Transmission eigenvalue problems with sign-changing coefficients

PICOF 2012

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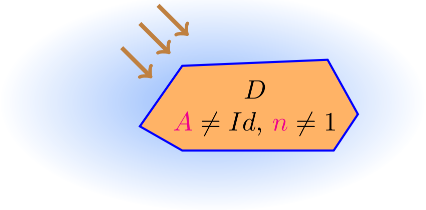
Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.



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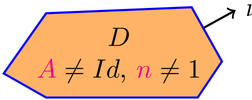

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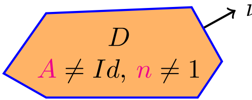

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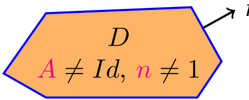

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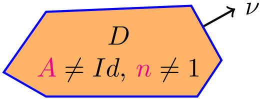

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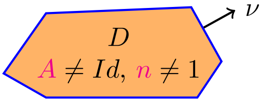

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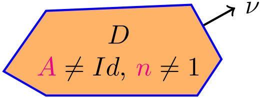

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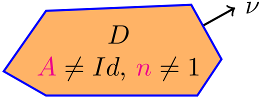

TRANSMISSION CONDITIONS ON ∂D

> **DEFINITION.** Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u,w) are called transmission eigenvalues.

- The goal in this talk is to prove that the set of transmission eigenvalues is at most discrete.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X} \backslash \{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_D A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \;=\; k^2 \int_D (n u \overline{u'} - w \overline{w'}),$$

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

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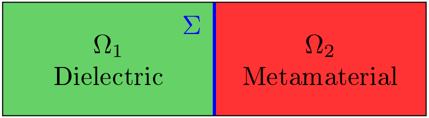
- In this talk, we want to highlight an

Idea 1: Analogy with another non standard transmission problem ...

Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:



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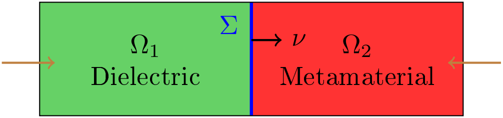

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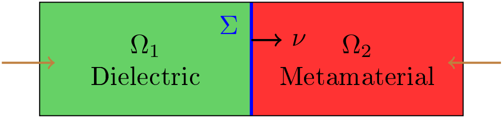

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- Eigenvalue problem for E_z in 2D:

Find $v \in \mathrm{H}_0^1(\Omega)$ such that:
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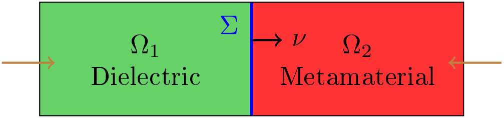

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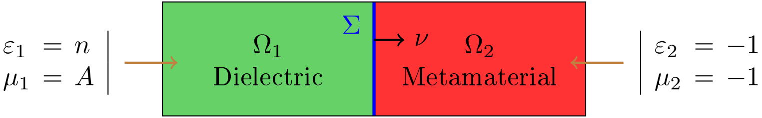
$$\int_{\Omega_1} \mu_1^{-1} \nabla v \cdot \overline{\nabla v'} - \int_{\Omega_2} |\mu_2|^{-1} \nabla v \cdot \overline{\nabla v'} = k^2 \left(\int_{\Omega_1} \varepsilon_1 v \overline{v'} - \int_{\Omega_2} |\varepsilon_2| v \overline{v'} \right).$$

Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



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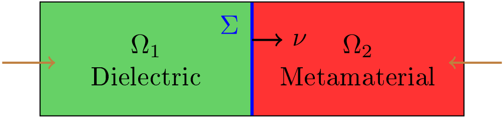
TRANSMISSION CONDITIONS ON Σ

$$\left.\begin{aligned}\varepsilon_1 &= n\\ \mu_1 &= A\end{aligned}\right|$$



$$\left|\begin{aligned}\varepsilon_2 &= -1\\ \mu_2 &= -1\end{aligned}\right.$$

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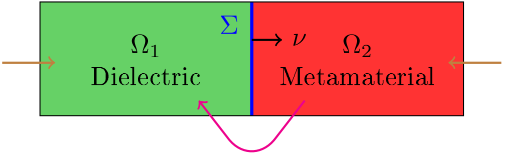

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Symmetry with respect to the interface Σ

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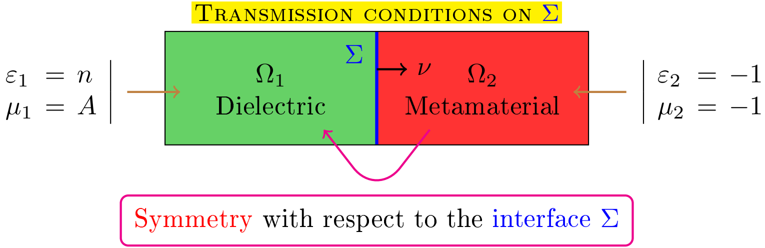

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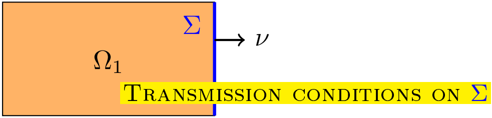

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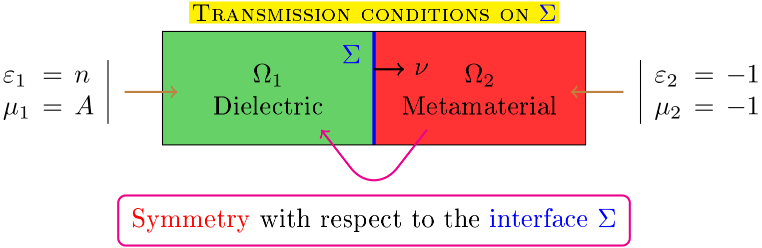

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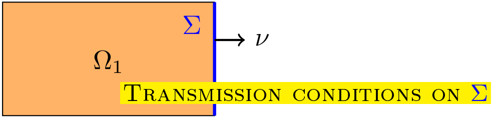


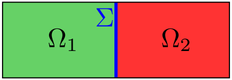
- The interface Σ in the DMTEP plays the role of the boundary ∂D in the ITEP.

Outline of the talk: three steps

1. An analogy between two transmission problems
2. The $\mathtt{T}$-coercivity method for the Dielectric/Metamaterial Transmission Problem
3. The $\mathtt{T}$-coercivity method for the Interior Transmission Problem

Study of the DMTP

- Problem for E_z in a symmetric 2D domain:



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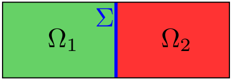

- We focus on the principal part:

$$(\mathscr{P}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1} \, \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

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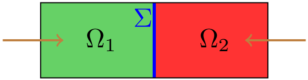
$\mu_1 > 0$ |  | $\mu_2 < 0$

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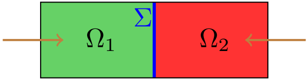
$\mu_1 > 0$ |  | $\mu_2 < 0$

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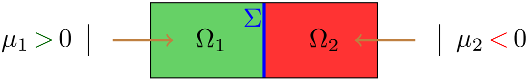

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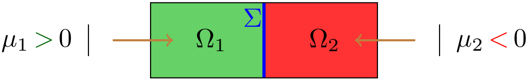

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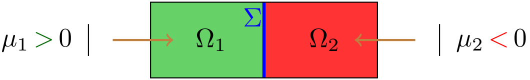

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Idea 2: Use the T-coercivity approach to deal with problem $(\mathscr{P}_V)$.

Idea of the T-coercivity 1/2

Let T be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

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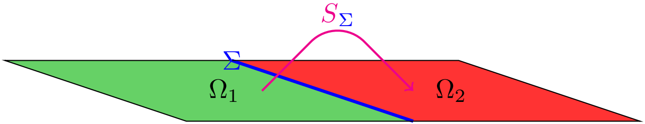

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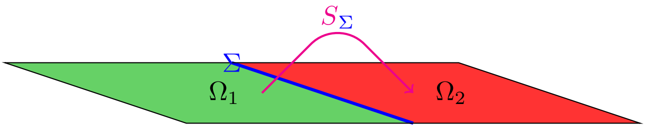

2. $\mathtt{T}_1 \circ \mathtt{T}_1 = Id$ so $\mathtt{T}_1$ is an isomorphism of $\mathrm{H}^1_0(\Omega)$

Idea of the $\mathtt{T}$-coercivity 2/2

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Theorem. The operator $\mathrm{div}\,(\mu^{-1} \nabla \cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$ if and only if the contrast $\kappa_\mu = \mu_1/\mu_2$ satisfies $\kappa_\mu \neq -1$.

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- This technique also allows to deal with non symmetric configurations.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

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Proposition. Suppose that $A > Id$ and $n > 1$. Then the set of transmission eigenvalues is discrete and countable.

- This result can be extended to situations where $A - Id$ and $n - 1$ change sign in Ω working with $\mathtt{T}(u,w) = (u - 2\chi w, w)$.

ITEP when $A = Id$

- When $A = Id$, the ITP is not of Fredholm type in X likewise the DMTP is not of Fredholm type in $H_0^1(\Omega)$ when $\mu_1 = -\mu_2$.

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- We change the functional framework working on the difference $v := u - w \in \mathrm{H}_0^2(D)$: k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}_0^2(D) \backslash \{0\}$ such that, for all $v' \in \mathrm{H}_0^2(D)$,

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- We focus on the principal part:

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Transmission problem with a possible sign-changing coefficient

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TRANSMISSION PROBLEM WITH A POSSIBLE SIGN-CHANGING COEFFICIENT

Idea 3: This transmission problem is very different from DMTP.

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THEOREM. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as $1 - n$ does not change sign in a neighbourhood of ∂D.

- Proof: T-coercivity or see J. Sylvester's work for a more precise study.

Generalizations

✔ T-coercivity approach can be used for non-constant coefficients (L^∞) and other problems (Maxwell's equations, elasticity, ...).

✔ It allows to justify the convergence of standard finite element methods.

♠ What happens when $A - Id$ change sign in a neighbourhood of the boundary?

☛ For the equivalent DMTP, strong singularities appear at the interface and H^1 is no longer the appropriate functional framework. We observe a black hole phenomenon (joint work with X. Claeys).

♠ We are not able to use the T-coercivity technique to prove existence of transmission eigenvalues.
⇒ T-coercivity gives positivity but operators are no longer symmetric.

Thank you for your attention.

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