Time harmonic acoustic scattering in presence of a shear flow and a Myers impedance condition

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Abstract. Noise reduction of aircraft engines can be achieved by a well-suited internal coating of the nacelle, which is generally modelized by the Myers impedance condition. Therefore there is a need of a numerical method for acoustics in presence of a complex flow and treated boundaries. We consider the time harmonic acoustic radiation in a confined flow in presence of treated boundaries. The case of a potential flow with Myers condition leads to a scalar formulation which is tractable by a standard finite element method. On the other hand, Galbrun's equation [1] seems to be well-suited for handling non potential flows [2, 3]. For a uniform flow we present a scalar problem in presence of PMLs (Perfectly Matched Layers). Well-posedness is proved which ensures the convergence of the finite element discretization. To extend to a shear flow, the vectorial Galbrun's equation is used and we show that Myers condition is natural and easy to incorporate in Galbrun's framework. We explain why the proof of the well-posedness is not so straighforward than in the scalar case, even for a uniform flow. Finally the difficulty is solved by enriching the Myers condition.

Key words: aeroacoustics, Myers condition, scattering of sound in flows, Galbrun's equation, finite element

1 The scalar formulation

We consider the radiation of an acoustic source f in an infinite 2D duct with treated boundaries (see Figure 1). In presence of a compressible fluid in parallel and subsonic uniform flow U_0 we can use a scalar

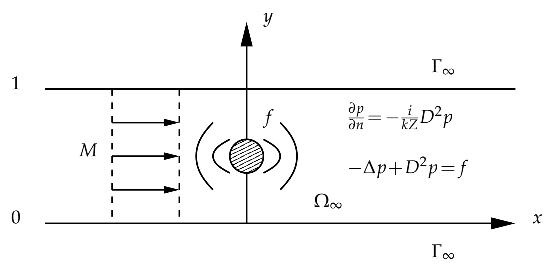


Figure 1: The pressure model

model. With time dependence $e^{-i\omega t}$ omitted and in a dimensionless form, the height h of the duct being the reference length, the pressure perturbation p satisfies the convected Helmholtz equation $-\Delta p + D^2 p = f$ in $\Omega_\infty = \{(x,y); 0<y<1\}$ where $D = M\partial/\partial x - ik$ is the convective derivative with $M = U_0/c$ the Mach number, $k = \omega h/c$ the dimensionless wave number and c the speed of sound. The upper and lower walls Γ are treated and modelized by the Myers boundary condition [4]: $\dfrac{\partial p}{\partial n} = -\dfrac{i}{kZ} D^2 p$ where the constant Z is the dimensionless impedance with $\Re e(Z) > 0$ (to get sound attenuation).

To close the problem some radiation conditions must be imposed. Using the guided modes we have built Dirichlet-to-Neuman operators (DtN) allowing to write exact radiation conditions at finite distance from the source [5]. Another easy way to select the outgoing solution consists in introducing PMLs: computations are done in a bounded domain Ω composed of the physical domain $\Omega_b = \{(x,y); 0<x<d, 0<y<1\}$ around the source and of surrounding layers $\Omega_\pm^L$ of length L (see Figure 2). The introduction of PMLs amounts to the transformation of the differential operator $\partial/\partial x \longrightarrow \alpha(x)\partial/\partial x$ in the governing equations

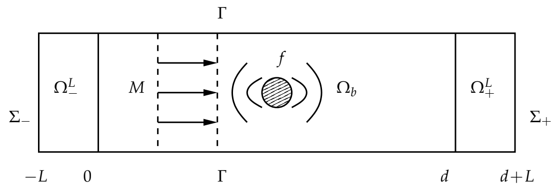


Figure 2:

of the problem. The complex function α is assumed to be unity in Ω_b and constant and equal to the complex scalar α^*, satisfying the following hypotheses $\mathrm{Re}(\alpha^*)>0, \mathrm{Im}(\alpha^*)<0$ in $\Omega\backslash\Omega_b$ (see [6] for a more thorough description and justification).

Therefore for a source $f \in L^2(\Omega)$ and $M<1$ the radiation problem reads:

$$\begin{cases} -\Delta_\alpha p + \mathrm{D}_\alpha^2 p = f \quad (\Omega), \\ \dfrac{\partial p}{\partial n} = -\dfrac{i}{kZ} D_\alpha^2 p \quad (\Gamma), \\ p = 0 \quad (\Sigma_\pm), \end{cases}$$

and has the equivalent variational form:

$$\begin{cases} \text{Find } p \in U = \{p \in H^1(\Omega), p|_\Gamma \in H^1(\Gamma)\} \text{ such that} \\ a(p,q) = \int_\Omega f\bar{q} \text{ for all } q \in U, \end{cases} \tag{1.1}$$

where the bilinear form $a(p,q)$ is defined as:

$$a(p,q) = \int_\Omega \frac{1}{\alpha}\left[\nabla_\alpha p \cdot \nabla_\alpha \bar{q} - \left(M\alpha\frac{\partial}{\partial x} - ik\right) p \left(M\alpha \frac{\partial}{\partial x} + ik\right)\bar{q}\right] - \frac{i}{kZ}\int_\Gamma \frac{1}{\alpha}\left[\left(M\alpha\frac{\partial}{\partial x} - ik\right) p \left(M\alpha\frac{\partial}{\partial x} + ik\right)\bar{q}\right].$$

To prove that problem 1.1 is well-posed, we use the Fredholm alternative which ensures the convergence of a finite element discretization. The most important step is to prove that $\exists C>0$ such that $\forall p \in U$:

$$\left|\int_\Omega (1-M^2)\alpha\left|\frac{\partial p}{\partial x}\right|^2 + \frac{1}{\alpha}\left|\frac{\partial p}{\partial y}\right|^2 - \frac{i}{kZ}\int_\Gamma M^2\alpha\left|\frac{\partial p}{\partial x}\right|^2\right| \geq C\left(\int_\Omega |\nabla p|^2 + \int_\Gamma \left|\frac{\partial p}{\partial x}\right|^2\right).$$

For $\arg(\alpha^*) = -i\pi/4$ (this is a value leading to good results in practice), this condition is fullfilled as soon as $\Im m(Z)<0$. On Figure 3 is represented the real part of the pressure radiated by a circular source with an upper treated boundary with $\Im m(Z)<0$. The finite element library MELINA is used [7]. The solution computed with the DtN is used as a reference solution to illustrate the efficiency of the use of PMLs. Similar results have been observed in [5] for a potential flow.

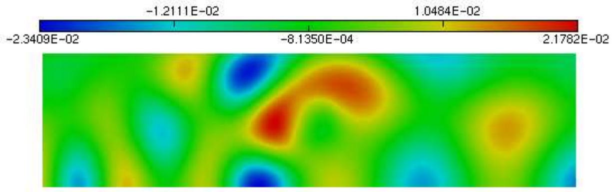
2 The Galbrun's framework

In presence of a shear flow there is no scalar model adapted to a finite element discretization. In the case of rigid boundaries, we have proved that the use of the vectorial Galbrun's equation is a very good alternative: we have developped numerical methods, using finite elements combined with perfectly Matched Layers, to determine acoustic radiation in presence of a confined shear flow [8,9] and a 2D complex flow [2,3].

We will now extend Galbrun's equation to the presence of treated boundaries. We consider for simplicity the case of a uniform flow. Galbrun's equation is well suited to take into account a Myers condition because the unknown is the displacement perturbation $\Re e\left(\boldsymbol{u}(x)e^{-i\omega t}\right)$ and Myers condition takes a very simple form when expressed versus $\boldsymbol{u}$. In particular it does not depend on the flow on the boundary.

Following [9], augmented Galbrun's equation reads $\mathrm{D}^2\boldsymbol{u} - \nabla(\mathrm{div}\boldsymbol{u}) + \mathbf{curl}(\mathrm{curl}\boldsymbol{u}) = \boldsymbol{f}$ and Myers condition takes the form $\mathrm{div}\boldsymbol{u} = ikZ\boldsymbol{u}\cdot\boldsymbol{n}$ ($p = -\mathrm{div}\boldsymbol{u}$). In presence of PMLs, the variational formulation associated to augmented Galbrun's equation is, introducing $V = \{\boldsymbol{u} \in H^1(\Omega)^2; \boldsymbol{u} = \boldsymbol{0} \text{ on } \Sigma_\pm\}$:

$$\begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that } \forall \boldsymbol{v} \in V \\ a(\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{v}}, \end{cases}$$



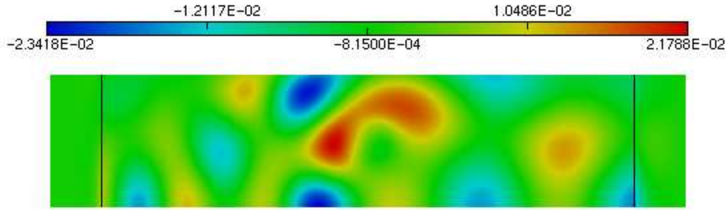


Figure 3: Real part of the acoustic pressure, DtN (top) PMLs (bottom), ($M=0.3$, $Z=1-i$, $k=7$)

where

$$a(\boldsymbol{u},\boldsymbol{v}) \;=\; \int_\Omega \frac{1}{\alpha}\left[\operatorname{div}_\alpha \boldsymbol{u}\,\operatorname{div}_\alpha \bar{\boldsymbol{v}} + \operatorname{curl}_\alpha \boldsymbol{u}\,\operatorname{curl}_\alpha \bar{\boldsymbol{v}} - \left(M\alpha\frac{\partial}{\partial x} - ik\right)\boldsymbol{u}\cdot\left(M\alpha\frac{\partial}{\partial x} + ik\right)\bar{\boldsymbol{v}}\right] - ikZ\int_\Gamma \frac{1}{\alpha}(\boldsymbol{u}\cdot\boldsymbol{n})(\bar{\boldsymbol{v}}\cdot\boldsymbol{n}).$$

Numerical implementation is straightforward but contrary to the situation with the scalar model, the theory is not satisfying. We will now clarify what are the difficulties to prove well-posedness and an enriched Myers condition which solves this difficulty will be proposed.

3 Theory for a simplified problem

3.1 Ill-posedness with the usual Myers condition

For the sake of clarity we consider a simpler problem: we take $\alpha=1$ everywhere and we only keep the terms with second order derivatives (other terms are compact perturbation and do not modify the conclusion):

Find $\boldsymbol{u}\in V=\{\boldsymbol{u}\in H^1(\Omega)^2; \boldsymbol{u}=\boldsymbol{0} \text{ on } \Sigma_\pm\}$ solution of:

$$\begin{cases} \text{Find } \boldsymbol{u}\in V \text{ such that } \forall \boldsymbol{v}\in V \\ a(\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{v}}, \end{cases} \tag{3.1}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) \;=\; \int_\Omega \left(\operatorname{div}\boldsymbol{u}\,\operatorname{div}\bar{\boldsymbol{v}} + \operatorname{curl}\boldsymbol{u}\,\operatorname{curl}\bar{\boldsymbol{v}} - M^2\frac{\partial \boldsymbol{u}}{\partial x}\cdot\frac{\partial \bar{\boldsymbol{v}}}{\partial x}\right) - ikZ\int_\Gamma (\boldsymbol{u}\cdot\boldsymbol{n})(\bar{\boldsymbol{v}}\cdot\boldsymbol{n}).$$

To prove that the problem 3.1 is well-posed we need to prove the following coerciveness property: $\exists C>0$ such that $\forall \boldsymbol{u}\in V$

$$|a(\boldsymbol{u},\boldsymbol{u})| \geq C\int_\Omega |\boldsymbol{\nabla}\boldsymbol{u}|^2.$$

In the rigid case $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ (or $|Z|=\infty$), this is easy because we have Costabel's identity [10]: $\forall \boldsymbol{u}\in W=\{\boldsymbol{u}\in H^1(\Omega)^2; \boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \Gamma \text{ and } \boldsymbol{u}=\boldsymbol{0} \text{ on } \Sigma_\pm\}$,

$$\int_\Omega |\operatorname{div}\boldsymbol{u}|^2 + |\operatorname{curl}\boldsymbol{u}|^2 = \int_\Omega |\boldsymbol{\nabla}\boldsymbol{u}|^2.$$

Therefore $\forall M<1$ and $\forall \boldsymbol{u}\in W$:

$$a(\boldsymbol{u},\boldsymbol{u}) = \int_\Omega |\operatorname{div}\boldsymbol{u}|^2 + |\operatorname{curl}\boldsymbol{u}|^2 - M^2\left|\frac{\partial \boldsymbol{u}}{\partial x}\right|^2 \geq (1-M^2)\int_\Omega |\boldsymbol{\nabla}\boldsymbol{u}|^2.$$

With Myers condition ($|Z|\neq\infty$), Costabel's identity is no longer true and despite the presence of the term $-ikZ\int_\Gamma (\boldsymbol{u}\cdot\boldsymbol{n})(\bar{\boldsymbol{v}}\cdot\boldsymbol{n})$ coerciveness fails.

3.2 Regularized Myers condition

We propose an enriched Myers condition:

$$\mathrm{div}\boldsymbol{u} = ikZ\left[\boldsymbol{u}\cdot\boldsymbol{n} - \beta\frac{\partial^2}{\partial x^2}(\boldsymbol{u}\cdot\boldsymbol{n})\right] \quad \text{on} \quad \Gamma,$$

with $\beta > 0$. Note that this new condition involves a second order tangential derivative, which was also the case for the "usual" Myers boundary condition when expressed versus the pressure. The new condition adds the term

$$\int_\Omega ikZ\beta\frac{\partial}{\partial x}(\boldsymbol{u}\cdot\boldsymbol{n})\frac{\partial}{\partial x}(\bar{\boldsymbol{v}}\cdot\boldsymbol{n}),$$

in the variational formulation 3.1.

Thanks to this new boundary condition, we are able to assert that the enriched problem is well-posed. The key point is to prove the following result: $\exists\delta > 0$ and $C > 0$ such that $\forall \boldsymbol{u} \in X = \left\{\boldsymbol{u} \in H^1(\Omega)^2; \boldsymbol{u} = \boldsymbol{0} \text{ on } \Sigma_\pm \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} \in H^1(\Gamma)\right\}$

$$\int_\Omega \left(|\mathrm{div}\boldsymbol{u}|^2 + |\mathrm{curl}\boldsymbol{u}|^2\right) - M^2\left|\frac{\partial \boldsymbol{u}}{\partial x}\right|^2 + k|Z|\beta\int_\Gamma\left|\frac{\partial}{\partial x}(\boldsymbol{u}\cdot\boldsymbol{n})\right|^2 \geq C\int_\Omega |\boldsymbol{\nabla}\boldsymbol{u}|^2 - \delta\int_\Omega |\boldsymbol{u}|^2.$$

β can be chosen small, the limit $\beta \to 0$ corresponding to approach the usual Myers condition. Finally, including PMLs in the enriched model, it is possible to extend the results of the scalar model to the enriched model and to determine conditions on the impedance to ensure the convergence of a finite element discretization. Numerical illustrations are in progress.

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