Transmission eigenvalue problems
with sign-changing coefficients

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Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion $D$ (coefficients $A$ and $n$) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.
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This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni’s talk):

\[ \text{Transmission conditions on } \partial D \]

Definition. Values of $k \in \mathbb{C}$ for which this problem has a non trivial solution $(u,w)$ are called transmission eigenvalues.

The goal in this talk is to prove that the set of transmission eigenvalues is at most discrete.
Presentation of the ITEP

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- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni’s talk):
  
  $u$ is the total field in $D$

  \[
  \text{div} (A \nabla u) + k^2 n u = 0 \quad \text{in } D
  \]
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  - $u$ is the total field in $D$
  - $w$ is the incident field in $D$

  
  \[
  \begin{align*}
  \text{div} (A \nabla u) + k^2 nu & = 0 \quad \text{in } D \\
  \Delta w + k^2 w & = 0 \quad \text{in } D
  \end{align*}
  \]
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$\bullet$ $u$ is the total field in $D$

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\[ \begin{align*}
\text{div} (A \nabla u) + k^2 nu &= 0 \quad \text{in } D \\
\Delta w + k^2 w &= 0 \quad \text{in } D \\
(u - w) &= 0 \quad \text{on } \partial D \\
\nu \cdot A \nabla u - \nu \cdot \nabla w &= 0 \quad \text{on } \partial D.
\end{align*} \]

Transmission conditions on $\partial D$
Presentation of the ITEP

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Definition. Values of $k \in \mathbb{C}$ for which this problem has a non-trivial solution $(u, w)$ are called transmission eigenvalues.

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**Transmission conditions on $\partial D$**

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\text{div} (A\nabla u) + k^2 n u &= 0 \quad \text{in } D \\
\Delta w + k^2 w &= 0 \quad \text{in } D \\
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Transmission conditions on $\partial D$

**Definition.** Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution $(u, w)$ are called transmission eigenvalues.

- The goal in this talk is to prove that the set of transmission eigenvalues is at most discrete.
Variational formulation for the ITEP

**k** is a *transmission eigenvalue* if and only if there exists \((u, w) \in X \setminus \{0\}\) such that, for all \((u', w') \in X\),

\[
\int_D A \nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' = k^2 \int_D (nuu' - w w'),
\]

with \(X = \{(u, w) \in H^1(D) \times H^1(D) | u - w \in H^1_0(D)\}\).
Variational formulation for the ITEP

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- This is a non standard eigenvalue problem.
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\int_D A \nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' = k^2 \int_D (nuu' - w w'),
\]

not coercive on \(X\)

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\int_D A \nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' = k^2 \int_D (nuu' - w\overline{w'}),
\]

not coercive on $X$ \hspace{1cm} not an inner product on $X$

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- This is a non standard eigenvalue problem.

- This problem has been widely studied since 1986-1988 (Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Sylvester, Païvärinta, Rynne, Sleeman...).
Variational formulation for the ITEP

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- In this talk, we want to highlight an

💡 Idea 1: Analogy with another non standard transmission problem ...
Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain \( \Omega \) of \( \mathbb{R}^2 \):

\[
\text{div} \left( \mu^{-1} \nabla v \right) + k^2 \epsilon v = 0 \quad \text{in} \quad \Omega.
\]

\( k \) is a transmission eigenvalue if and only if there exists \( v \in H^1_0(\Omega) \) such that, for all \( v' \in H^1_0(\Omega) \),

\[
\int_{\Omega_1} \mu^{-1} \nabla v \cdot \nabla v' - \int_{\Omega_2} |\mu_2|^{-1} \nabla v \cdot \nabla v' = k^2 (\int_{\Omega_1} \epsilon_1 v v' - \int_{\Omega_2} |\epsilon_2| v v').
\]
Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain $\Omega$ of $\mathbb{R}^2$:

$\varepsilon_1 := \varepsilon|_{\Omega_1} > 0$, $\mu_1 := \mu|_{\Omega_1} > 0$ $\rightarrow$ \(\Omega_1\) Dielectric $\rightarrow \nabla$ $\Omega_2$ Metamaterial $\leftarrow$ $\varepsilon_2 := \varepsilon|_{\Omega_2} < 0$, $\mu_2 := \mu|_{\Omega_2} < 0$
Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain $\Omega$ of $\mathbb{R}^2$:

\[
\begin{align*}
\varepsilon_1 &:= \varepsilon|_{\Omega_1} > 0 \\
\mu_1 &:= \mu|_{\Omega_1} > 0
\end{align*}
\]

\[\rightarrow\]

\[
\begin{align*}
\Omega_1 &\quad \text{Dielectric} \\
\Sigma &\quad \text{Metamaterial} \\
\Omega_2 &\quad \nu
\end{align*}
\]

\[
\begin{align*}
\varepsilon_2 &:= \varepsilon|_{\Omega_2} < 0 \\
\mu_2 &:= \mu|_{\Omega_2} < 0
\end{align*}
\]

- Eigenvalue problem for $E_z$ in 2D:

Find $v \in H^1_0(\Omega)$ such that:

\[
\text{div}(\mu^{-1} \nabla v) + k^2 \varepsilon v = 0 \quad \text{in} \ \Omega.
\]
Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain $\Omega$ of $\mathbb{R}^2$:

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Dielectric $\Omega_1$ $\Sigma$ Metamaterial $\Omega_2$

- Eigenvalue problem for $E_z$ in 2D:

\[
\text{Find } v \in H^1_0(\Omega) \text{ such that:} \\
\text{div}(\mu^{-1}\nabla v) + k^2 \varepsilon v = 0 \quad \text{in } \Omega.
\]

- $k$ is a transmission eigenvalue if and only if there exists $v \in H^1_0(\Omega) \setminus \{0\}$ such that, for all $v' \in H^1_0(\Omega)$,

\[
\int_{\Omega_1} \mu_1^{-1} \nabla v \cdot \nabla v' \, \text{d}x - \int_{\Omega_2} |\mu_2|^{-1} \nabla v \cdot \nabla v' \, \text{d}x = k^2 \left( \int_{\Omega_1} \varepsilon_1 vv' \, \text{d}x - \int_{\Omega_2} |\varepsilon_2| vv' \, \text{d}x \right).
\]
Equivalence DMTEP/ITEP

DMTEP in the domain $\Omega$:

$$\begin{align*}
e_1 &= n \\
\mu_1 &= A
\end{align*}$$

The interface $\Sigma$ in the DMTEP plays the role of the boundary $\partial D$ in the ITEP.

$$\begin{align*}
e_2 &= -1 \\
\mu_2 &= -1
\end{align*}$$
Equivalence DMTEP/ITEP

DMTEP in the domain $\Omega$:

Transmission conditions on $\Sigma$

$\varepsilon_1 = n$
$\mu_1 = A$
$\varepsilon_2 = -1$
$\mu_2 = -1$

$\Omega_1$
$\Omega_2$

Symmetry with respect to the interface $\Sigma$

The interface $\Sigma$ in the DMTEP plays the role of the boundary $\partial D$ in the ITEP.
Equivalence DMTEP/ITEP

- **DMTEP in the domain** $\Omega$: 

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DMTEP in the domain $\Omega$:

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Transmission conditions on $\Sigma$

Symmetry with respect to the interface $\Sigma$

We obtain a problem analogous to the ITEP in $\Omega_1$:
Equivalence DMTEP/ITEP

- **DMTEP** in the domain $\Omega$:

  $\varepsilon_1 = n$
  $\mu_1 = A$

  $\Omega_1$ Dielectric
  $\Sigma$
  $\nu$
  $\Omega_2$ Metamaterial

  Transmission conditions on $\Sigma$

  Symmetry with respect to the interface $\Sigma$

- We obtain a problem analogous to the **ITEP** in $\Omega_1$:

  Transmission conditions on $\Sigma$

- The interface $\Sigma$ in the DMTEP plays the role of the boundary $\partial D$ in the ITEP.
Outline of the talk: three steps

1. An analogy between two transmission problems

2. The T-coercivity method for the Dielectric/Metamaterial Transmission Problem

3. The T-coercivity method for the Interior Transmission Problem
1. An analogy between two transmission problems

2. The $T$-coercivity method for the Dielectric/Metamaterial Transmission Problem

3. The $T$-coercivity method for the Interior Transmission Problem
Study of the DMTP

- Problem for $E_z$ in a symmetric 2D domain:

Mathematical formulation:

Find $v \in H^1_0(\Omega)$ such that:

\[ \int_{\Omega} \mu^{-1} \nabla v \cdot \nabla v' = \langle f, v' \rangle_{\Omega} \quad \forall v' \in H^1_0(\Omega). \]

Definition. We will say that the problem $(PV)$ is well-posed if the operator $\text{div} (\mu^{-1} \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

The form $a$ is not coercive. For $\mu_2 = -\mu_1$, we can build a kernel of infinite dimension to $(PV)$.

Idea 2: Use the $T$-coercivity approach to deal with problem $(PV)$.
Study of the DMTP

- Problem for $E_z$ in a symmetric 2D domain:

$$\Omega_1 \quad \Sigma \quad \Omega_2$$

- We focus on the principal part:

$$\mathcal{P}_V$$

Find $v \in H^1_0(\Omega)$ such that:

$$\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H^1_0(\Omega).$$

Definition. We will say that the problem $(\mathcal{P}_V)$ is well-posed if the operator $\text{div}(\mu^{-1} \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

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For $\mu_2 = -\mu_1$, we can build a kernel of infinite dimension to $(\mathcal{P}_V)$.

Idea 2: Use the $T$-coercivity approach to deal with problem $(\mathcal{P}_V)$. 
Study of the DMTP

- Problem for $E_z$ in a symmetric 2D domain:

\[
\begin{align*}
\mu_1 > 0 & \quad \Omega_1 \quad \Sigma \quad \Omega_2 \quad \mu_2 < 0 \\
(\mu_1 \text{ and } \mu_2 \text{ constant to simplify})
\end{align*}
\]

- We focus on the principal part:

\[
(P_V) \quad \text{Find } v \in H^1_0(\Omega) \text{ such that:}
\]

\[
\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H^1_0(\Omega).
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Study of the DMTP

- Problem for $E_z$ in a symmetric 2D domain:

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- We focus on the principal part:

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\begin{align*}
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\text{Definition.} \quad \text{We will say that the problem } (\mathcal{P}_V) \text{ is well-posed if the operator} \\
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Study of the DMTP

Problem for $E_z$ in a symmetric 2D domain:

$\mu_1 > 0 \rightarrow \Omega_1 \rightarrow \Sigma \rightarrow \Omega_2 \leftarrow \mu_2 < 0$

($\mu_1$ and $\mu_2$ constant to simplify)

We focus on the principal part:

Find $v \in H^1_0(\Omega)$ such that:

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**Definition.** We will say that the problem $(\mathcal{P}_V)$ is well-posed if the operator $\text{div} (\mu^{-1} \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

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Study of the DMTP

Problem for $E_z$ in a symmetric 2D domain:

$\mu_1 > 0 \quad \Omega_1 \quad \Sigma \quad \Omega_2 \quad \mu_2 < 0$

($\mu_1$ and $\mu_2$ constant to simplify)

We focus on the principal part:

\[
\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H^1_0(\Omega).
\]

Definition. We will say that the problem $(PV)$ is well-posed if the operator $\text{div} (\mu^{-1} \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

- The form $a$ is not coercive.
- For $\mu_2 = -\mu_1$, we can build a kernel of infinite dimension to $(PV)$. 
Study of the DMTP

Problem for $E_z$ in a symmetric 2D domain:

$$\mu_1 > 0 \quad \sum \quad \Omega_1 \quad \Omega_2 \quad \mu_2 < 0$$

($\mu_1$ and $\mu_2$ constant to simplify)

We focus on the principal part:

$$\begin{align*}
\text{(PV)} & \quad \text{Find } v \in H^1_0(\Omega) \text{ such that:} \\
\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla v' &= \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H^1_0(\Omega).
\end{align*}$$

Definition. We will say that the problem (PV) is well-posed if the operator $\text{div} (\mu^{-1} \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

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Idea 2: Use the T-coercivity approach to deal with problem (PV).
Idea of the $T$-coercivity 1/2

Let $T$ be an isomorphism of $H^1_0(\Omega)$.

$$(PV) \left| \begin{array}{c}
\text{Find } v \in H^1_0(\Omega) \text{ such that:} \\
a(v, v') = l(v'), \forall v' \in H^1_0(\Omega).
\end{array} \right.$$
Idea of the $T$-coercivity 1/2

Let $T$ be an isomorphism of $\mathbb{H}^1_0(\Omega)$.

\[(\mathcal{P}_V) \iff (\mathcal{P}^T_V)\quad \text{Find } v \in \mathbb{H}^1_0(\Omega) \text{ such that:}\]
\[a(v, Tv') = l(Tv'), \forall v' \in \mathbb{H}^1_0(\Omega).\]
Idea of the T-coercivity 1/2

Let $T$ be an isomorphism of $H^1_0(\Omega)$.

$$(PV) \iff (PT_V) \bigg| \quad \text{Find } v \in H^1_0(\Omega) \text{ such that:}$$

$$a(v, Tv') = l(Tv'), \forall v' \in H^1_0(\Omega).$$

Goal: Find $T$ such that $a$ is $T$-coercive:

$$\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla (Tv) \geq C \|v\|_{H^1_0(\Omega)}^2.$$ 

In this case, Lax-Milgram $\Rightarrow (PT_V)$ (and so $(PV)$) is well-posed.
Idea of the $T$-coercivity 1/2

Let $T$ be an isomorphism of $H^1_0(\Omega)$.

Find $v \in H^1_0(\Omega)$ such that:

$$a(v, Tv') = l(Tv'), \ \forall v' \in H^1_0(\Omega).$$

Goal: Find $T$ such that $a$ is $T$-coercive:

$$\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla (Tv) \geq C \|v\|_{H^1_0(\Omega)}^2.$$

In this case, Lax-Milgram $\Rightarrow$ $(\mathcal{P}_V^T)$ (and so $(\mathcal{P}_V)$) is well-posed.

1. Define $T_1 v = v_1$ in $\Omega_1$
   $$-v_2 + \ldots$$ in $\Omega_2$
Idea of the T-coercivity 1/2

Let $T$ be an isomorphism of $H^1_0(\Omega)$.

$$(P_V) \iff (P^T_V) \bigg| \begin{align*} & \text{Find } v \in H^1_0(\Omega) \text{ such that:} \\ & a(v, T^t v') = l(T^t v'), \forall v' \in H^1_0(\Omega). \end{align*}$$

Goal: Find $T$ such that $a$ is $T$-coercive:
$$\int_\Omega \mu^{-1} \nabla v \cdot \nabla (Tv) \geq C \|v\|_{H^1_0(\Omega)}^2.$$ In this case, Lax-Milgram $\Rightarrow (P^T_V)$ (and so $(P_V)$ is well-posed.

1. Define $T_1 v = \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2S^\Sigma v_1 & \text{in } \Omega_2 \end{cases}$, where $S^\Sigma$ is the symmetry.
Idea of the $\text{T}$-coercivity 1/2

Let $T$ be an isomorphism of $H^1_0(\Omega)$.

$$(P_V) \iff (P^T_V) \quad \text{Find } v \in H^1_0(\Omega) \text{ such that: } a(v, Tv') = l(Tv'), \forall v' \in H^1_0(\Omega).$$

Goal: Find $T$ such that $a$ is $T$-coercive:

$$\int_{\Omega} \mu^{-1} \nabla v \cdot \nabla (Tv) \geq C \|v\|^2_{H^1_0(\Omega)}.$$

In this case, Lax-Milgram $\Rightarrow (P^T_V)$ (and so $(P_V)$) is well-posed.

1. Define $T_1 v = \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2S\Sigma v_1 & \text{in } \Omega_2 \end{cases}$, where $S\Sigma$ is the symmetry.

2. $T_1 \circ T_1 = Id$ so $T_1$ is an isomorphism of $H^1_0(\Omega)$.
Idea of the $T$-coercivity 2/2

One has $a(v, T_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma v_1)$
3 One has \( a(v, T_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma v_1) \)

Young’s inequality + \( \|S\Sigma\| = 1 \Rightarrow a \) is T-coercive when \( |\mu_2| > \mu_1 \).
Idea of the $T$-coercivity 2/2

3. One has $a(v, T_1v) = \int_{\Omega} |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_{\Sigma} v_1)$

Young’s inequality $+ \| S_{\Sigma} \| = 1 \Rightarrow a$ is T-coercive when $|\mu_2| > \mu_1$.

4. With $T_2v = \begin{cases} v_1 - 2S_{\Sigma}v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases}$, $a$ is T-coercive when $\mu_1 > |\mu_2|$. 

Conclusion: The operator $\text{div}(\mu/\mu - 1 \nabla \cdot)$ is an isomorphism from $H_{10}(\Omega)$ to $H_{-1}(\Omega)$ if and only if the contrast $\kappa \mu = \mu_1/\mu_2$ satisfies $\kappa \mu \neq -1$.

By a localization process, when $\mu_1$ and $\mu_2$ are not constant, we can prove that $\text{div}(\mu/\mu - 1 \nabla \cdot)$ is of Fredholm type when $\inf_{\Omega_1 \cap \Omega} \mu_1 / \inf_{\Omega_2 \cap \Omega} \mu_2 < -1$ or $\sup_{\Omega_1 \cap \Omega} \mu_1 / \sup_{\Omega_2 \cap \Omega} \mu_2 > -1$ where $\Omega$ is a neighborhood of $\Sigma$.

This technique also allows to deal with non-symmetric configurations.
Idea of the T-coercivity 2/2

3. One has \( a(v, T_1v) = \int_\Omega |\mu|^{-1}|\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma v_1) \)

Young’s inequality + \( \|S_\Sigma\| = 1 \Rightarrow \) a is T-coercive when \( |\mu_2| > \mu_1 \).

4. With \( T_2v = \begin{pmatrix} v_1 - 2S_\Sigma v_2 \\ -v_2 \end{pmatrix} \) in \( \Omega_1 \) \( \cap \) in \( \Omega_2 \), a is T-coercive when \( \mu_1 > |\mu_2| \).

5. Conclusion:

**Theorem.** The operator \( \text{div} (\mu^{-1} \nabla \cdot) \) is an isomorphism from \( H^1_0(\Omega) \) to \( H^{-1}(\Omega) \) if and only if the contrast \( \kappa_\mu = \mu_1/\mu_2 \) satisfies \( \kappa_\mu \neq -1 \).
Idea of the T-coercivity 2/2

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- By a localization process, when $\mu_1$ and $\mu_2$ are not constant, we can prove that $\text{div} (\mu^{-1} \nabla \cdot)$ is of Fredholm type when

  $$\inf_{\Omega_1 \cap \nabla} \mu_1 / \inf_{\Omega_2 \cap \nabla} \mu_2 < -1 \quad \text{or} \quad \sup_{\Omega_1 \cap \nabla} \mu_1 / \sup_{\Omega_2 \cap \nabla} \mu_2 > -1$$

  where $\nabla$ is a neighbourhood of $\Sigma$. 
Idea of the T-coercivity 2/2

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5 Conclusion:

**Theorem.** The operator $\text{div}(\mu^{-1} \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ if and only if the contrast $\kappa_\mu = \mu_1/\mu_2$ satisfies $\kappa_\mu \neq -1$.

- By a localization process, when $\mu_1$ and $\mu_2$ are not constant, we can prove that $\text{div}(\mu^{-1} \nabla \cdot)$ is of Fredholm type when
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where $\mathcal{V}$ is a neighbourhood of $\Sigma$.

- This technique also allows to deal with non symmetric configurations.
1. An analogy between two transmission problems

2. The T-coercivity method for the Dielectric/Metamaterial Transmission Problem

3. The T-coercivity method for the Interior Transmission Problem
Define on $X \times X$ the sesquilinear form

$$a((u, w), (u', w')) = \int_{\Omega} A \nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' - k^2 (nuu' - w\overline{w'}),$$

with $X = \{(u, w) \in H^1(\Omega) \times H^1(\Omega) \mid u - w \in H^1_0(\Omega)\}$. 
Study of the ITEP

Define on $X \times X$ the sesquilinear form

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with $X = \{(u, w) \in H^1(\Omega) \times H^1(\Omega) | u - w \in H^1_0(\Omega)\}$.

Introduce the isomorphism $T(u, w) = (u - 2w, w)$. 
Define on $X \times X$ the sesquilinear form

$$a((u, w), (u', w')) = \int_\Omega A \nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' - k^2 (nuu' - w w'),$$

with $X = \{(u, w) \in H^1(\Omega) \times H^1(\Omega) | u - w \in H^1_0(\Omega)\}$.

Introduce the isomorphism $T(u, w) = (u - 2w, w)$.

For $k \in \mathbb{R} \setminus \{0\}$, $A > Id$ and $n > 1$, one finds

$$\Re a((u, w), T(u, w)) \geq C (\|u\|_{H^1(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2), \quad \forall (u, w) \in X.$$
Study of the ITEP

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$$a((u, w), (u', w')) = \int_{\Omega} A \nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' - k^2 (nuu' - w w'),$$

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Using the analytic Fredholm theorem, one deduces the

PROPOSITION. Suppose that $A > Id$ and $n > 1$. Then the set of transmission eigenvalues is discrete and countable.
Study of the ITEP

Define on \( X \times X \) the sesquilinear form

\[
a((u, w), (u', w')) = \int_\Omega A\nabla u \cdot \nabla u' - \nabla w \cdot \nabla w' - k^2 (nuu' \circ w w'),
\]

with \( X = \{(u, w) \in H^1(\Omega) \times H^1(\Omega) \mid u - w \in H^1_0(\Omega)\} \).

Introduce the isomorphism \( T(u, w) = (u - 2w, w) \).

For \( k \in \mathbb{R} \setminus \{0\}, A > Id \) and \( n > 1 \), one finds

\[
\Re a((u, w), T(u, w)) \geq C (\|u\|_{H^1(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2), \quad \forall (u, w) \in X.
\]

Using the analytic Fredholm theorem, one deduces the

**Proposition.** Suppose that \( A > Id \) and \( n > 1 \). Then the set of transmission eigenvalues is discrete and countable.

This result can be extended to situations where \( A - Id \) and \( n - 1 \) change sign in \( \Omega \) working with \( T(u, w) = (u - 2\chi w, w) \).
ITEP when $A = Id$

- When $A = Id$, the ITP is not of Fredholm type in $X$ likewise the DMTP is not of Fredholm type in $H^1_0(\Omega)$ when $\mu_1 = -\mu_2$. 
ITEP when $A = Id$

- We change the functional framework working on the difference $v := u - w \in H^2_0(D)$: $k$ is a transmission eigenvalue if and only if there exists $v \in H^2_0(D) \setminus \{0\}$ such that, for all $v' \in H^2_0(D)$,

$$
\int_D \frac{1}{1 - n} (\Delta v + k^2 nv)(\Delta v' + k^2 v') = 0.
$$

- We focus on the principal part:

\[
(\mathcal{F}_V) \quad \begin{array}{c}
\text{Find } v \in H^2_0(D) \text{ such that:} \\
\int_D \frac{1}{1 - n} \Delta v \Delta v' = \langle f, v' \rangle_D, \quad \forall v' \in H^2_0(D).
\end{array}
\]

\[
\begin{array}{c}
\underbrace{a(v,v')}_{} \quad \underbrace{l(v')}_{}
\end{array}
\]
ITEP when $A = Id$

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\end{aligned}
\] 

\begin{itemize}
    \item Transmission problem with a possible sign-changing coefficient
    \item Idea 3: This transmission problem is very different from DMTP.
    \item Theorem. The problem $(F_V)$ is well-posed in the Fredholm sense as soon as $1 - n$ does not change sign in a neighborhood of $\partial D$.
    \item Proof: $T$-coercivity or see J. Sylvester's work for a more precise study.
\end{itemize}
ITEP when $A = Id$

- We change the functional framework working on the difference $v := u - w \in H^2_0(D)$: $k$ is a transmission eigenvalue if and only if there exists $v \in H^2_0(D) \setminus \{0\}$ such that, for all $v' \in H^2_0(D)$,

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Idea 3: This transmission problem is very different from DMTP.
ITEP when $A = \text{Id}$

- We change the functional framework working on the difference $v := u - w \in H^2_0(D)$: $k$ is a transmission eigenvalue if and only if there exists $v \in H^2_0(D) \setminus \{0\}$ such that, for all $v' \in H^2_0(D)$,

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- We focus on the principal part:

$$
(F_V) \quad \begin{cases}
\text{Find } v \in H^2_0(D) \text{ such that:} \\
\int_D \frac{1}{1 - n} \Delta v \Delta v' = \langle f, v' \rangle_D, \quad \forall v' \in H^2_0(D).
\end{cases}
$$

Idea 3: This transmission problem is very different from DMTP.

Theorem. The problem $(F_V)$ is well-posed in the Fredholm sense as soon as $1 - n$ does not change sign in a neighbourhood of $\partial D$.

- Proof: $T$-coercivity or see J. Sylvester’s work for a more precise study.
Generalizations

✓ T-coercivity approach can be used for non-constant coefficients ($L^\infty$) and other problems (Maxwell’s equations, elasticity, ...).

✓ It allows to justify the convergence of standard finite element methods.

♠ What happens when $A - Id$ change sign in a neighbourhood of the boundary?

♣ For the equivalent DMTP, strong singularities appear at the interface and $H^1$ is no longer the appropriate functional framework. We observe a black hole phenomenon (joint work with X. Claeys).

♠ We are not able to use the T-coercivity technique to prove existence of transmission eigenvalues.

⇒ T-coercivity gives positivity but operators are no longer symmetric.
Thank you for your attention.


