ELECTROMAGNETIC APPROXIMATE TRANSMISSION CONDITION FOR THIN PERIODIC LAYER

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Talk Abstract
This work deals with the construction of approximate models for the scattering of electromagnetic waves by a thin and periodic layer. With the help of an asymptotic expansion with respect to both the thickness and the period of the layer, we build an approximate model wherein the thin periodic layer is replaced by an effective transmission condition. This work is an extension of the scalar problem studied in [1] to the full electromagnetic problem. In this first work, we restrict ourselves to a cartesian configuration of the geometry. We pay particular attention to the analysis of the obtained approximate problems. In particular, to overcome the well-known compactness issues of the Maxwell equations, we build an adapted Helmholtz decomposition that allows us to prove the well-posedness and the stability of our model. Error estimates are provided and numerical experiments validating the accuracy of the model are summarized.

1 Problem Description

Let us first define the domain Ω as

$$\Omega := \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3, |x_i| < L_i/2, i \in \{1,2,3\}\Big\}. \tag{1}$$

Let $\Sigma^{\pm}_{x_i}$ ($i \in \{1,2,3\}$) denote the boundary of Ω with outgoing normal $\pm e_i$ (Fig. 1).
We are interested in the numerical computation of the electric Field $\mathbf{E}^\delta$ satisfying second order Maxwell Equations,

$$\operatorname{curl} \frac{1}{\mu^\delta} \operatorname{curl} \mathbf{E}^\delta - \omega^2 \epsilon^\delta \mathbf{E}^\delta = f \text{ in } \mathcal{D}'(\Omega), \tag{2}$$

with periodic boundary conditions on the lateral boundaries $\Sigma^{\pm}_{x_1}$ and $\Sigma^{\pm}_{x_2}$,

$$\left\{ \begin{array}{ll} \mathbf{E}^\delta \times e_{i|\Sigma^+_{x_i}} = \mathbf{E}^\delta \times e_{i|\Sigma^-_{x_i}}, & i \in \{1,2\}, \\ (\mu^\delta)^{-1} \operatorname{curl} \mathbf{E}^\delta \times e_{i|\Sigma^+_{x_i}} = (\mu^\delta)^{-1} \operatorname{curl} \mathbf{E}^\delta \times e_{i|\Sigma^-_{x_i}}, & \end{array} \right. \tag{3}$$

and an impedance condition on the lower and upper boundaries $\Sigma^{\pm}_{x_3}$:

$$\pm \operatorname{curl} \mathbf{E}^\delta \times e_3 - i\omega (\mathbf{E}^\delta)_T = 0 \quad \text{on} \quad \Sigma^{\pm}_{x_3}. \tag{4}$$

The domain Ω is made of a thin, periodic (in x_1 and x_2) dielectric layer placed into an homogeneous medium (see Fig.1): the electric permittivity ϵ^δ and the magnetic permeability μ^δ are δ periodic in x_1 and x_2, and constant for $|x_3| > \delta/2$. More precisely, we assume that there exist two positive bounded functions μ and ρ which satisfy

$$\begin{cases} \mu(X_1 + 1, X_2, X_3) = \mu(X_1, X_2, X_3), \\ \mu(X_1, X_2 + 1, X_3) = \mu(X_1, X_2, X_3), \\ \mu(X_1, X_2, X_3) = 1 \text{ si } |X_3| > \frac{1}{2}, \end{cases} \tag{5}$$

$$\begin{cases} \epsilon(X_1 + 1, X_2, X_3) = \epsilon(X_1, X_2, X_3), \\ \epsilon(X_1, X_2 + 1, X_3) = \epsilon(X_1, X_2, X_3), \\ \epsilon(X_1, X_2, X_3) = 1 \text{ si } |X_3| > \frac{1}{2}. \end{cases} \tag{6}$$

such that

$$\begin{cases} \mu^\delta(x_1, x_2, x_3) = \mu(x_1/\delta, x_2/\delta, x_3/\delta), \\ \epsilon^\delta(x_1, x_2, x_3) = \epsilon(x_1/\delta, x_2/\delta, x_3/\delta). \end{cases} \tag{7}$$

To simplify the analysis, we also assume that ϵ and μ are even functions in x_1, x_2 and x_3. The source term f is in $L^2(\Omega)$, satisfies $\operatorname{div} f = 0$ and is compactly supported. We further assume that its support does not intersect the mean interface Γ

$$\Gamma := \{(x_1, x_2, x_3) \in \Omega \text{ such that } x_3 = 0\} \tag{8}$$

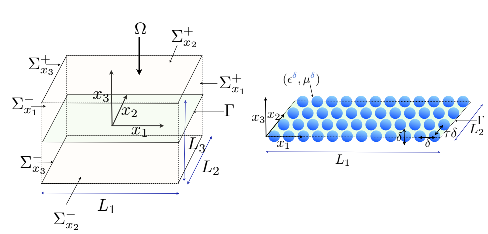

Figure 1: Domain Ω

Difficulties, Objectives and Methods

We are interested in the numerical computations of $\mathbf{E}^\delta$ in situations where the thickness and the period δ are much smaller than the wavelength λ. Under these circumstances, numerical computations of the solution would become prohibitively expensive as the small scale δ goes to 0, since the employed mesh needs to accurately follow the geometry of the inhomogeneities.

In order to overcome this difficulty, we build an approximate transmission model wherein the periodic thin layer is replaced by a transmission condition on Γ. 'Far' from Γ, the solution of the approximate model is close to the exact one. The numerical discretization of the approximate problem is expected to be much less expensive than the exact one, since the employed mesh is no longer constrained by the small scale.

The construction of this simplified transmission condition is based on an asymptotic expansion of $\mathbf{E}^\delta$ with respect to small scale δ. We employ for this purpose a method which combines the techniques of periodic-homogenization and matched asymptotic expansions (see [2]-[3]): without going into details, we can prove that the following expansion holds:

$$\mathbf{E}^\delta = \begin{cases} \sum_{n\in\mathbb{N}} \delta^n \mathbf{E}_n(x_1,x_2,x_3) & \text{far from } \Gamma \\ \sum_{n\in\mathbb{N}} \delta^n \mathcal{E}_n(\frac{x_1}{\delta},\frac{x_2}{\delta},\frac{x_3}{\delta};x_1,x_2) & \text{close to } \Gamma. \end{cases} \quad (9)$$

From the previous expansion it is then possible to build an approximate problem of order 1 by constructing a problem whose solution is close to the first two terms of the far field expansion, i.e. $\mathbf{E}_0 + \delta\mathbf{E}_1$, up to $O(\delta^2)$ terms (cf. [4]).

2 Construction and analysis of a first order approximate transmission condition

2.1 Preamble : periodicity cell problems

The transmission condition takes into account the periodicity and the thickness of the thin layer through constants which are computed by solving so-called periodicity cell problems. Let us define the infinite periodicity cell B

$$B := \big\{((X_1,X_2,X_3)\in\mathbb{R}^3,\, |X_1|\le 1/2, |X_2|\le 1/2\big\}$$

and the weighted Sobolev space $W_1(\mathbb{R}^3)$,

$$W_1(\mathbb{R}^3) := \big\{p\in\mathcal{D}'(\mathbb{R}^3), p\,\text{1-periodic in } X_1 \text{ and } X_2, \nabla p\in L^2(B), \frac{p}{\sqrt{1+X_3^2}}\in L^2(B)\big\}. \quad (10)$$

For $i = 1$, 2 or 3, and for $\sigma = \mu$ or ϵ, we consider $\tilde{p}_i^\sigma$, the solution of the well-posed cell problem:

$$\begin{cases} \tilde{p}_i^\sigma \in W^1(\mathbb{R}^3)|\mathbb{R}, \\ \operatorname{div}(\sigma\nabla p_i^\sigma) = -\partial_{X_i}\sigma \text{ in } \mathcal{D}'(\mathbb{R}^3). \end{cases} \quad (11)$$

Then, we define p_i^σ by

$$p_i^\sigma := \tilde{p}_i^\sigma + X_i. \quad (12)$$

As we shall see, these functions appear in the approximate transmission condition through integrals of their derivatives over the truncated periodicity cell B_0:

$$B_0 = \big\{(x,y,z)\in B, |z|\le 1/2\big\}. \quad (13)$$

2.2 First order approximate problem

We are now in a position to define our first order approximate problem. Let us introduce a new parameter $\alpha \ge 0$ and the domain $\Omega_\alpha^\delta := \Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-$, where

$$\Omega_{\alpha\delta}^\pm := \{(x_1,x_2,x_3)\in\Omega \text{ such that } \pm x_3 > \alpha\delta\}.$$

Our approximate transmission problem is defined as follows. Find $\mathbf{E}_1^\delta$ that satisfies the Maxwell equations

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_1^\delta - \omega^2\mathbf{E}_1^\delta = f \text{ in } \Omega_{\alpha\delta}^+\cup\Omega_{\alpha\delta}^-, \quad (14)$$

the boundary conditions (3-4) and the following non-trivial transmission condition

$$-[e_3\times\mathbf{E}_1^\delta]_\alpha = \delta\Big(\frac{A}{\omega^2}\,\vec{\operatorname{curl}}_\Gamma\operatorname{curl}_\Gamma\big((\langle\operatorname{curl}\mathbf{E}_1^\delta\rangle)_\alpha\big)_T - D_1\big(\langle\operatorname{curl}\mathbf{E}_1^\delta\rangle_\alpha\big)_T\Big), \quad (15)$$

$$-[e_3\times\operatorname{curl}\mathbf{E}_1^\delta]_\alpha = \delta\Big(B\,\vec{\operatorname{curl}}_\Gamma\operatorname{curl}_\Gamma\langle\mathbf{E}_{1T}^\delta\rangle_\alpha - \omega^2 D_2\langle\mathbf{E}_{1T}^\delta\rangle_\alpha\Big), \quad (16)$$

where, for any smooth function, the jump $[u]_\alpha$ and the mean-value $\langle u\rangle_\alpha$ are given by

$$[u]_\alpha = u_\alpha^+ - u_\alpha^-, \quad \langle u\rangle_\alpha = \frac{1}{2}\left(u_\alpha^+ + u_\alpha^-\right),$$

and $u_\alpha^\pm = u(x_1,x_2,\pm\alpha\delta)$. The constants A, B and the diagonal matrices D_1 and D_2 represent the periodic ring and are given by

$$\begin{array}{l} A := 2\alpha - 1 + \int_{B_0}\partial_{X_3}p_3^\epsilon\,dX, \\ B := 2\alpha - 1 + \int_{B_0}\partial_{X_3}p_3^\mu\,dX, \\ (D_1)_{1,1} := 2\alpha - 1 + \int_{B_0}\mu\partial_{X_1}p_1^\mu\,dX, \\ (D_1)_{2,2} := 2\alpha - 1 + \int_{B_0}\mu\partial_{X_2}p_2^\mu\,dX, \\ (D_2)_{1,1} := 2\alpha - 1 + \int_{B_0}\epsilon\partial_{X_1}p_1^\epsilon\,dX, \\ (D_2)_{2,2} := 2\alpha - 1 + \int_{B_0}\epsilon\partial_{X_2}p_2^\epsilon dX. \end{array} \quad (17)$$

Remark 2.1. The introduction of the parameter α is motivated by stability considerations (see [1]). Note that for α large enough, the previous constants are positive. In what follows, we assume that α is chosen large enough so that constants (17) are positive.

2.3 Analysis of the approximate model

We shall prove that the problem (14, 3, 4, 15, 16) is well posed. We first rewrite it in a variational form. Let us introduce the operator $\mathcal{G}$:

$$\begin{cases} TH^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to TH_\#(\mathrm{curl}_\Gamma,\Gamma), \\ g \mapsto \lambda \text{ satisfying } -g = A\vec{\mathrm{curl}}_\Gamma \mathrm{curl}_\Gamma \lambda - D_1\omega^2\omega^2\lambda. \end{cases} \tag{18}$$

where $TH_\#(\mathrm{curl}_\Gamma,\Gamma)$ denotes the set of $TH(\mathrm{curl}_\Gamma,\Gamma)$ functions which are L_1-periodic in x_1 and L_2-periodic in x_2. It is easily seen that $\mathcal{G}$ is well-defined except for a sequence of frequencies ω_n which tends toward $+\infty$. From now, we make the assumption that, in our case, ω is not in this set of forbidden frequencies. The first part of the transmission condition (15) can be rewritten with the help of the operator $\mathcal{G}$:

$$\langle(\mathrm{curl}_\Gamma \mathbf{E}_1^\delta)_T\rangle_\alpha = \frac{\omega^2}{\delta}\mathcal{G}([e_3 \times \mathbf{E}_1^\delta]_\alpha). \tag{19}$$

The natural variational space associated with problem (14, 3, 4, 16, 19) is

$$X^\delta := \big\{u \in H_\#(\mathrm{curl}\,,\Omega^+_{\alpha\delta}) \cap H_\#(\mathrm{curl}\,,\Omega^-_{\alpha\delta}), \\ \langle u_T\rangle_\alpha \in TH_{per}(\mathrm{curl}_\Gamma,\Gamma), (u_T)_{|\Sigma^\pm_{x_3}} \in TL^2(\Sigma^\pm_{x_3})\big\},$$

equipped with the norm

$$\|\mathbf{E}\|^2_{X^\delta} = \|\mathbf{E}\|^2_{H(\mathrm{curl}\,,\Omega^\pm_{\alpha\delta})} + \delta B\,\|\langle\mathbf{E}_T\rangle_\alpha\|^2_{H(\mathrm{curl}\,,\Gamma)} \\ + \omega\int_{\Sigma^\pm}|\mathbf{E}_T|^2\,ds + \frac{A\omega^2}{\delta}\,\|\mathrm{curl}_\Gamma\mathcal{G}([e_3\times\mathbf{E}]_\alpha)\|^2_{L^2(\Gamma)} \\ + \frac{\omega^4}{\delta}\int_\Gamma D_1\mathcal{G}([e_3\times\mathbf{E}]_\alpha)\cdot\overline{\mathcal{G}([e_3\times\mathbf{E}]_\alpha)}\,ds.$$

The variational form is then obtained by multiplying (14) by a test function $v \in X^\delta$ and by integrating by parts over the domain $\Omega_{\alpha\delta}$:

$$\forall\varphi \in X^\delta, \qquad a_\delta(\mathbf{E}_1^\delta,\varphi) = \int_{\Omega^+_{\alpha\delta}\cup\Omega^-_{\alpha\delta}} f\cdot\overline{\varphi}, \tag{20}$$

where,

$$a_\delta(\mathbf{E}_1^\delta,\varphi) := a_\delta^+(\mathbf{E}_1^\delta,\varphi) + b_\delta(\mathbf{E}_1^\delta,\varphi),$$

with,

$$a_\delta^+(\mathbf{E}_1^\delta,\varphi) := \int_{\Omega^+_{\alpha\delta}\cup\Omega^-_{\alpha\delta}} \mathrm{curl}\,\mathbf{E}_1^\delta\cdot\overline{\mathrm{curl}\,\varphi}\,dx_1 \\ - i\omega\int_{\Sigma^\pm}(\mathbf{E}_1^\delta)_T\cdot\overline{\varphi}\,ds \\ + \delta B\int_\Gamma \mathrm{curl}_\Gamma\langle(\mathbf{E}_1^\delta)_T\rangle_\alpha\cdot\overline{\mathrm{curl}_\Gamma\langle\varphi_T\rangle_\alpha}\,ds, \tag{21}$$

and

$$b_\delta(\mathbf{E}_1^\delta,\varphi) := -\int_{\Omega^+_{\alpha\delta}\cup\Omega^-_{\alpha\delta}}\omega^2\mathbf{E}_1^\delta\cdot\overline{\varphi}\,dx_1 \\ - \delta\int_\Gamma\omega^2 D_2\langle(\mathbf{E}_1^\delta)_T\rangle_\alpha\cdot\overline{\langle\varphi\rangle_\alpha}ds \\ + \frac{\omega^2}{\delta}\langle\mathcal{G}([e_3\times\mathbf{E}_1^\delta]_\alpha),\overline{[e_3\times\varphi]_\alpha}\rangle. \tag{22}$$

In the previous formula, $\langle\cdot,\cdot\rangle$ denotes a duality product defined on Γ whose pivot is $L^2(\Gamma)$. Note that a^+ is positive.

We can now formulate our main result:

Proposition 2.2. For δ small enough, Problem (20) is well posed. More precisely, there exist $\delta_0 > 0$ and a constant $C_1 > 0$ such that, for any $\delta < \delta_0$,

$$\left\|\mathbf{E}_1^\delta\right\|_{X^\delta} \le C_1\,\|f\|_{L^2(\Omega)}\,. \tag{23}$$

Moreover, there also exist a constant $\delta_\gamma > 0$ and a constant $C_2 > 0$, such that, for any $\delta < \delta_\gamma$,

$$\left\|\mathbf{E}^\delta - \mathbf{E}_1^\delta\right\|_{H(\mathrm{curl}\,,\Omega_\gamma)} \le C_2\,\delta^2\,\|f\|_{L^2(\Omega)}\,.$$

where $0 < \gamma < \frac{L_3}{2}$ and

$$\Omega_\gamma := \{(x_1,x_2,x_3)\in\Omega, |x_3| > \gamma\}\,.$$

Proof of Proposition 2.2

We shall not give the entire details but only explain the main steps of the well-posedness proof (see [4] for the convergency proof). The main idea is to use the Fredholm alternative framework. This cannot be done directly since X^δ is not compactly embedded in $L^2(\Omega_{\alpha\delta})$ (cf. [5]). Hence, the operator associated with the sesquilinear form b is not compact in X^δ. To overcome this difficulty, following a classical method (see [6]), we write a particular Helmholtz decomposition, which is completely adapted to our transmission problem. More precisely, it is possible to prove that

$$X^\delta = X_0^\delta \oplus \nabla S, \tag{24}$$

where

$$S^\delta := \{p \in H^1(\Omega^+_{\alpha\delta}) \cap H^1(\Omega^-_{\alpha\delta}),\ \langle p\rangle_\alpha \in H^1(\Gamma),$$
$$p_{|\Sigma^\pm_{x_3}} = 0, \nabla p \times e_{i|\Sigma^+_{x_i}} = \nabla p \times e_{1|\Sigma^-_{x_1}}, i \in \{1,2\}\}, \quad (25)$$

and

$$\begin{aligned} X_0^\delta &:= \{u \in X^\delta,\ b_\delta(u, \nabla p) = 0,\ \forall p \in S^\delta\} \quad (26)\\ &= \{u \in X^\delta,\ \operatorname{div} u = 0 \in \Omega^+_{\alpha\delta} \cup \Omega^-_{\alpha\delta},\\ &\quad u \cdot e_{i\Sigma^+_{x_i}} = u \cdot e_{i\Sigma^-_{x_i}}, i \in \{1,2\},\\ &\quad \langle u \cdot e_3\rangle_\alpha = \frac{1}{\delta}\mathrm{curl}_\Gamma \mathcal{G}([e_3 \times u]_\alpha)\ ^*,\\ &\quad -[u \cdot e_3]_\alpha = \delta\ \mathrm{div}_\Gamma D_2\langle u_T\rangle_\alpha\ ^* \quad \} \end{aligned}$$

- X_0^δ is included in $H^{1/2}(\Omega_{\alpha\delta})$: indeed, note that X_0^δ is a subset of $H(\mathrm{curl}\,, \Omega_{\alpha\delta}) \cap H(\mathrm{div}\,, \Omega_{\alpha\delta}) \cap TL^2(\Sigma^\pm_{x_3})$. In addition, by the transmission conditions $(*)$, we can also see that $(u_\alpha)^\pm \in H^{1/2}(\Gamma)$. These two arguments are sufficient to prove that X_0^δ is included in $H^{1/2}(\Omega_{\alpha\delta})$ (see [7]). Consequently, for a fixed δ, the operator associated with b^δ is compact in X_0^δ.

- Moreover it is easily seen that $\mathbf{E}_1^\delta$ is in X_0^δ. Consequently, Problem (20) is equivalent to the following one, where test functions are in X_0^δ:

$$\forall \varphi \in X_0^\delta, \qquad a_\delta(\mathbf{E}_1^\delta, \varphi) = \int_{\Omega^+_{\alpha\delta}\cup\Omega^-_{\alpha\delta}} f \cdot \overline{\varphi}, \qquad (27)$$

Since b^δ is compact in X_0^δ, the previous problem is a Fredholm type problem.

-To end the proof, it suffices to establish the uniqueness for δ small enough. This is done by establishing (by contradiction) the stability estimate (23).

3 Numerical validation

To end this abstract, we present some brief numerical results which confirm both the accuracy and the efficiency of our model. We are interested in the simulation of the scattering by a thin periodic layer ($\delta = 0.0125$ made of an array of 32×32 spherical dielectric heterogeneities ($\mu = \epsilon = 2$) placed into an homogeneous medium ($\mu = \epsilon = 1$). The source term is an incident plane wave $u_{inc}(x_1, x_2, x_3) = e^{-16i\pi x_3/2}e_1$. In figure 3, we can see the real part of the e_1 component of the scattered fields in the plane $x = 0$ obtained by the exact (computed with a strongly refined mesh) and approximate models: the two solutions are really close (relative error = 0.01 ($H(\mathrm{curl}\,, \Omega_\gamma)$ norm)) whereas the mesh used to obtain the approximate solution is much less refined that the exact one.

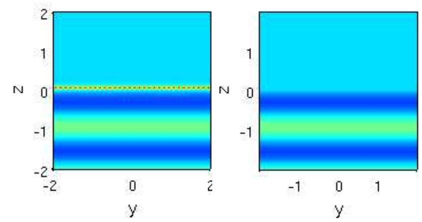

Figure 2: "Exact" and approximate solutions

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