambda^{\pm}(0)|^{-2}$. Moreover, noticing that $|\Theta_\lambda^{\pm}(z)| = |\Psi_\lambda^{\pm}(z)| \leq 1$, we deduce from (A.3) that

$$\left|\Phi_\lambda^{\pm}(z)\right|^2 \rho_\lambda^{\pm} \leq \frac{\left(|\Psi_\lambda^{\pm}(z)| + |\Theta_\lambda^{\pm}(z)|\right)^2}{4\pi\beta_\lambda} \leq \frac{1}{\pi\beta_\lambda}.$$

Similarly, we have

$$\left|\frac{\partial \Phi_\lambda^{\pm}}{\partial z}(z)\right|^2 \rho_\lambda^{\pm} \leq \frac{\left(\left|(\partial\Psi_\lambda^{\pm}/\partial z)(z)\right| + \left|(\partial\Theta_\lambda^{\pm}/\partial z)(z)\right|\right)^2}{4\pi\beta_\lambda} \leq \frac{\lambda + k^{\pm}(z)^2}{\pi\beta_\lambda},$$

since $|(\partial\Theta_\lambda^{\pm}/\partial z)(z)| = |(\partial\Psi_\lambda^{\pm}/\partial z)(z)| \leq \sqrt{\lambda + k^{\pm}(z)^2}$. ∎

The following lemma expresses a perturbation relation between Φ_λ^+ and Φ_λ^-.

Lemma A.2. *For $\lambda \in \Lambda_{\mathrm{c}}$ and $z \in \mathbb{R}^+$, define*

$$\text{(A.4)} \qquad \sigma_\lambda := \frac{\Theta_\lambda^+(0)}{\Theta_\lambda^-(0)} \quad \textit{and} \quad \widetilde{\Phi}_\lambda(z) := \Phi_\lambda^+(z) - \sigma_\lambda\, \Phi_\lambda^-(z).$$

Then both σ_λ and $\widetilde{\Phi}_\lambda(z)$ extend continuously to analytic functions in the quadrant $\mathcal{Q} := \{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > -k_\infty^2, \mathrm{Im}(\lambda) < 0\}$, and for all $\lambda \in \mathcal{Q}$, function $\widetilde{\Phi}_\lambda(z)$ is exponentially decreasing as $z \to +\infty$.

Proof. It is clear that $\beta_\lambda := (k_\infty^2 + \lambda)^{1/2}$ extends continuously to an analytic function in $\mathbb{C} \setminus (-k_\infty^2 + \mathrm{i}\mathbb{R}^+)$. Therefore, the same holds for $\Theta_\lambda^{\pm}(0)$, since $c_\lambda^{\pm}(0)$ and $s_\lambda^{\pm}(0)$ are entire functions of λ. Moreover, the zeros of $\Theta_\lambda^{\pm}(0)$, if any, are necessarily smaller than $-k_\infty^2$. Indeed, these are precisely the eigenvalues of $A^\pm$, that is, the roots of the dispersion equation (A.1), except if $\lambda = -k_\infty^2$ is a zero of $\Theta_\lambda^\pm(0)$, but this case is excluded by assumption (2.10). As a consequence, σ_λ and $\widetilde{\Phi}_\lambda(z)$ are analytic in $\mathcal{Q}$.

From (A.3), we have

$$\widetilde{\Phi}_\lambda := \frac{\Theta_\lambda^+(0)}{2\mathrm{i}\beta_\lambda} \left(\Psi_\lambda^+ - \Psi_\lambda^- + \frac{\Psi_\lambda^-(0)}{\Theta_\lambda^-(0)}\,\Theta_\lambda^- - \frac{\Psi_\lambda^+(0)}{\Theta_\lambda^+(0)}\,\Theta_\lambda^+ \right).$$

If $z \geq \max(h^-, h^+)$, this expression simplifies as

$$\widetilde{\Phi}_\lambda(z) = \frac{\Theta_\lambda^+(0)}{2\mathrm{i}\beta_\lambda} \left(\frac{\Psi_\lambda^-(0)}{\Theta_\lambda^-(0)} - \frac{\Psi_\lambda^+(0)}{\Theta_\lambda^+(0)} \right) \mathrm{e}^{-\mathrm{i}\beta_\lambda z},$$

since $\Theta_\lambda^\pm(z) = \exp(-\mathrm{i}\beta_\lambda z)$ and $\Psi_\lambda^\pm(z) = \exp(+\mathrm{i}\beta_\lambda z)$ if $z \geq \max(h^-, h^+)$. When $\lambda \in \mathcal{Q}$, this shows that $\widetilde{\Phi}_\lambda(z)$ is exponentially decreasing as $z \to +\infty$ (for $\operatorname{Im}\beta_\lambda < 0$). In other words, $\widetilde{\Phi}_\lambda$ can be interpreted as an *incoming* wave. $\square$

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