Revealing guided modes in a plasmonic waveguide using PMLs at corners

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KOZWaves 2014, 18\textsuperscript{th} February
Guided modes in a plasmonic waveguide (1)

Previously in Anne-Sophie Bonnet-Ben Dhia’s talk

2D Scattering problem
Artificial Boundary Condition
Black-hole waves captured by Perfectly Matched Layers

Dielectric $\Omega_1$ $\epsilon_1 > 0$

Metal $\Omega_2$ $\epsilon_2 < 0$

Metal permittivity modelized by the dissipationless Drude’s model

$$\epsilon(\omega) = \epsilon_\infty \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$
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In this presentation

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Bounded section

Looking for propagative waves along $z$

$$u(x, y, z, t) = \tilde{u}(x, y)e^{i(\beta z - \omega t)} \quad \beta, \omega \in \mathbb{R}$$
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2D Eigenproblem
Looking for the guided modes
Looking for the waves propagating along $z$

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We consider the time-harmonic scalar model of Maxwell’s equations (simplified model)
Guided modes in a plasmonic waveguide

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$\text{div} \left( \frac{1}{\varepsilon} \nabla \tilde{u} \right) - \frac{\beta^2}{\varepsilon} \tilde{u} + \omega^2 \mu \tilde{u} = 0 \quad \Omega \quad \mu > 0$

$\tilde{u} = 0 \quad \partial \Omega$
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Two ways to study the problem

- for a chosen frequency $\omega$, find the axial eigenvalues $A(\omega)\tilde{u} = \frac{\beta^2}{\epsilon} \tilde{u}$
Guided modes in a plasmonic waveguide (2)

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Formulation of the eigenvalue problem:

For $\beta \in \mathbb{R}$, Find $\tilde{u} \in V_0 \setminus \{0\}, \omega \in \mathbb{C}$ s.t.

$$-\frac{1}{\mu} \text{div} \left( \frac{1}{\varepsilon} \nabla \tilde{u} \right) + \frac{\beta^2}{\mu \varepsilon} \tilde{u} = \omega^2 \tilde{u} \quad V_0 = \{ u/ \int_{\Omega} |u|^2 + |\nabla u|^2 d\Omega < +\infty, u|_{\partial \Omega} = 0 \}$$
Some numerical experiments (1)

For $\beta \in \mathbb{R}$, Find $\tilde{u} \in V_0 \setminus \{0\}$, $\omega \in \mathbb{C}$ s.t.

$$A(\beta)\tilde{u} = \omega^2 \tilde{u}$$

Numerical illustrations with FE

Parameters $\epsilon_1 = 1 \quad \epsilon_2 = -\frac{10}{7} \quad \beta = 1$
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Coarse mesh
Some numerical experiments (1)

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Coarse mesh

Fine mesh

![Graph showing Im($\omega^2$) and Re($\omega^2$)]
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For a coarse mesh
No convergence due to black-hole waves

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Revealing guided modes in a plasmonic waveguide using Perfectly Matched Layers at the corners, CARVALHO Camille, KOZWaves, February 2014
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Coarse mesh

Fine mesh

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Fine mesh

No FEM convergence but some modes seem stable: guided modes are hidden in this spectrum!
Some numerical experiments (2)

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Numerical illustrations with FEM

Parameters $\epsilon_1 = 1$ $\epsilon_2 = -\frac{10}{7}$ $\beta = 1$

New numerical method involving PMLs: the method is stable, sorts the modes and reveals the guided modes
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$\omega^2 \in \mathbb{R}$

guided/evanescent modes
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New numerical method involving PMLs: the method is stable, sorts the modes and reveals the guided modes.

$\omega^2 \in \mathbb{R}$ guided/evanescent modes

$\omega^2 \in \mathbb{C} \setminus \mathbb{R}$ leaky modes?
* Introduction

* **Properties of the operator**

* Link between leakage and black-hole waves

* Numerical results
First steps: when $\epsilon > 0$

Variational formulation:

$$A(\beta)u = \omega^2 u \iff \int_{\Omega} \frac{1}{\epsilon} \nabla u \cdot \nabla v + \int_{\Omega} \frac{\beta^2}{\epsilon} u \overline{v} = \omega^2 \int_{\Omega} \mu u \overline{v} \quad \forall \, v \in V_0$$
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Good Properties: $A(\beta)$ is self-adjoint and has compact resolvent
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The spectrum of $A(\beta)$ is a sequence of positive real eigenvalues

Good Properties $\Rightarrow (\omega_n(\beta))_{n \in \mathbb{N}}$ with finite multiplicity, tending to $+\infty$

Approximation with Finite Elements converges (no spurious modes)

First steps: when $\epsilon > 0$

Variational formulation:

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Dispersion curve (for the 10 first ev.)

$\epsilon_1 = 1$, $\epsilon_2 = 2$

\[\omega^2 = \frac{\beta^2}{\epsilon_2 \mu}\]
First steps: when $\epsilon > 0$

Variational formulation:

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Do $A(\beta)$'s Good Properties remain when $\epsilon$ is sign-changing?

What impact have the corners?


Revealing guided modes in a plasmonic waveguide using Perfectly Matched Layers at the corners, CARVALHO Camille, KOZWaves, February 2014
Properties of $A(\beta)$ when $\epsilon$ changes sign

Variational formulation:

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Does $A(\beta)$ have Good Properties?
YES under some conditions on $\varepsilon$ and the geometry
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Does $A(\beta)$ have Good Properties?

YES under some conditions on $\epsilon$ and the geometry

In our case: YES iff $\kappa_\epsilon = \frac{\epsilon_1}{\epsilon_2} \not\in I_c \subset \mathbb{R}^-$ Critical interval

(cf. Anne-Sophie Bonnet-Ben Dhia’s talk)

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*T-coercivity for scalar interface problems between dielectrics and metamaterials,*
Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, Patrick Ciarlet, M2AN, 2012
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Outside $I_c$

Inside $I_c$

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<td>Real discrete spectrum tending to $\pm \infty$</td>
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Inside $I_c$

* * *

$T$-coercivity for scalar interface problems between dielectrics and metamaterials,
Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, Patrick Ciarlet, M2AN, 2012
Properties of $A(\beta)$ when $\epsilon$ changes sign

For a metal:

$$A(\beta) u = \int_\Omega \mu \tilde{u} \tilde{v} \quad \forall \tilde{v} \in V_0$$

Do $A(\beta)$ have Good Properties?

YES under some conditions on $\epsilon$ and the geometry

In our case

$T$-coercivity for scalar interface problems between dielectrics and metamaterials,
Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, Patrick Ciarlet, M2AN, 2012
For a metamaterial: is sign changing?

Variational formulation:

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(cf. Anne-Sophie Bonnet-Ben Dhia’s talk)

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Outside $I_c$

$A(\beta)$ has the Good Properties

Real discrete spectrum tending to $\pm \infty$

FEM converges (under some condition on the mesh)

Inside $I_c$

$A(\beta)$ has the Good Properties

No FEM convergence due to black-hole waves

T-coercivity for scalar interface problems between dielectrics and metamaterials, Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, Patrick Ciarlet, M2AN, 2012
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**Outside** $I_c$

(Good case)

$A(\beta)$ has the Good Properties

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Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, Patrick Ciarlet, M2AN, 2012
Introduction

Properties of the operator

Link between leakage and black-hole waves

Numerical results
Energy conservation with one corner (1)

In the **Bad case**, $V_0$ is **not** the adapted mathematical framework to find the modes. **Need to take into account the black-hole wave** $s^+ \notin V_0$

$$u = u_{\text{reg}} + c^+ s^+$$
In the Bad case, $V_0$ is not the adapted mathematical framework to find the modes. Need to take into account the black-hole wave $s^+ \not\in V_0$.

\[ u = u_{\text{reg}} + c^+ s^+ \]

\[-\text{div} \left( \frac{1}{\epsilon} \nabla u \right) + \frac{\beta^2}{\epsilon} u = \omega^2 \mu u \quad \Omega/B_{\rho} \]

\[ u = 0 \quad \partial \Omega \]

\[ \beta \in \mathbb{R} \]
Energy conservation with one corner (1)

In the **Bad case**, $V_0$ is not the adapted mathematical framework to find the modes. Need to take into account the black-hole wave $s^+ \notin V_0$

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$$u = 0 \quad \partial \Omega$$

By the energy technique, and taking the imaginary part

$$-|c^+|^2 \kappa \int_{\theta=0}^{2\pi} \frac{1}{\varepsilon} |\phi(\theta)|^2 \, d\theta \sim \Im(\omega^2) \int_\Omega \mu |u|^2 \, d\Omega$$
Energy conservation with one corner (1)

In the **Bad case**, $V_0$ is not the adapted mathematical framework to find the modes. Need to take into account the black-hole wave $s^+ \not\in V_0$

$$u = u_{\text{reg}} + c^+ s^+$$

$$-\text{div} \left( \frac{1}{\epsilon} \nabla u \right) + \frac{\beta^2}{\epsilon} u = \omega^2 \mu u \quad \Omega/B_\rho$$

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$$\geq 0 \quad \geq 0$$
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$$-|c^+|^2 \kappa \int_{\theta=0}^{2\pi} \frac{1}{\epsilon} |\phi(\theta)|^2 d\theta \sim \Re(\omega^2) \int_\Omega \mu |u|^2 d\Omega$$

$$\omega^2 \in \mathbb{R} \implies c^+ = 0 \text{ then } u = u_{\text{reg}} \in V_0 \text{ The real eigenvalues do not excite } s^+$$
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In the **Bad case**, \( V_0 \) is not the adapted mathematical framework to find the modes. Need to take into account the black-hole wave \( s^+ \not\in V_0 \)

\[
\begin{align*}
    u &= u_{\text{reg}} + c^+ s^+ \\
    -\text{div} \left( \frac{1}{\varepsilon} \nabla u \right) + \frac{\beta^2}{\varepsilon} u &= \omega^2 \mu u & \Omega / B_\rho \\
    u &= 0 & \partial \Omega \\
    \beta &\in \mathbb{R}
\end{align*}
\]

By the energy technique, and taking the imaginary part

\[
-|c^+|^2 \kappa \int_{\theta=0}^{2\pi} \frac{1}{\varepsilon} |\phi(\theta)|^2 \, d\theta \sim \mathbb{R}(\omega^2) \int_{\Omega} \mu |u|^2 \, d\Omega
\]

\( \omega^2 \in \mathbb{R} \implies c^+ = 0 \) then \( u = u_{\text{reg}} \in V_0 \) The **real** eigenvalues do not excite \( s^+ \)

\( \omega^2 \in \mathbb{C} \setminus \mathbb{R} \implies \mathbb{R}(\omega^2) \leq 0 \) The **complex** eigenvalues excite the black-hole wave and are located in the same complex half-plane
Energy conservation with one corner (1)

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\[
u(x, y)e^{i(\beta z - \omega t)}
\]

The energy is decaying in time because of leakage at the corners (leaky modes appear for open waveguides)
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By the energy technique, and taking the imaginary part

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$u(x, y) e^{i(\beta z - \omega t)}$ The energy is decaying in time because of leakage at the corners (leaky modes appear for open waveguides)

Need to take $s^+$ into account to compute the regular cavity modes \textit{i.e.} the guided/evanescent modes. \textbf{Plus it sorts} all the modes!!
In the **Bad case**, $V_0$ is not the adapted mathematical framework to find the modes. Need to take into account the black-hole wave $s^+ \not\in V_0$

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---

**Need to take $s^+$ into account to compute the regular cavity modes i.e. the guided/evanescent modes. Plus it sorts all the modes !!**

There is an efficient numerical method to capture $s^+$: using PMLs
Outline

- Introduction
- Properties of the operator
- Link between leakage and black-hole waves
- Numerical results
Numerical illustrations

P2 Finite Elements $\epsilon_1 = 1, \epsilon_2 = -7/10, \beta = 1$
Numerical illustrations

P2 Finite Elements $\epsilon_1 = 1$, $\epsilon_2 = -7/10$, $\beta = 1$ Animation in time $e^{-i\omega_j(\beta)t}$
Numerical illustrations

P2 Finite Elements $\epsilon_1 = 1$, $\epsilon_2 = -7/10$, $\beta = 1$  
Animation in time $e^{-i \omega_j(\beta) t}$

$\omega^2 \in \mathbb{R}^+$  
Guided modes

$\Omega_1$  
$\Omega_2$

$x$  
$\partial \Omega$  
y

Revealing guided modes in a plasmonic waveguide using Perfectly Matched Layers at the corners, CARVALHO Camille, KOZWaves, February 2014
Numerical illustrations

P2 Finite Elements  \( \epsilon_1 = 1, \epsilon_2 = -7/10, \beta = 1 \)  Animation in time  \( e^{-i \omega_j(\beta)t} \)

\( \omega^2 \in \mathbb{R}^- \)  
Evanescent modes

\( \omega^2 \in \mathbb{R}^+ \)  
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\( \omega^2 \in \mathbb{R}^- \) Evanescent modes

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P2 Finite Elements $\epsilon_1 = 1$, $\epsilon_2 = -7/10$, $\beta = 1$ Animation in time $e^{-i \omega_j(\beta) t}$

$\omega^2 \in \mathbb{R}^-$ Evanescent modes

$\omega^2 \in \mathbb{R}^+$ Guided modes

$\omega^2 \in \mathbb{C} \setminus \mathbb{R}$ Leaky modes
Numerical illustrations

$\Omega_1$  $\Omega_2$

P2 Finite Elements  $\epsilon_1 = 1$, $\epsilon_2 = -7/10$, $\beta = 1$  Animation in time $e^{-i \omega_j(\beta) t}$

$\omega^2 \in \mathbb{R}^-$

Evanescent modes

$\omega^2 \in \mathbb{R}^+$

Guided modes

The Energy conservation gives us:

$u = u_{reg} + c_{Top} s^{+}_{Top} + c_{Left} s^{+}_{Left} + c_{Right} s^{+}_{Right}$

$\mathcal{E}_{Top} + \mathcal{E}_{Left} + \mathcal{E}_{Right} \sim \Im(\omega^2) \int_{\Omega} \mu |u|^2 d\Omega$

$\omega^2 \in \mathbb{C} \setminus \mathbb{R}$  Leaky modes

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P2 Finite Elements \( \epsilon_1 = 1, \epsilon_2 = -7/10, \beta = 1 \) Animation in time \( e^{-i \omega_j(\beta) t} \)

\( \omega^2 \in \mathbb{R}^- \)
Evanescent modes

\( \omega^2 \in \mathbb{R}^+ \)
Guided modes

The Energy conservation gives us:
\[
u = u_{\text{reg}} + c_{\text{Top}} s_{\text{Top}}^+ + c_{\text{Left}} s_{\text{Left}}^+ + c_{\text{Right}} s_{\text{Right}}^+
\]
\[
\mathcal{E}_{\text{Top}} + \mathcal{E}_{\text{Left}} + \mathcal{E}_{\text{Right}} \sim \Im(\omega^2) \int_{\Omega} \mu |u|^2 d\Omega
\]

\( \mathcal{E}_{\text{Top}} = 0 \)

\( \mathcal{E}_{\text{Left}} = \mathcal{E}_{\text{Right}} = 0 \)

\( \omega^2 \in \mathbb{C} \setminus \mathbb{R} \) Leaky modes

Revealing guided modes in a plasmonic waveguide using Perfectly Matched Layers at the corners, CARVALHO Camille, KOZWaves, February 2014
Conclusion and prospects

Conclusion

* Need to take into account the black-hole wave to sort the guided modes and the leaky modes

* The PML method is an efficient method to reveal the guided modes

Prospects

* Maxwell 2D

* Non linear eigenproblem (dissipationless Drude’s model) $\epsilon(\omega) = \epsilon_{\infty} \left(1 - \frac{\omega_p^2}{\omega^2}\right)$

* What’s happening when $\mu < 0$ (metamaterials)?

* Open plasmonic waveguides
Thank you for your attention.