

Doubly probabilistic representation for the
stochastic porous media type equation.
Représentation doublement probabiliste pour l'
équation des milieux poreux stochastique.

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Summary: The purpose of the present paper consists in proposing and discussing a doubly probabilistic representation for a stochastic porous media equation in the whole space \mathbb{R}^1 perturbed by a multiplicative colored noise. For almost all random realizations ω , one associates a stochastic differential equation in law with random coefficients, driven by an independent Brownian motion.

Resumé: Ce papier propose et discute une représentation doublement probabiliste pour une équation des milieux poreux stochastique dans l'espace tout entier \mathbb{R}^1 , perturbée par un bruit multiplicatif coloré. Pour presque toute réalisation ω du hasard, on associe une équation différentielle stochastique en loi avec coefficients aléatoires, dirigée par un mouvement brownien indépendant.

Key words: stochastic partial differential equations; infinite volume; singular porous media type equation; doubly probabilistic representation; multiplicative noise; singular random Fokker-Planck type equation; filtering.

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1 Introduction

We consider a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ and real functions e^0, \dots, e^N on \mathbb{R} , for some strictly positive integer N . In the whole paper, the following assumption will be in force.

Assumption 1.1. • $|\psi(u)| \leq \text{const}|u|$, $u \geq 0$. In particular, $\psi(0) = 0$.

- $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that its restriction to \mathbb{R}_+ is monotone increasing. Moreover we also suppose that $\lim_{u \rightarrow 0} \frac{\psi(u)}{u}$ exists.
- Let $e^i \in C_b^2(\mathbb{R})$, $0 \leq i \leq N$.

Let $T > 0$ and (Ω, \mathcal{F}, P) , be a fixed probability space. A generic element of Ω will be denoted by ω . $(\mathcal{F}_t, t \in [0, T])$ will stand for a filtration, fulfilling the usual conditions and we suppose $\mathcal{F} = \mathcal{F}_T$. Let $\mu(t, \xi), t \in [0, T], \xi \in \mathbb{R}$, be a random field of the type

$$\mu(t, \xi) = \sum_{i=1}^N e^i(\xi) W_t^i + e^0(\xi)t, \quad t \in [0, T], \xi \in \mathbb{R},$$

where $W^i, 1 \leq i \leq N$, are independent continuous (\mathcal{F}_t) -Brownian motions on (Ω, \mathcal{F}, P) , which are fixed from now on until the end of the paper.

For technical reasons we will sometimes set $W_t^0 \equiv t$. We focus on a stochastic partial differential equation of the following type:

$$\begin{cases} \partial_t X(t, \xi) &= \frac{1}{2} \partial_{\xi\xi}^2 (\psi(X(t, \xi)) + X(t, \xi) \partial_t \mu(t, \xi)), \\ X(0, d\xi) &= x_0(d\xi), \end{cases} \quad (1.1)$$

which holds in the sense of Definition 2.9, where x_0 is a given probability measure on \mathbb{R} . The stochastic multiplication above is of Itô type. We look for a solution of (1.1) with time evolution in $L^1(\mathbb{R})$. Since ψ restricted to \mathbb{R}_+ is non-negative, Assumption 1.1 implies $\psi(u) = \Phi^2(u)u$, $u \geq 0$, $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ being a non-negative continuous function which is bounded on \mathbb{R}_+ .

Remark 1.2. 1. In the sequel we will consider, without further comments extensions of ψ (and Φ) to the real line which fulfill the first two items of Assumption 1.1 for $u \in \mathbb{R}$ instead of $u \geq 0$.

2. The restriction on $u \mapsto \Phi(u)$ introduced in Assumption 1.1 to be continuous is not always necessary, but here we assume this for simplicity.

When $\psi(u) = |u|^{m-1}u$, $m > 1$, (1.1) and $\mu \equiv 0$, (1.1) is nothing else but the classical porous media equation. When ψ is a general increasing function (and

$\mu \equiv 0$), there are several contributions to the analytical study of (1.1), starting from [12] for existence, [15] for uniqueness in the case of bounded solutions and [13] for continuous dependence on the coefficients. Those are the classical references when the space variable varies on the real line. For equations in a bounded domain and Dirichlet boundary conditions, for simplicity, we only refer to monographs, e.g. [28, 26, 1, 2].

As far as the stochastic porous media is concerned, most of the work for existence and uniqueness concerned the case of bounded domain, see for instance [4, 5, 3]. In the infinite volume case, i.e. when the underlying domain is \mathbb{R}^d , well-posedness was fully analyzed in [22], when ψ is polynomially bounded (including the fast diffusion case) when the space dimension is $d \geq 3$. [8] established existence and uniqueness for any dimension $d \geq 1$ and the authors obtained estimates for finite time extinction. To the best of our knowledge, except for [22] and [8], this seems to be the only work concerning a stochastic porous type equation in infinite volume.

We provide a probabilistic representation of solutions to (1.1) extending the results of [14, 6] which treated the deterministic case $\mu \equiv 0$. In the deterministic case, it seems that the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a non-linear deterministic PDE was McKean [19], particularly in relation with the so called propagation of chaos. In his case, however, the coefficients were smooth. From then on the literature steadily grew and nowadays there is a vast amount of contributions to the subject, see the reference list of [14, 6]. A probabilistic representation when $\psi(u) = |u|u^{m-1}$, $m > 1$, was provided for instance in [11], in the case of the classical porous media equation. When $m < 1$, i.e. in the case of the fast diffusion equation, [9] provides a probabilistic representation of the so called **Barenblatt solution**, i.e. the solution whose initial condition is concentrated at zero.

[14, 6] discussed the probabilistic representation when $\mu = 0$ in the so called non-degenerate and degenerate case respectively (see Definition 6.1), where ψ also may have jumps.

In the case $\mu = 0$, the equation (1.1) models a non-linear phenomenon macroscopically. Let us denote by $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ the solution of that equation. The idea of the probabilistic representation is to find a process $(Y_t, t \in [0, T])$ whose law at time t has $u(t, \cdot)$ for its density. In this case the equation (1.1) is conservative, in the sense that the integral (mass) of the solution is conserved along the time.

The process Y turns out to be the weak solution of the non-linear stochastic differ-

ential equation

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(u(s, Y_s)) dB_s, \\ \text{Law}(Y_t) &= u(t, \cdot), \quad t \geq 0, \end{cases} \quad (1.2)$$

where B is a classical Brownian motion. The behavior of Y is the microscopic counterpart of the phenomenon described by (1.1), describing the evolution of a single particle, whose law behaves according to (1.1).

The idea of this paper is to consider the case when $\mu \neq 0$. This includes the case when μ is not vanishing but it is deterministic; it happens when only e^0 is non-zero, and $e^i \equiv 0, 1 \leq i \leq n$. In this case our technique gives a sort of forward Feynman-Kac formula for a non-linear PDE. One of the main interests of this paper is that it provides a (forward) probabilistic representation for *non conservative* (random) PDE.

We introduce a doubly stochastic representation on which one can represent the solution of (1.1) as the weighted-law with respect to the random field μ (or simply the μ -weighted law) of a solution to a non-linear SDE.

Intuitively, it describes the microscopic aspect of the SPDE (1.1) for almost all quenched ω . The terminology strongly refers to the case where the probability space (Ω, \mathcal{F}, P) on which the SPDE is defined, remains fixed.

We represent a solution X to (1.1) making use of another independent source of randomness described by another probability space based on some set Ω_1 .

The analog of the process Y , obtained when μ is zero in [6, 14], is a doubly stochastic process, still denoted by Y defined on $(\Omega_1 \times \Omega, Q)$, for which, X constitutes the so-called family of μ -marginal weighted laws of Y , see Definition 2.4. Y is the solution of a *doubly stochastic non-linear diffusion* problem, see Definition 3.1. It will be a (doubly) stochastic process $(\omega_1, \omega) \mapsto Y(\omega_1, \omega)$ solution of

$$Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \omega)) dB_s, \quad (1.3)$$

and $B(\cdot, \omega)$ is a Brownian motion on Ω_1 for almost any fixed $\omega \in \Omega$. The solution of (1.3) is in the following sense: fixing a realization $\omega \in \Omega$, $Y(\cdot, \omega)$ is a weak solution to the first line of (1.2) with $u(t, \xi) = X(t, \xi, \omega)$. Moreover $X(t, \xi, \omega)$ is the μ -marginal weighted law of $Y_t(\cdot, \omega)$.

The paper includes the following main achievements.

1. If we replace in (1.3) $a(s, \xi, \omega) = \Phi(X(s, \xi, \omega))$ and a is bounded and non-degenerate, we show existence and uniqueness of the solution, strongly in ω , weakly in $\omega_1 \in \Omega_1$, see Proposition 4.1. We also show the existence of law densities, for P -almost all quenched ω , see Proposition 4.4.

2. Theorem 3.3 states that the μ -marginal weighted laws X of a solution Y of a *doubly stochastic non-linear diffusion* problem constitute a solution of the stochastic porous media equation (1.1).
3. Conversely, given a solution X of (1.1), under suitable conditions, there is a solution Y of the doubly stochastic non-linear diffusion. This is discussed in Theorem 6.3 and in Theorem 7.1, distinguishing respectively the cases when ψ is non-degenerate and degenerate, see Definition 6.1.
4. When ψ is non-degenerate, then the doubly stochastic non-linear diffusion problem also admits uniqueness, see Theorem 6.3.
5. Section 3.2 illustrates a filtering interpretation for a solution of SPDE (1.1). Indeed, the μ -marginal weighted laws X of a solution Y of a doubly stochastic non-linear diffusion problem (1.3) can be seen as *conditional densities* of $Y_t, t \in [0, T]$ with respect to some probability measure.
6. Uniqueness of the stochastic Fokker-Planck equation obtained replacing Φ^2 by a function $a(t, \omega, \xi)$ in (1.1), see Theorem 5.1.
7. Existence of a density to the solution of (1.3), see Proposition 4.4.

2 Preliminaries

2.1 Basic notations

First we introduce some basic recurrent notations. $\mathcal{M}(\mathbb{R})$ denotes the space of finite real measures.

We recall that $\mathcal{S}(\mathbb{R})$ is the space of the Schwartz fast decreasing test functions. $\mathcal{S}'(\mathbb{R})$ is its dual, i.e. the space of Schwartz tempered distributions. On $\mathcal{S}'(\mathbb{R})$, the map $(I - \Delta)^{\frac{s}{2}}, s \in \mathbb{R}$, is well-defined. For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the classical Sobolev space consisting of all functions $f \in \mathcal{S}'(\mathbb{R})$ such that $(I - \Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R})$. We introduce the norm

$$\|f\|_{H^s} := \|(I - \Delta)^{\frac{s}{2}} f\|_{L^2},$$

where $\|\cdot\|_{L^p}$ is the classical $L^p(\mathbb{R})$ -norm for $1 \leq p \leq \infty$. In the sequel, we will often simply denote $H^{-1}(\mathbb{R})$, by H^{-1} and $L^2(\mathbb{R})$ by L^2 . Furthermore, $W^{r,p}$ denote the classical Sobolev space of order $r \in \mathbb{N}$ in $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$.

Definition 2.1. *Given a function e belonging to $L^1_{\text{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, we say that it is an H^{-1} -multiplier, if the map $\varphi \mapsto \varphi e$ is continuous from $\mathcal{S}(\mathbb{R})$ to H^{-1} with respect to the H^{-1} -topology on both spaces.*

In the following lines we give some other sufficient conditions on a function e to be an H^{-1} -multiplier.

Lemma 2.2. *Let $e : \mathbb{R} \rightarrow \mathbb{R}$. If $e \in W^{1,\infty}$ (for instance if $e \in W^{2,1}$), then e is a $H^{-1}(\mathbb{R})$ -multiplier. In particular the functions $e^i, 0 \leq i \leq N$ of Definition 1.1 are $H^{-1}(\mathbb{R})$ -multipliers.*

Proof. By duality arguments, we observe that it is enough to show the existence of a constant $\mathcal{C}(e)$ such that

$$\|eg\|_{H^1} \leq \mathcal{C}(e) \|g\|_{H^1}, \quad \forall g \in \mathcal{S}(\mathbb{R}). \quad (2.1)$$

(2.1) follows by product derivation rules, with for instance $\mathcal{C}(e) = \sqrt{2} \left(\|e\|_\infty^2 + \|e'\|_\infty^2 \right)^{\frac{1}{2}}$. \square

With respect to the random field μ , we introduce a notation for the Itô type stochastic integral below.

Let $Z = (Z(s, \xi), s \in [0, T], \xi \in \mathbb{R})$ be a random field on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that $\int_0^T \left(\int_{\mathbb{R}} |Z(s, \xi)| d\xi \right)^2 ds < \infty$ a.s. and it is an $L^1(\mathbb{R})$ -valued (\mathcal{F}_s) -progressively measurable process. Then, the stochastic integral

$$\int_{[0,t] \times \mathbb{R}} Z(s, \xi) \mu(ds, \xi) d\xi := \sum_{i=0}^N \int_0^t \left(\int_{\mathbb{R}} Z(s, \xi) e^i(\xi) d\xi \right) dW_s^i,$$

is well-defined.

More generally, if $s \mapsto Z(s, \cdot)$ is a measurable map $[0, T] \times \Omega \mapsto \mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the space of signed finite measures, such that $\int_0^T \|Z(s, \cdot)\|_{\text{var}}^2 ds < \infty$, then the stochastic integral

$$\int_{[0,t] \times \mathbb{R}} Z(s, \xi) \mu(ds, \xi) d\xi := \sum_{i=0}^N \int_0^t \left(\int_{\mathbb{R}} Z(s, d\xi) \right) e^i(\xi) dW_s^i,$$

is well-defined.

We specify now better the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ of the introduction. We will consider a fixed filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$, where $(\mathcal{F}_t)_{t \in [0, T]}$ is the canonical filtration of a standard Brownian motion (W^1, \dots, W^N) enlarged with the σ -field generated by x_0 . We also suppose that \mathcal{F}_0 contains the P -null sets and $\mathcal{F} = \mathcal{F}_T$.

Let (Ω_1, \mathcal{H}) be a measurable space. In the sequel, we will also consider another filtered probability space $(\Omega_0, \mathcal{G}, \mathbf{Q}, (\mathcal{G}_t)_{t \in [0, T]})$, where $\Omega_0 = \Omega_1 \times \Omega$, $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$.

Clearly any random element Z on (Ω, \mathcal{F}) will be implicitly extended to (Ω_0, \mathcal{G}) setting $Z(\omega_1, \omega) = Z(\omega)$. In particular $W^i, i = 1 \dots N$ will be extended in that way.

Here we fix some conventions concerning measurability. Any topological space E is naturally equipped with its Borel σ -algebra $\mathcal{B}(E)$. For instance $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}([0, T])$) denotes the Borel σ -algebra of \mathbb{R} (resp. $[0, T]$).

Given any probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, the σ -field \mathcal{F} will always be omitted. When we will say that a map $T : \Omega \times E \rightarrow \mathbb{R}$ is measurable, we will implicitly suppose that the corresponding σ -algebras are $\mathcal{F} \otimes \mathcal{B}(E)$ and $\mathcal{B}(\mathbb{R})$.

All the processes on any generic measurable space $(\Omega_2, \mathcal{F}_2)$ will be considered to be measurable with respect to both variables (t, ω) . In particular any processes on $\Omega_1 \times \Omega$ is supposed to be measurable with respect to $([0, T] \times \Omega_1 \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{H} \otimes \mathcal{F})$.

A function $(A, \omega) \mapsto Q(A, \omega)$ from $\mathcal{H} \times \Omega \rightarrow \mathbb{R}_+$ is called **random kernel** (resp. **random probability kernel**) if for each $\omega \in \Omega$, $Q(\cdot, \omega)$ is a finite positive (resp. probability) measure and for each $A \in \mathcal{H}$, $\omega \mapsto Q(A, \omega)$ is \mathcal{F} -measurable. The finite measure $Q(\cdot, \omega)$ will also be denoted by Q^ω . To that random kernel we can associate a specific finite measure (resp. probability) denoted by \mathbf{Q} on (Ω_0, \mathcal{G}) setting $\mathbf{Q}(A \times F) = \int_F Q(A, \omega) P(d\omega) = \int_F Q^\omega(A) P(d\omega)$, for $A \in \mathcal{H}, F \in \mathcal{F}$. The probability Q from above will be supposed here and below to be associated with a random probability kernel.

Definition 2.3. *If there is a measurable space (Ω_1, \mathcal{H}) and a random kernel Q as before, then the probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$ will be called **suitable enlarged probability space** (of (Ω, \mathcal{F}, P)).*

As said above, any random variable on (Ω, \mathcal{F}) will be considered as a random variable on $\Omega_0 = \Omega_1 \times \Omega$. Then, obviously, W^1, \dots, W^N are independent Brownian motions also $(\Omega_0, \mathcal{G}, Q)$.

Given a local martingale M on any filtered probability space, the process $Z := \mathcal{E}(M)$ denotes its Doléans exponential, which is a local martingale. In particular it is the unique solution of $dZ_t = Z_{t-} dM_t$, $Z_0 = 1$. When M is continuous we have $Z_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$.

2.2 The concept of marginal weighted laws

Let us consider a suitably enlarged probability space as in Definition 2.3.

Definition 2.4. *Let $Y : \Omega_1 \times \Omega \times [0, T] \rightarrow \mathbb{R}$ be a measurable process, progressively measurable on $(\Omega_0, \mathcal{G}, \mathbf{Q}, (\mathcal{G}_t))$, where (\mathcal{G}_t) is some filtration on $(\Omega_0, \mathcal{G}, \mathbf{Q})$ such that W^1, \dots, W^N are (\mathcal{G}_t) -Brownian motions on $(\Omega_0, \mathcal{G}, \mathbf{Q})$. We will make use of the*

stochastic integral notation

$$\int_0^t \mu(ds, Y_s) = \sum_{i=0}^N \int_0^t e^i(Y_s) dW_s^i, t \in [0, T]. \quad (2.2)$$

As we shall see below in Proposition 2.6, for every $t \in [0, T]$

$$E^{\mathbf{Q}} \left(\mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s) \right) \right) < \infty. \quad (2.3)$$

To Y , we will associate its **family of μ -marginal weighted laws**, (or simply **family of μ -weighted laws**) i.e. the family of random kernels ($t \in [0, T]$),

$$\Gamma_t = (\Gamma_t^Y(A, \omega), A \in \mathcal{B}(\mathbb{R}), \omega \in \Omega)$$

defined by

$$\varphi \mapsto E^{Q^\omega} \left(\varphi(Y_t(\cdot, \omega)) \mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s)(\cdot, \omega) \right) \right) = \int_{\mathbb{R}} \varphi(r) \Gamma_t^Y(dr, \omega), \quad (2.4)$$

where φ is a generic bounded real Borel function. We will also say that for fixed $t \in [0, T]$, Γ_t is the **μ -marginal weighted law** of Y_t .

Remark 2.5. i) If Ω is a singleton $\{\omega_0\}$, $e^i = 0$, $1 \leq i \leq N$, the μ -marginal weighted laws coincide with the weighted laws

$$\varphi \mapsto E^{\mathbf{Q}} \left(\varphi(Y_t) \exp \left(\int_0^t e^0(Y_s) ds \right) \right),$$

with $\mathbf{Q} = Q^{\omega_0}$. In particular if $\mu \equiv 0$ then the μ -marginal weighted laws are the classical laws.

ii) By (2.3), for any $t \in [0, T]$, for P almost all $\omega \in \Omega$,

$$E^{Q^\omega} \left(\mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s)(\cdot, \omega) \right) \right) < \infty.$$

iii) The function $(t, \omega) \mapsto \Gamma_t(A, \omega)$ is measurable, for any $A \in \mathcal{B}(\mathbb{R})$, because Y is a measurable process.

iv) In the case $e^0 = 0$, the situation is the following. For each fixed $\omega \in \Omega$, (2.4) is a (random) non-negative measure which is not a probability. However the expectation of its total mass is indeed 1.

Proposition 2.6. Consider the situation of Definition 2.4. Then we have the following.

i) The process $M_t := \mathcal{E}_t \left(\sum_{i=1}^N \int_0^t e^i(Y_s) dW_s^i \right)$ is a martingale. We emphasize that the sum starts indeed at $i = 1$.

ii) The quantity (2.3) is bounded by $\exp(T \|e^0\|_\infty)$.

iii) $E^{\mathbf{Q}}(M_t^2) \leq \exp(3T \sum_{i=1}^N \|e^i\|_\infty^2), t \in [0, T]$. Consequently M is a uniformly integrable martingale.

iv) For P -a.e. $\omega \in \Omega$, $\sup_{0 \leq t \leq T} \|\Gamma_t(\cdot, \omega)\|_{\text{var}} < \infty$, where we remind that $\|\cdot\|_{\text{var}}$ stands for the total variation.

Remark 2.7. Proposition 2.6 ii) yields in particular that Y always admits μ -marginal weighted laws.

Proof. i) The result follows since the Novikov condition $E^{\mathbf{Q}}\left(\exp\left(\frac{1}{2} \sum_{i=1}^N \int_s^t e^i(Y_s)^2 ds\right)\right) < \infty$ is verified, because the functions $e^i, i = 1 \dots N$, are bounded.

ii) This follows because $E^{\mathbf{Q}}(M_t) = 1, \forall t \in [0, T]$.

iii) M_t^2 is equal to $N_t \exp\left(3 \sum_{i=1}^N \int_0^t (e^i)^2(Y_s) ds\right)$, where N is a positive martingale with $N_0 = 1$.

iv) For $t \in [0, T]$,

$$\begin{aligned} \sup_{t \leq T} \|\Gamma_t(\cdot, \omega)\|_{\text{var}} &= \sup_{t \leq T} E^{\mathbf{Q}^\omega} \left(M_t \exp \left(\int_0^t e^0(Y_s) ds \right) \right) \\ &\leq \exp(T \|e^0\|_\infty) \sup_{t \leq T} E^{\mathbf{Q}^\omega}(M_t). \end{aligned}$$

Taking the expectation with respect to P it implies

$$\begin{aligned} E^P \left(\sup_{t \leq T} \|\Gamma_t^Y(\cdot, \omega)\|_{\text{var}} \right) &\leq \exp(T \|e^0\|_\infty) E^P \left(\sup_{t \leq T} E^{\mathbf{Q}^\omega}(M_t) \right) \\ &\leq \exp(T \|e^0\|_\infty) E^P \left(E^{\mathbf{Q}^\omega} \left(\sup_{t \leq T} M_t \right) \right). \end{aligned}$$

By the Burkholder-Davis-Gundy (BDG) inequality this is bounded by

$$\begin{aligned} 3 \exp(T \|e^0\|_\infty) E^{\mathbf{Q}} \left(\langle M \rangle_T^{\frac{1}{2}} \right) &\leq 3 \exp(T \|e^0\|_\infty) E^{\mathbf{Q}} \left(\left[\int_0^T ds \sum_{i=1}^N M_s^2 e^i(Y_s)^2 \right]^{\frac{1}{2}} \right) \\ &\leq C(e, N, T) \left\{ E^{\mathbf{Q}} \left(\int_0^T ds M_s^2 \right) \right\}^{\frac{1}{2}}, \end{aligned}$$

where the last inequality is due to Jensen's inequality; $C(e, N, T)$ is a constant depending on N, T and $e^i, i = 0 \dots N$. By Fubini's Theorem and item iii), we have

$$E^{\mathbf{Q}} \left(\int_0^T ds M_s^2 \right) \leq T \exp(3T \sum_{i=1}^N \|e^i\|_\infty^2).$$

□

The lemma below gives a characterization of the μ -weighted laws of a process Y living on an enlarged probability space.

Lemma 2.8. *Let Y (resp. \tilde{Y}) be a process on a suitable enlarged probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$ (resp. $(\tilde{\Omega}_0, \tilde{\mathcal{G}}, \tilde{\mathbf{Q}})$). Set $W = (W^1, \dots, W^N)$. Suppose that the law of (Y, W) under \mathbf{Q} and the law of (\tilde{Y}, W) under $\tilde{\mathbf{Q}}$ are the same. Then, the μ -marginal weighted laws of Y under \mathbf{Q} coincide a.s. with the μ -marginal weighted laws of \tilde{Y} under $\tilde{\mathbf{Q}}$.*

Proof. Let $0 \leq t \leq T$. Using the assumption, we deduce that for any bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, and every $F \in \mathcal{F}_t$, we have

$$E^{\mathbf{Q}} \left(1_F f(Y_t) \mathcal{E}_t \left(\sum_{i=0}^N \int_0^\cdot e^i(Y_s) dW_s^i \right) \right) = E^{\tilde{\mathbf{Q}}} \left(1_F f(\tilde{Y}_t) \mathcal{E}_t \left(\sum_{i=0}^N \int_0^\cdot e^i(\tilde{Y}_s) dW_s^i \right) \right). \quad (2.5)$$

To show this, using classical regularization properties of Itô integral, see e.g. Theorem 2 in [25], and uniform integrability arguments, we first observe that

$$\mathcal{E}_t \left(\sum_{i=0}^N \int_0^\cdot e^i(Y_s) dW_s^i \right)$$

is the limit in $L^2(\Omega_0, \mathbf{Q})$ of

$$\mathcal{E}_t \left(\sum_{i=0}^N \int_0^\cdot e^i(Y_s) \frac{W_{s+\varepsilon}^i - W_s^i}{\varepsilon} ds \right).$$

A similar approximation property arises replacing Y with \tilde{Y} and \mathbf{Q} with $\tilde{\mathbf{Q}}$. Then (2.5) easily follows.

To conclude, it will be enough to show the existence of a countable family $(f_j)_{j \in \mathbb{N}}$ of bounded continuous real functions for which, for P almost all $\omega \in \Omega$, for any $j \in \mathbb{N}$, we have $R_j = \tilde{R}_j$ where

$$\begin{aligned} R_j(\omega) &= E^{\mathbf{Q}^\omega} \left(f_j(Y_t(\cdot, \omega)) \mathcal{E}_t \left(\sum_{i=0}^N \int_0^\cdot e^i(Y_s(\cdot, \omega)) dW_s^i \right) \right) \\ \tilde{R}_j(\omega) &= E^{\tilde{\mathbf{Q}}^\omega} \left(f_j(\tilde{Y}_t(\cdot, \omega)) \mathcal{E}_t \left(\sum_{i=0}^N \int_0^\cdot e^i(\tilde{Y}_s(\cdot, \omega)) dW_s^i \right) \right). \end{aligned}$$

This will follow, since applying (2.5), for any $F \in \mathcal{F}_t$, we have $E^P(1_F R_j) = E^P(1_F \tilde{R}_j)$. \square

2.3 SPDE, weak-strong existence of SDEs

In this section we introduce the basic concepts related to the stochastic porous media equation and the related non-linear diffusion.

Definition 2.9. A random field $X = (X(t, \xi, \omega), t \in [0, T], \xi \in \mathbb{R}, \omega \in \Omega)$ is said to be a solution to (1.1) if P a.s. we have the following.

1. $X \in C([0, T]; \mathcal{S}'(\mathbb{R})) \cap L^2([0, T]; L^1_{\text{loc}}(\mathbb{R}))$.
2. X is an $\mathcal{S}'(\mathbb{R})$ -valued (\mathcal{F}_t) -progressively measurable process.
3. For any test function $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support, $t \in]0, T]$ a.s. we have

$$\begin{aligned} \int_{\mathbb{R}} X(t, \xi) \varphi(\xi) d\xi &= \int_{\mathbb{R}} x_0(d\xi) \varphi(\xi) + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}} \psi(X(s, \xi, \cdot)) \varphi''(\xi) d\xi \\ &+ \int_{[0, t] \times \mathbb{R}} X(s, \xi) \varphi(\xi) \mu(ds, \xi) d\xi. \end{aligned}$$

At Definition 3.1, we will present the concept of *double stochastic non-linear diffusion* which is a McKean type equation with a supplementary source of randomness. Before this, as a first step, we will introduce a particular the case of simple *double stochastic differential equation* (DSDE). Let $\gamma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be an (\mathcal{F}_t) -progressively measurable random fields and x_0 be a probability on $\mathcal{B}(\mathbb{R})$.

Definition 2.10. a) We say that $(DSDE)(\gamma, x_0)$ admits **weak-strong existence** if there is a suitable extended probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$, i.e. a measurable space (Ω_1, \mathcal{H}) , a probability kernel $(Q(\cdot, \omega), \omega \in \Omega)$ on $\mathcal{H} \times \Omega$, two \mathbf{Q} -a.s. continuous processes Y, B on (Ω_0, \mathcal{G}) where $\Omega_0 = \Omega_1 \times \Omega$, $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ such that the following holds.

- 1) For almost all ω , $Y(\cdot, \omega)$ is a (weak) solution to

$$\begin{cases} Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) dB_s(\cdot, \omega), \\ \text{Law}(Y_0) = x_0, \end{cases} \quad (2.7)$$

with respect to Q^ω , where $B(\cdot, \omega)$ is a Q^ω -Brownian motion for almost all ω .

- 2) We denote (\mathcal{Y}_t) the canonical filtration associated with $(Y_s, 0 \leq s \leq t)$ and $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset, \Omega_1\} \otimes \mathcal{F}_t)$. W^1, \dots, W^N is a (\mathcal{G}_t) -martingale under \mathbf{Q} .
- 3) For every $0 \leq s \leq T$, for every bounded continuous $\mathcal{A} : C([0, s]) \rightarrow \mathbb{R}$, the r.v. $\omega \mapsto E^{Q^\omega}(\mathcal{A}(Y_r(\cdot, \omega), r \in [0, s]))$ is \mathcal{F}_s -measurable.

b) We say that $(DSDE)(\gamma, x_0)$ admits **weak-strong uniqueness** if the following holds. Consider a measurable space (Ω_1, \mathcal{H}) (resp. $(\tilde{\Omega}_1, \tilde{\mathcal{H}})$), a probability kernel $(Q(\cdot, \omega), \omega \in \Omega)$ (resp. $(\tilde{Q}(\cdot, \omega), \omega \in \Omega)$), with processes (Y, B) (resp. (\tilde{Y}, \tilde{B})) such that (2.7) holds (resp. (2.7) holds with $(\Omega_0, \mathcal{G}, \mathbf{Q})$ replaced with

$(\tilde{\Omega}_0, \tilde{\mathcal{G}}_0, \tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}$ being associated with $(\tilde{Q}(\cdot, \omega))$). Moreover we suppose that item 2) is verified for Y and \tilde{Y} .

Then (Y, W^1, \dots, W^N) and $(\tilde{Y}, W^1, \dots, W^N)$ have the same law.

c) A process Y fulfilling items 1) and 2) under (a) will be called **weak-strong solution of (DSDE)** (γ, x_0) .

Remark 2.11. Let Y be a weak-strong solution of (DSDE) (γ, x_0) with corresponding B .

a) Since for almost all $\omega \in \Omega$, $B(\cdot, \omega)$ is a Brownian motion under Q^ω , it is clear that B is a Brownian motion under Q , which is independent of \mathcal{F}_T , i.e. independent of W^1, \dots, W^N .

Indeed let $\mathcal{A} : C([0, T]) \rightarrow \mathbb{R}$ be a continuous bounded functional, and denote by \mathcal{W} the Wiener measure on $C([0, T])^N$. Let F be a bounded \mathcal{F}_T -measurable r.v. Since for each ω , $B(\cdot, \omega)$ is a Wiener process with respect to Q^ω , we get

$$\begin{aligned} E^{\mathbf{Q}}(F\mathcal{A}(B)) &= \int_{\Omega} F E^{Q^\omega}(\mathcal{A}(B(\cdot, \omega))) dP(\omega) = \int_{\Omega} F(\omega) dP(\omega) \int_{\Omega_1} \mathcal{A}(\omega_1) d\mathcal{W}(\omega_1) \\ &= \int_{\Omega_0} F(\omega) d\mathbf{Q}(\omega_0) \int_{\Omega_0} \mathcal{A}(\omega_1) d\mathbf{Q}(\omega_0). \end{aligned}$$

This shows that (W^1, \dots, W^N) and B are independent. Taking $F = 1_\Omega$ in previous expression, the equality between the left-hand side and the third term, shows that B is a Brownian motion under Q .

b) Since for any $1 \leq i, j \leq N$,

$$[W^i, W^j]_t = \delta_{ij}t, \quad [W^i, B] = 0, \quad [B, B]_t = t,$$

Lévy's characterization theorem, implies that (W^1, \dots, W^N, B) is a \mathbf{Q} -Brownian motion.

c) An equivalent formulation to 1) in item a) of Definition 2.10 is the following. For P a.e., $\omega \in \Omega$, $Y(\cdot, \omega)$ solves the Q^ω -martingale problem with respect to the (random) PDE operator

$$L_t^\omega f(\xi) = \frac{1}{2} \gamma^2(t, \xi, \omega) f''(\xi),$$

and initial distribution x_0 . Indeed, we remark that the construction can be performed on the canonical space $\Omega_1 = C([0, T]; \mathbb{R})$.

Proposition 2.12. Let Y be a process as in Definition 2.10 a). We have the following.

1. Y is a (\mathcal{G}_t) -martingale on the product space $(\Omega_0, \mathcal{G}, \mathbf{Q})$.

2. $[Y, W^i] = 0, \forall 1 \leq i \leq N$.

Proof. Let $0 \leq s < t \leq T$, $F_s \in \mathcal{F}_s$ and $G : C([0, s]) \rightarrow \mathbb{R}$ be continuous and bounded. We will prove below that, for $1 \leq i \leq N + 1$, setting $W_t^{N+1} = 1$, for all $t \geq 0$,

$$E^{\mathbf{Q}}(Y_t W_t^i G(Y_r, r \leq s) 1_{F_s}) = E^{\mathbf{Q}}(Y_s W_s^i 1_{F_s} G(Y_r, r \leq s)). \quad (2.8)$$

Then (2.8) with $i = N + 1$ shows item 1. Considering (2.8) with $1 \leq i \leq N$, shows that $Y W^i$ is a (\mathcal{G}_t) -martingale, which shows item 2. Therefore, it remains to show (2.8).

The left-hand side of that equality gives

$$\begin{aligned} \int_{\Omega} dP(\omega) \quad & W_t^i(\omega) 1_{F_s}(\omega) E^{Q^\omega}(Y_t(\cdot, \omega) G(Y_r(\cdot, \omega), r \leq s)) \\ &= \int_{\Omega} dP(\omega) 1_{F_s}(\omega) W_t^i(\omega) E^{Q^\omega}(Y_s(\cdot, \omega) G(Y_r(\cdot, \omega), r \leq s)), \end{aligned}$$

because $Y(\cdot, \omega)$ is a Q^ω -martingale for P -almost all ω . To obtain the right-hand side of (2.8) it is enough to remember that W^i are (\mathcal{G}_t) -martingales and that item a) 3) in Definition 2.10 holds. This concludes the proof of Proposition 2.12. \square

Remark 2.13. *Lemma 2.8 shows that, whenever weak-strong uniqueness holds, then the μ -weighted marginal laws of any weak solution Y are uniquely determined.*

3 The concept of doubly probabilistic representation

3.1 The doubly stochastic non-linear diffusion.

We come back to the notations and conventions of the introduction and of Section 2. Let x_0 be a probability on \mathbb{R} . The doubly probabilistic representation is based on the following idea. Let $Y : \Omega_0 \times [0, T] \rightarrow \mathbb{R}$ be a measurable process where $\Omega_0 = \Omega_1 \times \Omega$ is the usual enlarged probability space as introduced in Definition 2.3. Let \mathbf{Q} be a probability inherited from a random kernel Q^ω as before Definition 2.3. Let (\mathcal{G}_t) , where (\mathcal{G}_t) is some filtration on (Ω_0, \mathcal{G}) such that W^1, \dots, W^N are (\mathcal{G}_t) -Brownian motions on $(\Omega_0, \mathcal{G}, \mathbf{Q})$.

Suppose that

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(X(s, Y_s)) dB_s, \\ \mu - \text{Weighted Law}(Y_t) &= X(t, \xi) d\xi, \quad t \in]0, T], \\ \mu - \text{Weighted Law}(Y_0) &= x_0(d\xi), \end{cases} \quad (3.1)$$

where B is a Q -standard Brownian motion with respect to (\mathcal{G}_t) . Then X solves the SPDE (1.1). This will be the object of Theorem 3.3. Vice versa, if X is a solution of (1.1) then there is a process Y solving (3.1), see Theorem 7.1.

Definition 3.1. 1) We say that the doubly stochastic non-linear diffusion (DSNLD) driven by Φ (on the space (Ω, \mathcal{F}, P) with initial condition x_0 , related to the random field μ (shortly (DSNLD) (Φ, μ, x_0)) admits **weak existence** if there is a measurable random field $X : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ with the following properties.

a) The problem (DSDE) (γ, x_0) with $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$ admits weak-strong existence.

b) $X = X(t, \xi, \cdot)d\xi, t \in]0, T]$, is the family of μ -marginal weighted laws of Y , where Y is the solution of (2.7) in Definition 2.10. In other words X constitutes the densities of those μ -marginal weighted laws.

2) A couple (Y, X) , such that Y is a (weak-strong) solution to the (DSDE) (γ, x_0) , is called **weak solution** to the (DSNLD) (Φ, μ, x_0) . Y is also called doubly stochastic representation of the random field X .

3) Suppose that, given two measurable random fields $X^i : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}, i = 1, 2$ on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$, and Y^i , on extended probability space $(\Omega^i, \mathbf{Q}^i), i = 1, 2$, such that (Y^i, X^i) is a weak-strong solution of (DSDE) $(\Phi(X^i), x_0), i = 1, 2$, where we always have that (Y^1, W^1, \dots, W^N) and (Y^2, W^1, \dots, W^N) have the same law. Then we say that the (DSNLD) (Φ, μ, x_0) admits **weak uniqueness**.

Remark 3.2. If (DSNLD) (Φ, μ, x_0) admits **weak uniqueness** then the μ -marginal weighted laws of Y are uniquely determined, P -a.s., see Lemma 2.8.

Theorem 3.3. Let (Y, X) be a solution of (DSNLD) (Φ, μ, x_0) . Then X is a solution to the SPDE (1.1).

Remark 3.4. 1. Let $t \in [0, T]$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be Borel and bounded. Then

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi, \omega) d\xi = E^{Q^\omega} \left(\varphi(Y_t(\omega)) \mathcal{E}_t \left(\int_0^t \mu(ds, Y_s(\omega)) \right) \right).$$

So

$$\int_{\mathbb{R}} X(t, \xi, \omega) d\xi = E^{Q^\omega} \left(\mathcal{E}_t \left(\int_0^t \mu(ds, Y_s(\omega)) \right) \right).$$

Even though for a.e. $\omega \in \Omega$, the previous expression is not necessarily a probability measure, of course,

$$\nu_\omega : \varphi \mapsto \frac{\int_{\mathbb{R}} \varphi(\xi) X(t, \xi, \omega) d\xi}{\int_{\mathbb{R}} X(t, \xi, \omega) d\xi}$$

is one. It can be expressed as

$$\nu_\omega(A) = \frac{E^{Q^\omega}(1_A(Y_t)\mathcal{E}_t(M(\cdot, \omega)))}{E^{Q^\omega}\mathcal{E}_t(M(\cdot, \omega))},$$

where $M_t(\cdot, \omega) = \int_0^t \mu(ds, Y_s(\cdot, \omega))$, $t \in [0, T]$, is defined in (2.2).

2. Consider the particular case $e_0 = 0, e_1 = c$, c being some constant. In this case, the μ -marginal laws are given by

$$A \mapsto E^{Q^\omega}(1_A(Y_t)c\mathcal{E}_t(W)) = c\mathcal{E}_t(W)E^{Q^\omega}(1_A(Y_t)) = c\mathcal{E}_t(W)\nu_\omega(t, A)$$

and $\nu_\omega(t, \cdot)$ is the law of $Y_t(\cdot, \omega)$ under Q^ω .

Proof. Let B denote the Brownian motion associated to Y as a solution to (DSDE)(γ, x_0), mentioned in item a)1) of Definition 3.1. For $t \in [0, T]$, we set

$$Z_t = \mathcal{E}_t\left(\int_0^\cdot \mu(ds, Y_s)\right), \quad M_t = Z_t \exp\left(-\int_0^t e^0(Y_s)ds\right), \quad t \in [0, T].$$

1. Proof of Definition 2.9 1. By Proposition 2.6, $(M_t, t \in [0, T])$ is a uniformly integrable martingale. Consequently $t \mapsto Z_t$ is continuous in $L^1(\Omega_0, \mathbf{Q})$. On the other hand the process Y is continuous. This implies that P a.e. $\omega \in \Omega$, $X \in C([0, T]; \mathcal{M}(\mathbb{R}))$, where $\mathcal{M}(\mathbb{R})$ is equipped with the weak topology. This implies that $X \in C([0, T]; \mathcal{S}'(\mathbb{R}))$. Furthermore, for P a.e. $\omega \in \Omega$, and $t \in]0, T]$, $X(t, \cdot, \omega) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} X(t, \xi, \omega)d\xi = \|\Gamma(t, \cdot, \omega)\|_{\text{var}}$. By Proposition 2.6 iv), it follows that P -a.s. $X(\cdot, \cdot, \omega) \in L^\infty([0, T]; L^1(\mathbb{R})) \subset L^2([0, T]; L^1_{\text{loc}}(\mathbb{R}))$.
2. Definition 2.9 2. follows from Remark 3.4 1) and Definition 2.10 a) 3).
3. Proof of Definition 2.9 3. Let $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support. Taking into account Proposition 2.12, we apply Itô's formula to get

$$\begin{aligned} \varphi(Y_t)Z_t &= \varphi(Y_0) + \int_0^t \varphi'(Y_s)Z_s dY_s + \int_0^t \varphi(Y_s)Z_s \left(\mu(ds, Y_s) - \frac{1}{2} \sum_{i=1}^N (e^i(Y_s))^2 ds \right) \\ &+ \frac{1}{2} \int_0^t \varphi''(Y_s)\Phi^2(X(s, Y_s))Z_s ds + \frac{1}{2} \int_0^t \varphi(Y_s)Z_s \left(\sum_{i=1}^N (e^i(Y_s))^2 \right) ds. \end{aligned}$$

Indeed we remark that $\int_0^t \varphi'(Y_s)d[Z, Y]_s = 0$, because $[Z, Y]_t = \sum_{i=1}^N \int_0^t e^i(Y_s)Z_s d[W^i, Y]_s = 0$; in fact $[W^i, Y] = 0$ by Proposition 2.12. So

$$\begin{aligned} \varphi(Y_t)Z_t &= \varphi(Y_0) + \int_0^t \varphi'(Y_s)Z_s \Phi(X(s, Y_s))dB_s \\ &+ \int_0^t \varphi(Y_s)Z_s \mu(ds, Y_s) + \frac{1}{2} \int_0^t \varphi''(Y_s)\Phi^2(X(s, Y_s))Z_s ds. \end{aligned}$$

Taking the expectation with respect to Q^ω we get dP -a.s.,

$$\begin{aligned} \int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi) &= \int_{\mathbb{R}} \varphi(\xi) x_0(d\xi) + \sum_{i=0}^N \int_0^t dW_s^i \left(\int_{\mathbb{R}} d\xi \varphi(\xi) e^i(\xi) X(s, \xi) \right) \\ &\quad + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}} d\xi \varphi''(\xi) \Phi^2(X(s, \xi)) X(s, \xi), \end{aligned}$$

which implies the result. Indeed, in the previous equality, we have used Lemma 3.5 below. □

Lemma 3.5. *Let $1 \leq i \leq N$. For P a.e. $\omega \in \Omega$, we have*

$$E^{Q^\omega} \left(\int_0^t \varphi(Y_s) Z_s e^i(Y_s) dW_s^i \right) (\cdot, \omega) = \int_0^t dW_s^i(\omega) \int_{\mathbb{R}} \varphi(\xi) e^i(\xi) X(s, \xi, \omega) d\xi.$$

Proof. Since the Brownian motions W^i are not random for Q^ω , it is possible to justify the permutation of the stochastic integral with respect to W^i and E^{Q^ω} by a Fubini argument approximating the stochastic integrals via Lebesgue integral, see e.g. Theorem 2 of [25]. A complete proof is given in [7]. □

3.2 Filtering interpretation

Item 1. of Remark 3.4 has an interpretation in the framework of filtering theory, see e.g. [20] for a comprehensive introduction on that subject.

Suppose $e^0 = 0$. Let $\hat{\mathbf{Q}}$ be a probability on some probability space $(\Omega_0, \mathcal{G}_T)$, and consider the non-linear diffusion problem (1.2) as a basic dynamical phenomenon. We suppose now that there are N observations Y^1, \dots, Y^N related to the process Y generating a filtration (\mathcal{F}_t) . We suppose in particular that $dY_t^i = dW_t^i + e^i(Y_t)dt$, $1 \leq i \leq N$, and W^1, \dots, W^N be (\mathcal{F}_t) -Brownian motions. Consider the following dynamical system of non-linear diffusion type:

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(X(s, Y_s)) dB_s \\ dY_t^i &= dW_t^i + e^i(Y_t)dt, 1 \leq i \leq N, \\ X(t, \cdot) &: \text{conditional law of } Y_t \text{ under } \mathcal{F}_t. \end{cases} \quad (3.1)$$

The third equality of (3.1) means, under $\hat{\mathbf{Q}}$, that we have,

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi) d\xi = E(\varphi(Y_t) | \mathcal{F}_t). \quad (3.2)$$

We remark that, under the new probability Q defined by $d\mathbf{Q} = d\hat{\mathbf{Q}} \mathcal{E}(\int_0^T \mu(ds, Y_s))$, Y^1, \dots, Y^N are standard (\mathcal{F}_t) -independent Brownian motions. Then (3.2) becomes

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi) d\xi = E^{\hat{\mathbf{Q}}}(\varphi(Y_t) | \mathcal{F}_t) = \frac{E^{\mathbf{Q}}(\varphi(Y_t) \mathcal{E}_t(\int_0^t \mu(ds, Y_s) | \mathcal{F}_t))}{E^{\mathbf{Q}}(\mathcal{E}_t(\int_0^t \mu(ds, Y_s) | \mathcal{F}_t))}.$$

Consequently, by Theorem 3.3, X will be the solution of the SPDE (1.1), with x_0 being the law of Y_0 ; so (1.1) constitutes the Zakai type equation associated with our filtering problem.

4 The densities of the μ -marginal weighted laws

This section constitutes an important step towards the doubly probabilistic representation of a solution to (1.1), when ψ is non-degenerate. Let x_0 be a fixed probability on \mathbb{R} . We recall that a process Y (on a suitable enlarged probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$), which is a weak solution to the (DSNLD) (Φ, μ, x_0) , is in particular a weak-strong solution of a (DSDE) (γ, x_0) where $\gamma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is some suitable progressively measurable random field on (Ω, \mathcal{F}, P) . The aim of this section is twofold.

- A) To show that whenever γ is a.s. bounded and non-degenerate, (DSDE) (γ, x_0) admit weak-strong existence and uniqueness.
 - B) The marginal μ -laws of the solution to (DSDE) (γ, x_0) admit a density for P a.s.
- A) We start discussing well-posedness.

Proposition 4.1. *We suppose the existence of random variables A_1, A_2 such that*

$$0 < A_1(\omega) \leq \gamma(t, \xi, \omega) \leq A_2(\omega), \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad dP\text{-a.s.}$$

Then (DSDE) (γ, x_0) admits weak-strong existence and uniqueness.

Proof. Uniqueness. This is the easy part. Let Y and \tilde{Y} be two solutions. Then for ω outside a P -null set N_0 , $Y(\cdot, \omega)$ and $\tilde{Y}(\cdot, \omega)$ are solutions to the same one-dimensional classical SDE with measurable bounded and non-degenerate (i.e. greater than a strictly positive constant) coefficients. Then, by Exercise 7.3.3 of [27] the law of $Y(\cdot, \omega)$ equals the law of $\tilde{Y}(\cdot, \omega)$. Then obviously the law of Y equals the law of \tilde{Y} .

Existence. This point is more delicate. In fact one needs to solve the random SDE for P almost all ω but in such a way that the solution produces bimeasurable processes Y and B .

First we regularize the coefficient γ . Let ϕ be a mollifier with compact support; we set $\phi_n(x) = n\phi(nx)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. We consider the random fields $\gamma_n : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ by $\gamma_n(t, x, \omega) := \int_{\mathbb{R}} \gamma(t, x - y, \omega) \phi_n(y) dy$.

Let $(\tilde{\Omega}_1, \tilde{\mathcal{H}}_1, \tilde{P})$ be a probability space where we can construct a random variable Y_0 distributed according to x_0 and an independent Brownian motion B .

In this way on $(\tilde{\Omega}_1 \times \Omega, \tilde{\mathcal{H}}_1 \otimes \mathcal{F}, \tilde{P} \otimes P)$ we dispose of a random variable Y_0 and a Brownian motion independent of $\{\emptyset, \tilde{\Omega}_1\} \otimes \mathcal{F}$. By usual fixed point techniques, it is possible to exhibit a (strong) solution of (DSDE) (γ_n, x_0) on the over mentioned product probability space. We can show that there is a unique solution $Y = Y^n$ of $Y_t = Y_0 + \int_0^t \gamma_n(s, Y_s, \cdot) dB_s$. In fact, the maps $\Gamma_n : Z \mapsto \int_0^t \gamma_n(s, Z_s, \omega) dB_s + Y_0$, where $\Gamma_n : L^2(\tilde{\Omega}_1 \times \Omega; \tilde{P} \otimes P) \rightarrow L^2(\tilde{\Omega}_1 \times \Omega, \tilde{P} \otimes P)$ are Lipschitz; by usual Picard fixed point arguments one can show the existence of a unique solution $Z = Z^n$ in $L^2(\tilde{\Omega}_1 \times \Omega; \tilde{P} \otimes P)$. We observe that, by usual regularization arguments for Itô integral as in Lemma 3.5, for ω -a.s., $Y(\cdot, \omega)$ solves for P a.e. $\omega \in \Omega$, equation

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma_n(s, Y_s(\cdot, \omega), \omega) dB_s, \quad (4.1)$$

on $(\tilde{\Omega}_1, \tilde{\mathcal{H}}_1, \tilde{P})$. We consider now the measurable space $\Omega_0 = \Omega_1 \times \Omega$, where $\Omega_1 = C([0, T], \mathbb{R})$, equipped with product σ -field $\mathcal{G} = \mathcal{B}(\Omega_1) \otimes \mathcal{F}$. On that measurable space, we introduce the probability measures \mathbf{Q}_n where $\mathbf{Q}_n(d\omega_1, \omega) = Q_n(d\omega_1, \omega)P(d\omega)$ and $Q_n(\cdot, \omega)$ being the law of $Y^n(\cdot, \omega)$ for almost all fixed ω . We set $Y_t(\omega_1, \omega) = \omega_1(t)$, where $\omega_1 \in C([0, T]; \mathbb{R})$. We denote by $(\mathcal{Y}_t, t \in [0, T])$ (resp. (\mathcal{Y}_t^1)) the canonical filtration associated with Y on Ω_0 (resp. Ω_1). The next step will be the following.

Lemma 4.2. *For almost all ω dP a.s. $Q_n(\omega, \cdot)$ converges weakly to $Q(\omega, \cdot)$, where under $Q(\cdot, \omega)$, $Y(\cdot, \omega)$ solves the SDE*

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) dB_s(\cdot, \omega),$$

where $B(\cdot, \omega)$ is an (\mathcal{Y}_t^1) -Brownian motion on Ω_1 .

Proof. It follows directly from Proposition A.1 of the Appendix. \square

This shows the validity of 1) if Definition 2.10 a).

Remark 4.3. 1) *Since $Q_n(\cdot, \omega)$ converges weakly to $Q(\cdot, \omega)$, ω dP a.s., then the limit (up to an obvious modification) is a measurable random kernel.*

2) *This also implies that $Y_n(\cdot, \omega)$ converges stably to $Q(\cdot, \omega)$. For details about the stable convergence the reader can consult [17, section VIII 5. c] and the recent monograph [16].*

The considerations above allow to complete the proof of Proposition 4.1. By Lemma 4.2, $Q^\omega = Q(\cdot, \omega)$ is a random kernel, being a limit of random kernels. Let us

consider the associated probability measure on the suitable enlarged probability space $(\Omega_0, \mathcal{G}, Q)$. We observe that Y on (Ω_0, \mathcal{G}) is obviously measurable, because it is the canonical process $Y(\omega_1, \omega) = \omega_1$. Setting

$$B_t(\cdot, \omega) = \int_0^t \frac{dY_s(\cdot, \omega)}{\gamma(s, Y_s(\cdot, \omega), \omega)},$$

we get $[B]_t(\cdot, \omega) = t$ under $Q(\cdot, \omega)$, so, by Lévy characterization theorem, it is a Brownian motion. Moreover B is bimeasurable.

Let $G = \mathcal{A}(Y_r(\cdot, \omega), r \in [0, s])$, where \mathcal{A} is a bounded functional $C([0, s]) \rightarrow \mathbb{R}$. We first observe that the r.v. $\omega \mapsto E^{Q^\omega}(G)$ is \mathcal{F}_s -measurable. This happens because Y is, under Q^ω , a martingale with quadratic variation $\left(\int_0^t \gamma^2(s, Y_s(\cdot, \omega), \omega) ds, 0 \leq t \leq T\right)$, i.e. with (random) coefficient which is (\mathcal{F}_t) -progressively measurable. This shows item 3) of Definition 2.10 a).

The last point to check is that W^1, \dots, W^N are (\mathcal{G}_t) -martingales, where $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset, \Omega_1\} \otimes \mathcal{F}_t)$, $0 \leq t \leq T$, i.e. item 2) of Definition 2.10.

Indeed, we justify this immediately. Consider $0 \leq s \leq t \leq T$. Taking into account monotone class arguments, given $F \in \mathcal{F}_s$, $G \in \mathcal{Y}_s^1$, $1 \leq i \leq N$, it is enough to prove that

$$E^{\mathbf{Q}}(FGW_t^i) = E^{\mathbf{Q}}(FGW_s^i). \quad (4.2)$$

Using the fact that W^i is an (\mathcal{F}_t) -martingale and that $E^{Q^\omega}(G)$ is \mathcal{F}_s -measurable by item a) 3) of Definition 2.10 (established above), the left-hand side of (4.2) gives

$$E^P(FW_t^i E^{Q^\omega}(G)) = E^P(FW_s^i E^{Q^\omega}(G)),$$

which constitutes the right-hand side of (4.2). This concludes the proof of the proposition. \square

We go on now with step B) of the beginning of Section 4.

Proposition 4.4. *We suppose the existence of r.v. A_1, A_2 such that*

$$0 < A_1(\omega) \leq \gamma(t, \xi, \omega) \leq A_2(\omega), \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad a.s.$$

Let Y be a weak-strong solution to (DSDE)(γ, x_0) and we denote by $(\nu_t(dy, \cdot), t \in [0, T])$, the μ -marginal weighted laws of process Y .

1. *There is a measurable function $q : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ such that $dtdP$ a.e., $\nu_t(dy, \cdot) = q_t(y, \cdot)dy$. In other words the μ -marginal weighted laws admit densities.*
2. $\int_{[0, T] \times \mathbb{R}} q_t^2(y, \cdot) dtdy < \infty \quad dP$ -a.s..

3. q is an $L^2(\mathbb{R})$ -valued progressively measurable process.

Proof. By 3) of Definition 2.10, the μ -marginal laws constitute an $\mathcal{S}'(\mathbb{R})$ -valued progressively measurable process. Consequently 3. holds if 1. and 2. hold.

Let

$$B_t(\cdot, \omega) := \int_0^t \frac{dY_s(\cdot, \omega)}{\gamma(s, Y_s(\cdot, \omega), \omega)}.$$

We denote again $Q^\omega := Q(\cdot, \omega)$ according to Definition 2.10, $\omega \in \Omega$.

Let $\omega \in \Omega$ be fixed. Let $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with compact support. We need to evaluate

$$E^{Q^\omega} \left(\int_0^T \varphi(s, Y_s) Z_s ds \right), \quad (4.3)$$

where $Z_s = M_s \exp \left(\int_0^s e^0(Y_r) dr \right)$ where $M_s = \mathcal{E}_s \left(\sum_{i=1}^N \int_0^s e^i(Y_r) dW_r^i \right)$.

M_s is smaller or equal than $\exp \left(\sum_{j=1}^N \int_0^s e^j(Y_r) dW_r^j \right)$ which equals

$$\exp \left(\sum_{j=1}^N \left\{ W_s^j e^j(Y_s) - \int_0^s W_r^j (e^j)'(Y_r) dY_r \right\} - \frac{1}{2} \int_0^s \left\{ \sum_{j=1}^N W_r^j (e^j)''(Y_r) \gamma^2(r, Y_r, \cdot) \right\} dr \right), \quad (4.4)$$

taking into account the fact that $[Y, W^j] = 0$ for any $1 \leq j \leq n$, by Proposition 2.12.

Denoting $\|g\|_\infty := \sup_{t \in [0, T]} |g(t)|$, for a function $g : [0, T] \rightarrow \mathbb{R}$, (4.4) is smaller or equal than

$$\exp \left(\sum_{j=1}^N \|W^j\|_\infty (\|e^j\|_\infty + \frac{T}{2} \|(e^j)''\|_\infty A_2^2) \right) \exp \left(- \int_0^s \left[\sum_{j=1}^N W_r^j (e^j)'(Y_r) \gamma(r, Y_r, \cdot) \right] dB_r \right).$$

So (4.3) is bounded by

$$\varrho(\omega) E^{Q^\omega} \left(\int_0^T |\varphi|(s, Y_s(\cdot, \omega)) R_s(\cdot, \omega) ds \right), \quad (4.5)$$

where

$$\begin{aligned} \varrho(\omega) &= \exp \left(T \|e_0\|_\infty + \sum_{i=1}^N \|W^i\|_\infty \|e^i\|_\infty \right. \\ &\quad \left. + T \frac{A_2^2(\omega)}{2} \sum_{i=1}^N (\|W^i\|_\infty^2 \|(e^i)'\|_\infty^2 + \|W^i\|_\infty \|(e^i)''\|_\infty) \right) \end{aligned}$$

and R is the Q^ω -exponential martingale

$$R_t(\cdot, \omega) = \exp \left(- \int_0^t \delta(r, \cdot, \omega) dB_r - \frac{1}{2} \int_0^t \delta^2(r, \cdot, \omega) dr \right).$$

where $\delta(r, \cdot, \omega) = \sum_{j=1}^N W_r^j (e^j)'(Y_r(\cdot, \omega)) \gamma(r, Y_r(\cdot, \omega), \omega)$. So there is a random (depending on $\omega \in \Omega$) constant

$$\varrho_1(\omega) := \text{const} (T, W^j, \|e^j\|_\infty, \|(e^j)'\|_\infty, \|(e^j)''\|_\infty, 1 \leq j \leq N, A_2(\omega)), \quad (4.6)$$

so that (4.5) is smaller than

$$\varrho_1(\omega) E^{Q^\omega} \left(\int_0^T |\varphi(s, Y_s(\cdot, \omega))| ds R_T(\cdot, \omega) \right), \quad (4.7)$$

where we remind that $R(\cdot, \omega)$ is a Q^ω -martingale. By Girsanov theorem, $\tilde{B}_t(\cdot, \omega) = B_t(\cdot, \omega) + \int_0^t \delta(r, \cdot, \omega) dr$ is a \tilde{Q}^ω -Brownian motion with $d\tilde{Q}^\omega = R_T(\cdot, \omega) dQ^\omega$. At this point, the expectation in (4.7) gives

$$E^{\tilde{Q}^\omega} \left(\int_0^T |\varphi|(s, Y_s(\cdot, \omega)) ds \right), \quad (4.8)$$

where

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) d\tilde{B}_s - \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) \delta(s, \cdot, \omega) ds.$$

For fixed $\omega \in \Omega$, δ is bounded by a random constant $\varrho_2(\omega)$ of the type (4.6). Moreover we keep in mind assumption (4.1) on γ . By Exercise 7.3.3 of [27], (4.8) is bounded by $\varrho_3(\omega) \|\varphi\|_{L^2([0, T] \times \mathbb{R})}$, where $\varrho_3(\omega)$ again depends on the same quantities as in (4.6) and Φ . So for ω dP -a.s., the map $\varphi \mapsto E^{Q^\omega} \left(\int_0^T \varphi(s, Y_s(\cdot, \omega)) Z_s(\cdot, \omega) ds \right)$ prolongs to $L^2([0, T] \times \mathbb{R})$. Using Riesz' theorem it is not difficult to show the existence of an $L^2([0, T] \times \mathbb{R})$ function $(s, y) \mapsto q_s(y, \omega)$ which constitutes indeed the density of the family of the μ -marginal weighted laws. \square

5 On the uniqueness of a Fokker-Planck type SPDE

The next result is an extension of Theorem 3.8 of [14] to the stochastic case. It has an independent interest since it is a Fokker-Planck SPDE with possibly degenerate measurable coefficients.

Theorem 5.1. *Let z_0 be a distribution in $\mathcal{S}'(\mathbb{R})$. Let z^1, z^2 be two measurable random fields belonging ω a.s. to $C([0, T], \mathcal{S}'(\mathbb{R}))$ such that $z^1, z^2 :]0, T] \times \Omega \rightarrow \mathcal{M}(\mathbb{R})$. Let $a : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ be a bounded measurable random field such that, for any $t \in [0, T]$, $a(t, \cdot)$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ -measurable. We suppose moreover the following.*

$$i) \quad z^1 - z^2 \in L^2([0, T] \times \mathbb{R}) \text{ a.s.}$$

ii) $t \mapsto (z^1 - z^2)(t, \cdot)$ is an (\mathcal{F}_t) -progressively measurable $\mathcal{S}'(\mathbb{R})$ -valued process.

iii) $\int_0^T \|z^i(s, \cdot)\|_{\text{var}}^2 ds < \infty$ a.s.

iv) z^1, z^2 are solutions to

$$\begin{cases} \partial_t z(t, \xi) = \partial_{\xi\xi}^2((az)(t, \xi)) + z(t, \xi)\mu(dt, \xi), \\ z(0, \cdot) = z_0. \end{cases} \quad (5.1)$$

Then $z^1 \equiv z^2$.

Remark 5.2. By solution of equation (5.1) we intend, as expected, the following: for every $\varphi \in \mathcal{S}(\mathbb{R})$, $\forall t \in [0, T]$,

$$\int_{\mathbb{R}} \varphi(\xi) z(t, d\xi) = \langle z_0, \varphi \rangle + \int_0^t ds \int_{\mathbb{R}} a(s, \xi) \varphi''(\xi) z(s, d\xi) + \sum_{j=0}^N \int_0^t dW_s^j \int_{\mathbb{R}} \varphi(\xi) e^j(\xi) z(s, d\xi).$$

Proof of Theorem 5.1. The proof makes use of the similar arguments as in Theorem 3.8 of [14] or Theorem 3.1 in [10], in a randomized form. The full proof is given in [24] Theorem 4.2, see also [7].

□

6 The non-degenerate case

We are now able to discuss the doubly probabilistic representation of a solution to (1.1) when ψ is non-degenerate provided that its solution fulfills some properties.

Definition 6.1. • We will say that equation (1.1) (or ψ) is **non-degenerate** if on each compact, there is a constant $c_0 > 0$ such that $\Phi \geq c_0$.

• We will say that equation (1.1) or ψ is **degenerate** if $\lim_{u \rightarrow 0^+} \Phi(u) = 0$.

One of the typical examples of degenerate ψ is the case of ψ being **strictly increasing after some zero**. This notion was introduced in [6] and it means the following. There is $0 \leq u_c$ such that $\psi_{[0, u_c]} \equiv 0$ and ψ is strictly increasing on $]u_c, +\infty[$.

Remark 6.2. 1. ψ is non-degenerate if and only if $\lim_{u \rightarrow 0^+} \Phi(u) > 0$.

2. Of course, if ψ is strictly increasing after some zero, with $u_c > 0$ then ψ is degenerate.

3. If ψ is degenerate, then $\psi^\kappa(u) = (\Phi^2(u) + \kappa)u$, for every $\kappa > 0$, is non-degenerate.

As announced the theorem below also holds when ψ is multi-valued.

Theorem 6.3. *We suppose the following assumptions.*

1. x_0 is a real probability measure.
2. ψ is non-degenerate.
3. There is only one random field $X : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ solution of (1.1) (see Definition 2.9) such that

$$\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi < \infty \quad \text{a.s.} \quad (6.1)$$

Then there is a unique weak solution to the (DSNLD)(Φ, μ, x_0).

Remark 6.4. 1. An easy adaptation of Theorem 3.4 of [8] (taking into account e^0), when ψ is Lipschitz and e^0, \dots, e^N belong to H^1 allows to show that there is a solution to (1.1) such that

$$E \left(\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi \right) < \infty. \text{ This holds even if } x_0 \text{ belongs to } H^{-1}(\mathbb{R}).$$

According to Theorem B.1, that solution is unique. In particular item 3. in Theorem 6.3 statement holds.

2. Theorem 6.3 constitutes the converse of Theorem 3.3 when ψ is non-degenerate.
3. The theorem also holds if ψ is multi-valued. For implementing this, we need to adapt the techniques of [14].
4. As side-effect of the proof of the weak-strong existence Proposition 4.1, the space $(\Omega_0, \mathcal{G}, \mathbf{Q})$ can be chosen as $\Omega_0 = \Omega_1 \times \Omega$, $\Omega_1 = C([0, T]; \mathbb{R}) \times \mathbb{R}$, $\mathcal{G} = \mathcal{B}(\Omega_1) \times \mathcal{F}$, $\mathbf{Q}(H \times F) = \int_{\Omega_1 \times \Omega} dP(\omega) 1_F(\omega) Q(d\omega_1, \omega)$.

Proof. 1) We set $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$. According to Proposition 4.1 there is a weak-strong solution Y to (DSDE)(γ, x_0). By Proposition 4.4 ω a.s. the μ -marginal weighted laws of Y admit densities $(q_t(\xi, \omega), t \in]0, T], \xi \in \mathbb{R}, \omega \in \Omega)$ such that dP -a.s. $\int_{[0, T] \times \mathbb{R}} ds d\xi q_s^2(\xi, \cdot) < \infty$ a.s.

2) Setting

$$\nu_t(\xi, \omega) = \begin{cases} q_t(\xi, \omega) d\xi & : t \in]0, T], \\ x_0 & : t = 0, \end{cases}$$

ν is a solution to (5.1) with $\nu_0 = x_0$, $a(t, \xi, \omega) = \Phi^2(X(t, \xi, \omega))$. This can be shown applying Itô's formula similarly as in the proof of Theorem 3.3.

3) On the other hand X is obviously also a solution of (5.1), which in particular verifies (6.1). Consequently $z^1 = \nu$, $z^2 = X$ verify items i), ii), iii) of Theorem 5.1. So Theorem 5.1 implies that $\nu \equiv X$; this shows that Y provides a solution to (DSNLD)(Φ, μ, x_0).

4) Concerning uniqueness, let Y^1, Y^2 be two solutions to the (DSNLD) related to (Φ, μ, x). The corresponding random fields X^1, X^2 constitute the μ -marginal laws of Y^1, Y^2 respectively.

Now Y^i , $i = 1, 2$, is a weak-strong solution of (DSDE)(γ_i, x) with $\gamma_i(t, \xi, \omega) = \Phi(X_i(t, \xi, \omega))$, so by Proposition 4.4 X_i , $i = 1, 2$ fulfills (6.1). By Theorem 3.3, X_1 and X_2 are solutions to (1.1). By assumption 3. of the statement, $X_1 = X_2$. The conclusion follows by Proposition 4.1, which guarantees the uniqueness of the weak-strong solution of (DSDE)(γ, x_0) with $\gamma = \gamma_1 = \gamma_2$. \square

Remark 6.5. *One side-effect of Theorem 6.3 is the following. Suppose ψ to be non-degenerate. Let $X : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a solution such that dP -a.s.*

$$\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi < \infty \quad \text{a.s. We have the following for } \omega \text{ } dP\text{-a.s.}$$

$$i) X(t, \cdot, \omega) \geq 0 \text{ a.e. } \forall t \in [0, T].$$

$$ii) E \left(\int_{\mathbb{R}} X(t, \xi) d\xi \right) = 1, \forall t \in [0, T] \text{ if } e_0 = 0.$$

Remark 6.6. *If (1.1) has a solution, not necessarily unique, then (DSNLD) with respect to (Φ, μ, x_0) still admits existence.*

7 The degenerate case

The idea consists in proceeding similarly to [6], which treated the case $\mu = 0$ and the case when x_0 is absolutely continuous with bounded density. ψ will be assumed to be strictly increasing after some zero $u_c \geq 0$, see Definition 6.1. We recall that if ψ is degenerate, then necessarily $\Phi(0) := \lim_{x \rightarrow 0} \Phi(x) = 0$.

The theorem below concerns existence, we do not know any uniqueness result in the degenerate case.

Theorem 7.1. *We suppose the following.*

1. *The functions $e^i, 1 \leq i \leq N$ belong to $H^1(\mathbb{R})$.*
2. *We suppose that $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing, Lipschitz and strictly increasing after some zero.*
3. *x_0 belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

Then there is a weak solution to the (DSNLD)(Φ, μ, x_0).

Remark 7.2. *If $u_c > 0$ then ψ is necessarily degenerate and also Φ restricted to $[0, u_c]$ vanishes.*

Proof (of Theorem 7.1).

- 1) We proceed by approximation rendering Φ non-degenerate. Let $\kappa > 0$. We define $\Phi_\kappa : \mathbb{R} \rightarrow \mathbb{R}_+$ by $\Phi_\kappa(u) = \sqrt{\Phi^2(u) + \kappa}$, $\psi_\kappa(u) = \Phi_\kappa^2(u) \cdot u$. Let X^κ be the solution so (1.1) with ψ_κ instead of ψ . According to Theorem 6.3 and Remark 6.4 4., setting

$$\tilde{\Omega}_1 = C([0, T], \mathbb{R}) \times \mathbb{R}, \quad Y(\omega_1, \omega) = \omega_1, \quad (7.1)$$

\mathcal{H} the Borel σ -algebra of $\tilde{\Omega}_1$, there are families of probability kernels Q^κ on $\mathcal{H} \times \Omega$, and measurable processes B^κ on $\tilde{\Omega}_0 = \tilde{\Omega}_1 \times \Omega$ such that

- i) $B^\kappa(\cdot, \omega)$ is a $Q^\kappa(\cdot, \omega)$ -Brownian motion;
- ii) Y is a (weak) solution, on $(\tilde{\Omega}_1, Q^\kappa(\cdot, \omega))$, of
$$Y_t = Y_0 + \int_0^t \Phi_\kappa(X^\kappa(s, Y_s, \omega)) dB_s^\kappa(\cdot, \omega), \quad t \in [0, T];$$
- iii) Y_0 is distributed according to $x_0 = X^\kappa(0, \cdot)$.
- iv) The μ -marginal weighted laws of Y under \mathbf{Q}^κ are $(X^\kappa(t, \cdot))$.

In agreement with Definition 3.1 and Definition 2.10, we need to show the existence of a suitable measurable space (Ω_1, \mathcal{H}) , a probability kernel Q on $\mathcal{H} \times \Omega$, a process B on $\Omega_0 := \Omega_1 \times \Omega$ such that the following holds.

- i) $B(\cdot, \omega)$ is a $Q(\cdot, \omega)$ -Brownian motion.
- ii) Y is a (weak) solution on $(\Omega_1, Q(\cdot, \omega))$ of
$$Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \omega)) dB_s(\cdot, \omega), \quad t \in [0, T],$$
i.e. item 1) of Definition 2.10. Moreover items 2), 3) of the same Definition are fulfilled.
- iii) Y_0 is distributed according to x_0 .

iv) For every $t \in]0, T]$, $\varphi \in C_b(\mathbb{R})$, if we denote $Q^\omega = Q(\cdot, \omega)$, we have

$$\int_{\mathbb{R}} X(t, \xi) \varphi(\xi) d\xi = E^{Q^\omega} \left(\varphi(Y_t) \mathcal{E}_t \left(\int_0^t \mu(ds, Y_s) X(s, Y_s) \right) \right).$$

2) We show now that X^κ approaches X in some sense when $\kappa \rightarrow 0$, where X is the solution to (1.1). This is given in the Lemma 7.3 below.

Lemma 7.3. *Under the assumptions of Theorem 7.1, according to Remark B.2, let X (resp. X^κ) be a solution of (1.1) verifying (2.1) with $\psi(u) = u\Phi^2(u)$ (resp. $\psi_\kappa(u) = u(\Phi^2(u) + \kappa)$), for $u > 0$. We have the following.*

- a) $\lim_{\kappa \rightarrow 0} \sup_{t \in [0, T]} E \left(\|X^\kappa(t, \cdot) - X(t, \cdot)\|_{H^{-1}}^2 \right) = 0$;
- b) $\lim_{\kappa \rightarrow 0} E \left(\int_0^T dt \|\psi(X^\kappa(t, \cdot)) - \psi(X(t, \cdot))\|_{L^2}^2 \right) = 0$;
- c) $\lim_{\kappa \rightarrow 0} \kappa E \left(\int_{[0, T] \times \mathbb{R}} dt d\xi (X^\kappa(t, \xi) - X(t, \xi))^2 \right) = 0$.

Remark 7.4. 1) a) implies of course

$$\lim_{\kappa \rightarrow 0} E \left(\int_0^T dt \|X^\kappa(t, \cdot) - X(t, \cdot)\|_{H^{-1}}^2 \right) = 0.$$

2) In particular Lemma 7.3 b) implies that for each sequence $(\kappa_n) \rightarrow 0$ there is a subsequence, still denoted by the same notation, that

$$\int_{[0, T] \times \mathbb{R}} (\psi(X^{\kappa_n}(t, \xi)) - \psi(X(t, \xi)))^2 dt d\xi \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

3) For every $t \in [0, T]$ $X(t, \cdot) \geq 0$ $d\xi \otimes dP$ a.e. Indeed, for this it will be enough to show that a.s.

$$\int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi) \geq 0 \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}), \quad (7.2)$$

for every $t \in [0, T]$. Since $X \in C([0, T]; \mathcal{S}'(\mathbb{R}))$ it will be enough to show (7.2) for almost all $t \in [0, T]$. This holds true since item 1) in this Remark 7.4, implies the existence of a sequence (κ_n) such that

$$\int_0^T dt \|X^{\kappa_n}(t, \cdot) - X(t, \cdot)\|_{H^{-1}}^2 \xrightarrow{n \rightarrow \infty} 0, \quad \text{a.s.}$$

4) Since ψ is strictly increasing after u_c , then, for P almost all ω , for almost all $(t, \xi) \in [0, T] \times \mathbb{R}$, there is a sequence (κ_n) such that $(X^{\kappa_n}(t, \xi) - X(t, \xi)) \mathbf{1}_{\{X(t, \xi) > u_c\}} \xrightarrow{n \rightarrow \infty} 0$. This follows from item 2) of Remark 7.4.

Since $\Phi^2(u) = 0$ for $0 \leq u \leq u_c$ and X is a.e. non-negative, this implies that $dt d\xi dP$ a.e. we have

$$\Phi^2(X(t, \xi)) (X^{\kappa_n}(t, \xi) - X(t, \xi)) \xrightarrow{n \rightarrow \infty} 0. \quad (7.3)$$

Proof (of Lemma 7.3). By Remark B.2 3. we can write dP -a.s. the following $H^{-1}(\mathbb{R})$ -valued equality.

$$(X^\kappa - X)(t, \cdot) = \int_0^t ds (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)))'' + \sum_{i=0}^N \int_0^t (X^\kappa(s, \cdot) - X(s, \cdot)) e^i dW_s^i.$$

So

$$\begin{aligned} (I - \Delta)^{-1}(X^\kappa - X)(t, \cdot) &= - \int_0^t ds (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))) \\ &\quad + \int_0^t ds (I - \Delta)^{-1} (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))) \\ &\quad + \sum_{i=0}^N \int_0^t (I - \Delta)^{-1} (e^i (X^\kappa(s, \cdot) - X(s, \cdot))) dW_s^i. \end{aligned}$$

After regularization and application of Itô calculus with values in H^{-1} , we will be able to estimate $g^\kappa(t) = \|(X^\kappa - X)(t, \cdot)\|_{H^{-1}}^2$. Taking advantage of the form of $\psi_\kappa - \psi$, we obtain

$$\begin{aligned} g^\kappa(t) &= \sum_{i=1}^N \int_0^t \|e^i (X^\kappa - X)(s, \cdot)\|_{H^{-1}}^2 ds \tag{7.4} \\ &\quad - 2 \int_0^t \langle (X^\kappa - X)(s, \cdot), \psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} \\ &\quad + 2 \int_0^t ds \langle (X^\kappa - X)(s, \cdot), (I - \Delta)^{-1} (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))) \rangle_{L^2} \\ &\quad + 2 \int_0^t ds \langle (X^\kappa - X)(s, \cdot), (I - \Delta)^{-1} e^0 (X^\kappa - X)(s, \cdot) \rangle_{L^2} + M_t^\kappa, \end{aligned}$$

where M^κ is the local martingale

$$M_t^\kappa = 2 \sum_{i=1}^N \int_0^t \langle (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot), (X^\kappa - X)(s, \cdot) e^i \rangle_{L^2} dW_s^i.$$

Indeed, M^κ is a well-defined local martingale because, taking into account (B.1) and Remark B.2, using classical arguments, we can prove that

$$\sum_{i=1}^N \int_0^t |\langle (X^\kappa - X)(s, \cdot), (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot) e^i \rangle_{L^2}|^2 ds < \infty \text{ a.s.}$$

(7.4) gives

$$\begin{aligned}
& g^\kappa(t) + 2 \int_0^t \langle (X^\kappa - X)(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} ds \\
& + 2\kappa \int_0^t \langle (X^\kappa - X)(s, \cdot), (X^\kappa - X)(s, \cdot) \rangle_{L^2} ds \\
& \leq -2\kappa \int_0^t ds \langle (X^\kappa - X)(s, \cdot), X(s, \cdot) \rangle_{L^2} ds + \sum_{i=1}^N \int_0^t \|e^i(X^\kappa - X)(s, \cdot)\|_{H^{-1}}^2 ds \\
& + 2 \int_0^t ds \langle (I - \Delta)^{-1}(X^\kappa - X)(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} \\
& + 2\kappa \int_0^t ds \langle (I - \Delta)^{-1}(X^\kappa - X)(s, \cdot), (X^\kappa - X)(s, \cdot) \rangle_{L^2} \\
& + 2\kappa \int_0^t ds \langle (I - \Delta)^{-1}(X^\kappa - X)(s, \cdot), X(s, \cdot) \rangle_{L^2} \\
& + 2 \int_0^t ds \langle (I - \Delta)^{-1}(X^\kappa - X)(s, \cdot), (e^0(X^\kappa - X)(s, \cdot)) \rangle_{L^2} + M_t^\kappa.
\end{aligned}$$

We use Cauchy-Schwarz and the inequality $2\sqrt{\kappa}b\sqrt{\kappa}c \leq \kappa b^2 + \kappa c^2$, with first $b = \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}$, $c = \|X(s, \cdot)\|_{L^2}$ and then $b = \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{H^{-2}}$, $c = \|X(s, \cdot)\|_{L^2}$. We also take into account the property of H^{-1} -multiplier for e^i , $0 \leq i \leq N$. Consequently there is a constant

$\mathcal{C}(e)$ depending on $(e^i, 0 \leq i \leq N)$ such that

$$\begin{aligned}
g^\kappa(t) &+ 2 \int_0^t \langle (X^\kappa - X)(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} ds \\
&+ 2\kappa \int_0^t \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 ds \\
&\leq \kappa \int_0^t \|(X^\kappa - X)(s, \cdot)\|_{L^2}^2 ds + \kappa \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 \\
&+ C(e) \int_0^t ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{H^{-1}}^2 \\
&+ 2 \int_0^t \|(X^\kappa - X)(s, \cdot)\|_{H^{-2}} \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2} \\
&+ 2\kappa \int_0^t ds g^\kappa(s) + \kappa \int_0^t ds \|(X^\kappa - X)(s, \cdot)\|_{H^{-2}}^2 + \kappa \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 + M_t^\kappa.
\end{aligned} \tag{7.5}$$

Since ψ is Lipschitz, it follows $(\psi(r) - \psi(r_1))(r - r_1) \geq \alpha(\psi(r) - \psi(r_1))^2$, for any $r, r_1 \geq 0$, for some $\alpha > 0$. Consequently, the inequality $2bc \leq b^2\alpha + \frac{c^2}{\alpha}$, with $b, c \in \mathbb{R}$ and the fact that $\|\cdot\|_{H^{-2}} \leq \|\cdot\|_{H^{-1}}$ give

$$\begin{aligned}
&2 \int_0^t ds \|(X^\kappa - X)(s, \cdot)\|_{H^{-2}} \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2} \\
&\leq \int_0^t ds \alpha g^\kappa(s, \cdot) + \int_0^t ds \langle \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)), X^\kappa(s, \cdot) - X(s, \cdot) \rangle_{L^2}.
\end{aligned}$$

So (7.5) yields

$$\begin{aligned}
g^\kappa(t) &+ \int_0^t \langle X^\kappa(s, \cdot) - X(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} ds \\
&+ \kappa \int_0^t ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 ds \\
&\leq 2\kappa \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 + M_t^\kappa + (C(e) + \alpha + 3\kappa) \int_0^t g^\kappa(s) ds.
\end{aligned} \tag{7.6}$$

Taking the expectation we get

$$E(g^\kappa(t)) \leq (C(e) + \alpha + 3\kappa) \int_0^t E(g^\kappa(s)) ds + 2\kappa \int_0^t E(\|X(s, \cdot)\|_{L^2}^2) ds,$$

for every $t \in [0, T]$. By Gronwall lemma we get

$$E(g^\kappa(t)) \leq 2\kappa E \left\{ \int_0^T ds \|X(s, \cdot)\|_{L^2}^2 \right\} e^{(C(e) + \alpha + 3\kappa)T}, \quad \forall t \in [0, T]. \quad (7.7)$$

Taking the supremum and letting $\kappa \rightarrow 0$, item a) of Lemma 7.3 is now established.

We go on with item b). Since ψ is Lipschitz, (7.6) implies that, for $t \in [0, T]$,

$$\begin{aligned} & \int_0^t ds \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2}^2 \\ & \leq \frac{1}{\alpha} ds \left\langle \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)), X^\kappa(s, \cdot) - X(s, \cdot) \right\rangle_{L^2} \\ & \leq \frac{\kappa}{2\alpha} \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 + C(e, \alpha) \int_0^t g^\kappa(s) ds + M_t^\kappa, \end{aligned}$$

where $C(e, \alpha)$ is a constant depending on $e^i, 0 \leq i \leq N$ and α . Taking the expectation for $t = T$, we get

$$E \left(\int_0^T ds \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2}^2 \right) \leq \frac{\kappa}{2\alpha} E \left(\int_0^T ds \|X(s, \cdot)\|_{L^2}^2 \right) + C(e, \alpha) \int_0^T E(g^\kappa(s)) ds.$$

Taking $\kappa \rightarrow 0$, (2.1) and (7.7) provide the conclusion of item b) of Lemma 7.3.

c) Coming back to (7.6), and $t = T$, we have

$$\kappa \int_0^T ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 \leq 2\kappa \int_0^T ds \|X(s, \cdot)\|_{L^2}^2 + M_T^\kappa + (C(e) + \alpha + 3\kappa) \int_0^T ds g^\kappa(s).$$

Taking the expectation we have

$$\kappa E \left(\int_0^T ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 \right) \leq 2\kappa E \left(\int_0^T ds \|X(s, \cdot)\|_{L^2}^2 \right) + (C(e) + \alpha + 3\kappa) E \left(\int_0^T g^\kappa(s) ds \right).$$

Using item a) and the fact that $E \left(\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi \right) < \infty$, the result follows.

Lemma 7.3 is finally completely established.

□

We need now another intermediate lemma concerning the paths of a solution to (1.1).

Lemma 7.5. *For almost all $\omega \in \Omega$, almost all $t \in [0, T]$,*

- 1) $\xi \mapsto \psi(X(t, \xi, \omega)) \in H^1(\mathbb{R})$,
- 2) $\xi \mapsto \Phi(X(t, \xi, \omega))$ is continuous.

Proof. Item 1) is established in [8], see Definition 3.2 and Theorem 3.4. 1) implies that $\xi \mapsto \psi(X(t, \xi, \omega))$ is continuous. See also Remark B.2 1. By the same arguments as in Proposition 4.22 in [6], we can deduce item 2). \square

- 3) We go on with the proof of Theorem 7.1. We keep in mind i), ii), iii), iv) at the beginning of item 1) of the proof. Since Φ is bounded, for P -almost all ω , using Burkholder-Davis-Gundy inequality one obtains

$$E^{Q^\kappa(\cdot, \omega)} (Y_t - Y_s)^4 \leq \text{const}(t - s)^2, \quad (7.8)$$

where const does not depend on ω . On the other hand, for all $Q^\kappa(\cdot, \omega)$, Y_0 is distributed according to x_0 .

At this point, we need a version of Kolmogorov-Centsov theorem for the stable convergence. Let $\tilde{\Omega}_0 = \tilde{\Omega}_1 \times \Omega$ as at the beginning of the proof of Theorem 7.1. We recall that $\tilde{\Omega}_1 = C([0, T]) \times \mathbb{R}$, $Y(\omega_1, \omega) = \omega_1$, \mathcal{H} is the Borel σ -field on $\tilde{\Omega}_1$.

Lemma 7.6. *Let be a sequence $Q^\kappa(\cdot, \omega)$ of random kernel on $\mathcal{H} \times \Omega$. Let us denote by \mathbf{Q}^κ the sequence of marginal laws of the probabilities on $(\tilde{\Omega}_0, \mathcal{H} \otimes \mathcal{F})$ given by $Q^\kappa(\cdot, \omega)P(d\omega)$. Suppose the following.*

- The sequences of marginal laws of the probabilities \mathbf{Q}^κ at zero are tight.
- There are $\alpha, \beta > 0$ such that

$$E^{Q^\kappa(\cdot, \omega)} |Y_t - Y_s|^\alpha \leq C(\omega)(t - s)^{1+\beta}, \quad 0 \leq s \leq t \leq T,$$

for some positive P -integrable random constant C .

Then there is a random kernel Q^∞ on $\mathcal{H} \times \Omega$ and a subsequence (κ_n) such that for every bounded continuous functional $G : \tilde{\Omega}_1 \rightarrow \mathbb{R}$, for every bounded \mathcal{F} -measurable r.v. $F : \Omega \rightarrow \mathbb{R}$, we have

$$\int_{\Omega} F(\omega) dP(\omega) \int_{\tilde{\Omega}_1} G(Y(\omega_1)) Q^{\kappa_n}(d\omega_1, \omega) \xrightarrow{n \rightarrow \infty} \int_{\Omega} F(\omega) dP(\omega) \int_{\tilde{\Omega}_1} G(Y(\omega_1)) Q^\infty(d\omega_1, \omega). \quad (7.9)$$

Proof. Taking the expectation with respect to P we obtain

$$E^{\mathbf{Q}^\kappa} (Y_t - Y_s)^\alpha \leq C_0(t - s)^{1+\beta}, \quad 0 \leq s \leq T,$$

where C_0 is the expectation of C . First, by usual arguments as Chebyshev inequality, one can show the following:

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \sup_{\kappa} \mathbf{Q}^\kappa \{(\omega_1, \omega) \mid |(W^1, \dots, W^N)(\omega)(0)| > \lambda; |\omega_1(0)| > \lambda\} &= 0, \\ \lim_{\delta \rightarrow 0} \sup_{\kappa} \mathbf{Q}^\kappa \{(\omega_1, \omega) \mid m((W^1, \dots, W^N, \omega_1); \delta) > \varepsilon\} &= 0, \forall \varepsilon > 0, \end{aligned}$$

where m denotes the modulus of continuity. By Theorem 4.10 of [18], the sequences of probabilities $\mathbf{Q}^\kappa, \kappa > 0$, on $\tilde{\Omega}_1 \times \Omega$ are tight. Let \mathbf{Q}^{κ_n} be a sequence converging weakly to a probability \mathbf{Q}^∞ on $\mathcal{H} \otimes \mathcal{F}$. Since \mathcal{F} is separable and $C([0, T])^N$, which is space value of process W , is a Polish space equipped with its Borel σ -algebra, according to [23], it is possible to desintegrate \mathbf{Q}^∞ , i.e. there is random kernel $Q^\infty(\cdot, \omega)$ such that for every bounded continuous functional $G : \tilde{\Omega}_1 \rightarrow \mathbb{R}$, for every bounded continuous $\tilde{F} : C([0, T])^N \rightarrow \mathbb{R}$ such that (7.9) holds for every $F = \tilde{F}(W)$, where $W = (W^1, \dots, W^N)$. Since continuous bounded functionals \tilde{F} are dense in $L^2(C([0, T])^N)$ equipped with Wiener measure, (7.9) holds also for any F bounded \mathcal{F} -measurable r.v. with $\mathbf{Q}^\infty(d\omega_1, d\omega) = Q^\infty(d\omega_1, \omega)P(d\omega)$. \square

By (7.8), we apply Lemma 7.6 with $\alpha = 2, \beta = 1$ and we consider the corresponding $Q^{\kappa_n}(\cdot, \omega)$ and the limit random kernel $Q(\cdot, \omega) := Q^\infty(\cdot, \omega)$. We define also the probability $\mathbf{Q} := \mathbf{Q}^\infty$ on $\tilde{\Omega}_0 = \tilde{\Omega}_1 \times \Omega$ according to the conventions introduced before Definition 2.3. In the sequel we denote again by $d\mathbf{Q}^\kappa(\omega_1, \omega) := dP(\omega)Q^\kappa(d\omega_1, \omega)$ and also $Q^{\omega, \kappa} := Q^\kappa(\cdot, \omega), Q^\omega := Q(\cdot, \omega)$.

From Lemma 7.6 derives the following.

Corollary 7.7. *For any bounded random element $F : \tilde{\Omega}_1 \times \Omega \rightarrow \mathbb{R}$ such that for almost all $\omega \in \Omega, F(\cdot, \omega) \in C(\tilde{\Omega}_1)$. Then*

$$\int_{\Omega} dP(\omega) \int_{\tilde{\Omega}_1} (dQ^{\omega, \kappa_n}(\omega_1) - dQ^\omega(\omega_1)) F(Y, \omega) \text{ converges to zero.}$$

Proof. See Appendix A. \square

We need here a technical lemma.

Lemma 7.8. *Let $t \in [0, T], p \in \mathbb{R}$.*

1. *There is $C(p) > 0$ such that*

$$E^{\mathbf{Q}^\kappa} \left(\mathcal{E}_t \left(\int_0^t \mu(ds, Y_s) \right)^p \right) \leq C(p), \quad \forall \kappa > 0.$$

2. For almost all $\omega \in \Omega$, and every $p \in \mathbb{R}$ there is a random constant $C(p, \omega)$ such that the random variables

$$E^{Q^{\omega, \kappa}} \left(\mathcal{E}_t \left(\int_0^t \mu(ds, Y_s) \right)^p \right) \leq C(p, \omega), \quad \forall \kappa > 0. \quad (7.10)$$

Proof. Without restriction of generality we can of course suppose $e^0 = 0$.

1. We can write

$$\begin{aligned} \mathcal{E}_t \left(\int_0^t \mu(ds, Y_s) \right)^p &= \mathcal{E}_t \left(p \int_0^t \mu(ds, Y_s) \right) \exp \left(\frac{p^2 - p}{2} \sum_{i=1}^N \left(\int_0^t e^i(Y_s)^2 ds \right) \right) \\ &\leq \mathcal{E}_t \left(p \int_0^t \mu(ds, Y_s) \right) \exp \left(T \frac{p^2 - p}{2} \sum_{i=1}^N \|e^i\|_\infty^2 \right). \end{aligned}$$

Since $p \int_0^t \mu(ds, Y_s)$ is a (\mathcal{G}_t) - Q^κ -martingale, the result follows.

2. Let $\omega \in \Omega$ excepted on a P -null set. The integrand of the expectation in (7.10) equals $\exp(J_1(n) + J_2(n))$, where

$$J_1(n) := p \sum_{i=1}^N \left(W_t^i e^i(Y_t) - \frac{1}{2} \int_0^t e^i(Y_s)^2 ds - \frac{1}{2} \int_0^t W_s^i (e^i)''(Y_s) \Phi^2(X(s, Y_s, \omega)) ds \right)$$

and $J_2(n) = -p \sum_{i=1}^N \int_0^t W_s^i (e^i)'(Y_s) dY_s$. For each ω , $\exp((J_1(n)))$ is bounded, so it remains to prove the existence of a random constant $C(p, \omega)$ such that for every $0 \leq i \leq N$

$$E^{Q^{\omega, \kappa}} \left(\exp \left(-p \int_0^t W_s^i (e^i)'(Y_s) dY_s \right) \right) \leq C(p, \omega). \quad (7.11)$$

Since $-p \int_0^t W_s^i (e^i)'(Y_s) dY_s$ is a $Q^{\omega, \kappa}$ -martingale,

$$\mathcal{E}_t^\kappa := \exp \left(-p \int_0^t W_s^i (e^i)'(Y_s) dY_s - \frac{p^2}{2} \int_0^t (W_s^i)^2 (e^i)'^2(Y_s) \Phi_\kappa^2(X^\kappa(s, Y_s, \omega)) ds \right)$$

is an (exponential) martingale, with respect to $Q^{\omega, \kappa}$. Consequently the left-

hand side of (7.11) is bounded by

$$\begin{aligned} & E^{Q^{\omega, \kappa}} \left(\mathcal{E}_t^\kappa \exp \left(\frac{p^2}{2} \int_0^t (W_s^i)^2 ((e^i)')^2(Y_s) \Phi_\kappa^2(X^\kappa(s, Y_s, \omega)) ds \right) \right) \\ & \leq C(p, \cdot) := \exp \left(\frac{p^2}{2} \|(e^i)'\|_\infty^2 \left(\|\Phi\|_\infty^2 + 1 \right) \int_0^T (W_s^i)^2 ds \right). \end{aligned}$$

This concludes the proof of Lemma 7.8. □

Lemma 7.9. *We fix $\omega \in \Omega$ excepted on some P -null set. Let $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ continuous with compact support. The random variables*

$$E^{Q^{\omega, \kappa}} \left(\int_0^T |\Phi_\kappa(X^\kappa(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega))| \varphi(r, Y_r) dr \right)$$

converge to zero a.s. and in $L^p(\Omega, P)$ for every $p \geq 1$, when $\kappa \rightarrow 0$.

Proof. Let $\omega \in \Omega$. Since φ has compact support, by Cauchy-Schwarz with respect to the measure $\varphi(r, Y(r))dr$ on $[0, T]$, it is enough to prove that

$$E^{Q^{\omega, \kappa}} \left(\int_0^T (\Phi_\kappa(X^\kappa(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)))^2 \varphi(r, Y_r) dr \right) \quad (7.12)$$

converges to zero. Since Φ is bounded it is enough to prove the convergence to zero for almost all $\omega \in \Omega$. In order not to overcharge the notation, in this proof we will omit the argument of ω of Y . By Fubini's theorem the left-hand side of (7.12) equals

$$\int_0^T dr E^{Q^{\omega, \kappa}} \left((\Phi_\kappa(X^\kappa(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)))^2 \varphi(r, Y_r) \right).$$

Using also Lebesgue dominated convergence theorem, given a sequence (κ_n) , when $n \rightarrow \infty$, it is enough to find a subsequence (κ_{n_ℓ}) such that for all $r \in [0, T]$ outside a possible Lebesgue null set

$$E^{Q^{\omega, \kappa_{n_\ell}}} \left\{ \left(\Phi_{\kappa_{n_\ell}}(X^{\kappa_{n_\ell}}(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)) \right)^2 \varphi(r, Y_r) \right\} \rightarrow_{\ell \rightarrow \infty} 0.$$

We set $Z_r(\omega_1, \omega) = \mathcal{E}_r \left(\int_0^\cdot \mu(\omega)(ds, Y_s(\omega_1)) \right)$. We will substitute from now on (n_ℓ) with n .

Taking into account Lemma 7.8 and Cauchy-Schwarz with respect to the finite measure $Z_r(\omega_1, \omega)Q^{\omega, \kappa_n}(d\omega_1)$, it is enough to prove that for r a.e.

$$E^{Q^{\omega, \kappa_n}} \left\{ (\Phi_{\kappa_n}(X^{\kappa_n}(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)))^2 \varphi(r, Y_r) Z_r(\cdot, \omega) \right\}$$

converges to zero when n goes to infinity.

Since X^κ constitutes the family of μ -marginal weighted laws of Y under $Q^{\omega, \kappa}$, previous expression gives

$$\begin{aligned} & \int_{\mathbb{R}} |\varphi(r, y) (\Phi_{\kappa_n}(X^{\kappa_n}(r, y, \omega)) - \Phi(X(r, y, \omega)))^2 X^{\kappa_n}(r, y, \omega) dy \\ & \leq I_{11}(\kappa_n, r) + I_{12}(\kappa_n, r) + I_{13}(\kappa_n, r) + I_{14}(\kappa_n, r), \end{aligned} \quad (7.13)$$

where we have developed the square in the first line of (7.13) using the definition of ψ and Φ_κ . Indeed we get

$$\begin{aligned} I_{11}(\kappa, r) &= \int_{\mathbb{R}} dy |\varphi(r, y)| |\psi(X^\kappa(r, y, \omega)) - \psi(X(r, y, \omega))|, \\ I_{12}(\kappa, r) &= \int_{\mathbb{R}} dy |\varphi(r, y)| \Phi^2(X^\kappa(r, y, \omega)) |(X - X^\kappa)(r, y, \omega)|, \\ I_{13}(\kappa, r) &= \int_{\mathbb{R}} dy |\varphi(r, y)| \kappa |X^{\kappa_n} - X|(r, y, \omega), \\ I_{14}(\kappa, r) &= \int_0^T dr \int_{\mathbb{R}} dy \kappa |X(r, y, \omega)| |\varphi(r, y)|. \end{aligned}$$

We denote $I_{1j}(\kappa) := \int_0^T I_{1j}(\kappa, r) dr$, $j = 1, 2, 3, 4$. It is of course enough to prove that, up to a subsequence $I_{1j}(\kappa_n) \rightarrow 0$, $j = 1, 2, 3, 4$, where $n \rightarrow \infty$. By Cauchy-Schwarz, $I_{11}^2(\kappa)$ is bounded by

$$\|\varphi\|_{L^2([0, T] \times \mathbb{R})}^2 \int_0^T dr \int_{\mathbb{R}} (\psi(X^\kappa(r, y, \omega)) - \psi(X(r, y, \omega)))^2 dy.$$

This converges to zero according to Remark 7.4 2), after extracting a subsequence κ_n (not depending on ω). The square of the expectation of $I_{12}(\kappa)$ is bounded by

$$\|\varphi\|_{L^2([0, T] \times \mathbb{R})}^2 \int_{[0, T] \times \mathbb{R}} dr dy \Phi^4(X(r, y, \omega)) |X^\kappa - X|^2(r, y, \omega).$$

The expectation of previous expression is indeed uniformly bounded in κ because of (7.6) and (7.7). So the family of r.v.

$\Phi^2(X^{\kappa_n}(r, y, \omega)) |(X - X^\kappa)(r, y, \omega)|$ is uniformly integrable with respect to the finite measure $dP(\omega)|\varphi|(t, y)dtdy$. Consequently $I_{12}(\kappa)$ goes to zero because of (7.3) in Remark 7.4 4).

$I_{13}^2(\kappa)$ is bounded by $\kappa \|\varphi\|_{L^2([0, T] \times \mathbb{R})}^2 \int_{[0, T] \times \mathbb{R}} drdy |X^\kappa - X|^2(r, y, \omega)$. After extracting a subsequence κ_n , previous expression converges to zero because of Lemma 7.3 c). Finally $I_{14}(\kappa) \xrightarrow{n \rightarrow \infty} 0$ by Cauchy-Schwarz and the fact that

$\int_{[0, T] \times \mathbb{R}} drdy X^2(r, y, \omega) < \infty$ dP -a.s. This establishes the proof of Lemma 7.9. \square

Let (κ_n) be the sequence introduced by the statement of Lemma 7.6. Previous Corollary 7.7 and Lemma 7.9 have the following consequences. Let $Q(d\omega_1, \omega)$ be the random kernel introduced in Lemma 7.5 and the related probability $\mathbf{Q}(d\omega_1, d\omega) = dP(\omega)Q(d\omega_1, \omega)$.

Corollary 7.10. *Let $R : \Omega \rightarrow \mathbb{R}$ be a bounded measurable r.v. Let $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function with compact support. The sequence*

$$\int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} dQ^{\omega, \kappa_n}(\omega_1) \int_0^T \varphi(r, Y_r) \Phi_{\kappa_n}^2(X^{\kappa_n}(r, Y_r, \omega)) dr \quad (7.14)$$

converges, when $n \rightarrow \infty$, to

$$\int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} dQ(\omega_1, \omega) \int_0^T \varphi(r, Y_r) \Phi^2(X(r, Y_r, \omega)) dr. \quad (7.15)$$

Proof. We split the difference between (7.14) and (7.15) which gives $I_1(n) + I_2(n)$ where

$$I_1(n) = \int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} dQ^{\omega, \kappa_n}(\omega_1) \left(\int_0^T \varphi(r, Y_r) (\Phi_{\kappa_n}^2(X^{\kappa_n}(r, Y_r, \omega)) dr - \Phi^2(X(r, Y_r, \omega)) dr) \right),$$

and

$$I_2(n) = \int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} (Q^{\omega, \kappa_n}(d\omega_1) - Q(d\omega_1, \omega)) \left(\int_0^T \varphi(r, Y_r) \Phi^2(X(r, Y_r, \omega)) dr \right).$$

We have

$$|I_1(n)| \leq 2 \|\Phi\|_{\infty} \|R\|_{\infty} \int_{\Omega} dP(\omega) \int_{\tilde{\Omega}_1} dQ^{\omega, \kappa_n}(\omega_1) \left(\int_0^T |\varphi(r, Y_r)| |\Phi_{\kappa_n}(X^{\kappa_n}(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega))| dr \right).$$

$I_1(n)$ converges to zero by Lemma 7.9. Concerning $I_2(n)$, by Fubini's theorem, we first observe that

$$I_2(n) = \int_0^T dr \int_{\Omega} dP(\omega) \left(\int_{\tilde{\Omega}_1} (Q^{\omega, \kappa_n}(d\omega_1) - Q(d\omega_1, \omega)) \varphi(r, Y_r) \Phi^2(X(r, Y_r, \omega)) R(\omega) \right).$$

We apply now Corollary 7.7, setting for fixed r , $F(\omega_1, \omega) = R(\omega)\varphi(r, \omega_1(r))\Phi^2(X(r, \omega_1(r), \omega))$ and the result follows. □

5) We go on with the proof of Theorem 7.1.

We want now to prove that $Y(\cdot, \omega)$ is a (weak) solution of

$$Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \cdot)) d\beta_s^\omega,$$

for some Brownian motion β^ω . This is related to item 1) of Definition 2.10 with $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$. According to Remark 2.11 c), for this it is enough to show that for dP -a.s. ω $Y(\cdot, \omega)$ is a solution of the following (local) martingale problem. For every $f \in C^{1,2}([0, T] \times \mathbb{R})$ with compact support, the process

$$Z_t^f := f(t, Y_t) - f(0, Y_0) - \frac{1}{2} \int_0^t \partial_{xx}^2 f(r, Y_r) \Phi^2(X(r, Y, \omega)) dr - \int_0^t \partial_r f(r, Y_r) dr,$$

is a (local) martingale under Q^ω .

This will be a consequence of the lemma below.

Lemma 7.11. *Let F be a bounded \mathcal{F}_s -measurable, let $\mathcal{A} : C([0, s]) \rightarrow \mathbb{R}$ bounded continuous functional. Let $G = \mathcal{A}(Y_r, r \leq s)$. Then, for $0 \leq s \leq t \leq T$ we have*

$$E(FE^{Q^\omega}(GZ_t^f)) = E(FE^{Q^\omega}(GZ_s^f)). \quad (7.16)$$

Proof. We set

$$Z_t^{\kappa, f} = f(t, Y_t) - f(0, Y_0) - \frac{1}{2} \int_0^t \partial_{xx}^2 f(r, Y_r) \Phi_\kappa^2(X^\kappa(r, Y, \omega)) dr - \int_0^t \partial_r f(r, Y_r) dr.$$

Let (κ_n) be the sequence introduced by Lemma 7.6. The difference of the right and left-hand side of (7.16) is the sum $(I_1 + I_2 + I_3)(\kappa_n)$ where

$$\begin{aligned} I_1(\kappa) &= E\left(F(E^{Q^\omega}(GZ_t^f) - E^{Q^{\omega, \kappa}}(GZ_t^{\kappa, f}))\right) \\ I_2(\kappa) &= E\left(FE^{Q^{\omega, \kappa}}(G(Z_t^{\kappa, f} - Z_s^{\kappa, f}))\right) \\ I_3(\kappa) &= E\left(F(E^{Q^{\omega, \kappa}}(GZ_s^{\kappa, f}) - E^{Q^\omega}(GZ_s^f))\right). \end{aligned}$$

$I_1(\kappa_n) + I_3(\kappa_n)$ converges to zero by Lemma 7.6, Corollary 7.10 and Lemma 7.9. $I_2(\kappa_n) = 0$ since $Z^{\kappa, f}$ is a $Q^{\kappa, \omega}$ -martingale. □

6) After previous intermediary step 5) we need to show that Y defined in (7.1) is a weak-strong solution of DSDE(γ, x_0) with $\gamma(s, \xi, \omega) = \Phi(X(s, \xi, \omega))$ and X is a solution of (1.1). We recall that the kernel $Q(\cdot, \omega)$ has been introduced through Lemma 7.6 on $(\tilde{\Omega}_1 \times \Omega, \mathcal{H} \otimes \mathcal{F})$. So, according to step 5), under $Q^\omega := Q(\cdot, \omega)$, Y is a martingale with $[Y]_t = \int_0^t \Phi^2(X(s, Y_s, \omega)) ds$. To conclude the proof of item 1) in Definition 2.10, it remains to construct the suitable required process B . For this, we need to enlarge the probability space $\tilde{\Omega}_1$ as follows. We set $\Omega_1 = \tilde{\Omega}_1 \times C([0, T]; \mathbb{R})$; the second component allows to define a Brownian motion. By an abuse of notation, we set again $Y_t(\omega_1, \omega) = \omega_1^0(t)$, this time with $\omega_1 = (\omega_1^0, \omega_1^1)$. In spite of adding the component ω_1^1 , in step 5) we have already shown $Q^\omega := Q(\cdot, \omega)$, is by construction the law of $Y(\cdot, \omega)$. We need to construct a process B on $\Omega \times \Omega_1$, such that for almost all ω , $B(\cdot, \omega)$ is a Q^ω -Brownian motion and (2.7) holds for $\gamma(t, \cdot, \omega) = \Phi(X(t, \cdot, \omega))$.

On Ω_1 we set $\beta_t(\omega_1) = \omega_1^1(t)$. We equip $C([0, T]; \mathbb{R})$ in Ω_1 with the Wiener measure \mathcal{W} so that β is a standard Brownian motion on Ω_1 . β can also be considered to be a Brownian motion on $\Omega_0 = \Omega_1 \times \Omega$ which is Q^ω -independent of Y for P -almost all $\omega \in \Omega$. Of course β is also independent of Y on the probability space $(\Omega_1 \times \Omega, \mathcal{B}(\Omega_1) \times \mathcal{F}, d\mathbf{Q}(\omega_1, \omega) := Q^\omega(d\omega_1)dP(\omega))$. β is also independent of (\mathcal{F}_t) .

We set now

$$B_t(\cdot, \omega) = \int_0^t dY_s(\cdot, \omega) 1_{\{\gamma(s, \xi, \omega) \neq 0\}} \frac{1}{\gamma(s, \xi, \omega)} + \int_0^t 1_{\{\gamma(s, \xi, \omega) = 0\}} d\beta_s.$$

Now for Q^ω -a.s. the quadratic variation of the Q^ω -martingale $B(\cdot, \omega)$ is t , so that, by Lévy characterization theorem, $B(\cdot, \omega)$ is a Brownian motion under Q^ω .

It remains to show items 2) and 3) of the definition of weak-strong solution. Let (\mathcal{Y}_t) be the canonical filtration of the process $Y(\cdot, \omega)$. Item 3) follows because of item 1) and because $\gamma(t, \cdot, \omega) = \Phi(X(t, \cdot, \omega))$ is progressively measurable. Concerning item 2) we see that under \mathbf{Q} defined by P and the kernel $Q(\cdot, \omega)$, W^1, \dots, W^N are \mathbf{Q} -martingales with (\mathcal{G}_t) as defined in Definition 2.10. Indeed let F be a bounded \mathcal{F}_s -measurable random variable and G be a bounded \mathcal{Y}_s -measurable r.v. Let $1 \leq i \leq N$. By item 3) $E^{Q^\omega}(G)$ is \mathcal{F}_s -measurable, so

$$E^{\mathbf{Q}}((W_t^i - W_s^i)FG) = E((W_t^i - W_s^i)FE^{Q^\omega}(G)) = 0,$$

since W^i is an \mathcal{F}_s -martingale.

7) The final step consists in proving that X is the family of μ -marginal weighted laws of Y . We need to show that for almost all ω , for every $t \in [0, T]$, $\varphi \in \mathcal{S}(\mathbb{R})$, that

$$\int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi, \omega) = E^{Q^\omega} \left(\varphi(Y_t) \mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s) \right) \right).$$

Since both sides of previous equality are \mathcal{F}_t -measurable, given a bounded \mathcal{F}_t -measurable random variable R it will be enough to show that

$$\int_{\Omega} dP(\omega) R(\omega) \int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi, \omega) = \int_{\Omega} dP(\omega) R(\omega) E^{Q^\omega} \left(\varphi(Y_t) \mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s) \right) \right). \quad (7.17)$$

Let $\omega \in \Omega$ outside some P -null set.

By step 1) of the proof of this Theorem 7.1, we know that X^κ fulfills, for almost all ω ,

$$\int_{\mathbb{R}} d\xi X^\kappa(t, \xi) \varphi(\xi) = E^{Q^{\kappa(\cdot, \omega)}} \left(\varphi(Y_t) \mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s) \right) \right),$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. Consequently if (κ_n) is the sequence obtained via Lemma 7.5, we have

$$\int_{\Omega} dP(\omega) R(\omega) \int_{\mathbb{R}} d\xi X^{\kappa_n}(t, \xi) \varphi(\xi) = \int_{\Omega} dP(\omega) R(\omega) E^{Q^{\omega, \kappa_n}} \left(\varphi(Y_t) \mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y) \right) \right), \quad (7.18)$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$.

Since $t \mapsto X(t, \cdot)$ is continuous from $[0, T]$ to $\mathcal{S}'(\mathbb{R})$ and the right-hand side of (7.17) is continuous on $[0, T]$ for fixed $\varphi \in \mathcal{S}(\mathbb{R})$, it is enough to show (7.17) for almost all $t \in [0, T]$.

Now for almost all t , the left-hand side of (7.17) is approached by the left-hand side of (7.18). Let us fix $t \in [0, T]$. It remains to show that the right-hand side of (7.17) is the limit of the right-hand side of (7.18). We fix $\omega \in \Omega$ outside a null set. We set $\mathcal{E}_t := \mathcal{E}_t \left(\int_0^\cdot \mu(ds, Y_s) \right)$, $t \in [0, T]$. By Theorem 2 of [25] and uniform integrability arguments, similarly as after (2.5), we have

$$\mathcal{E}_t = \exp(\psi_\omega(Y)),$$

where $\psi_\omega : \tilde{\Omega}_1 \rightarrow \mathbb{R}$ is a continuous modification of

$$\omega \mapsto \left(\eta \mapsto \int_0^\eta e^i(\eta_s) dW_s^i - \frac{1}{2} \int_0^\eta e^i(\eta_s)^2 ds \right).$$

Indeed, previous random field, indexed by $\eta \in \tilde{\Omega}_1$, admits a continuous modification; to prove this we make use of Kolmogorov-Centsov theorem and Doob's inequality, which says that for any $0 \leq i \leq N$, there is a constant $\text{const} = \text{const}((e^i)')$ with

$$E \left(\left| \int_0^t (e^i(\eta_s^1) - e^i(\eta_s^2)) dW_s^i \right|^4 \right) \leq \text{const} \sup_{s \in [0, T]} |\eta^1 - \eta^2|^2(s), \quad \eta^1, \eta^2 \in \tilde{\Omega}_1.$$

At this point we fix $M > 0$. We decompose the difference of the right-hand sides of (7.18) and (7.17) as

$$J_1(n, M) + J_2(n, M) + J_3(n, M), \quad (7.19)$$

where

$$\begin{aligned} J_1(n, M) &= \int_{\Omega} dP(\omega) R(\omega) E^{Q^{\omega, \kappa_n}} (\varphi(Y_t) \mathcal{E}_t - \varphi(Y_t)(\mathcal{E}_t \wedge M)), \\ J_2(n, M) &= \int_{\Omega} dP(\omega) R(\omega) (E^{Q^{\omega, \kappa_n}} - E^{Q^\omega}) (\varphi(Y_t)(\mathcal{E}_t \wedge M)), \\ J_3(n, M) &= \int_{\Omega} dP(\omega) R(\omega) E^{Q^\omega} (\varphi(Y_t)(\mathcal{E}_t \wedge M) - \varphi(Y_t) \mathcal{E}_t). \end{aligned}$$

Setting $\mathbf{Q}^{\kappa_n}(d\omega, d\omega_1) = dP(\omega) Q^{\omega, \kappa_n}(d\omega_1)$, by Cauchy-Schwarz and Chebyshev inequalities, for every $p > 1$, we have

$$|J_1(n, M)| = \left| \int_{\Omega_1 \times \Omega} d\mathbf{Q}^{\kappa_n} \varphi(Y_t) \mathcal{E}_t 1_{\{\mathcal{E}_t > M\}} \right| \leq \|\varphi\|_{\infty} \frac{E^{Q^{\kappa_n}}(\mathcal{E}_t^p)}{M^{p-1}}.$$

By Lemma 7.8, we get $\sup_n |J_1(n, M)| \rightarrow 0$ if $M \rightarrow \infty$. By a similar reasoning, replacing $Q^{\kappa_n}(d\omega, d\omega_1)$ with $Q(d\omega, d\omega_1) = dP(\omega) Q^\omega(d\omega_1)$, we can prove that $\sup_n |J_3(n, M)| \rightarrow 0$. Let $\varepsilon > 0$. Let M such that $\sup_n |J_1(n, M) + J_3(n, M)| \leq \varepsilon$. On the other hand we have

$$J_2(n, M) = \int_{\Omega} dP(\omega) R(\omega) \left(E^{Q^{\omega, \kappa_n}} - E^{Q^\omega} \right) (\varphi(Y_t)(\psi_\omega(Y) \wedge M)).$$

Since for almost all ω , $F(\eta, \omega) := R(\omega) \varphi(\eta(t)) \psi_\omega(\eta)$ is bounded and continuous, Corollary 7.7 implies that $J_2(n, M)$ goes to zero when $n \rightarrow \infty$.

Taking the limsup in (7.19) we get

$$\limsup_{n \rightarrow \infty} |J_1(n, M) + J_2(n, M) + J_3(n, M)| \leq \varepsilon.$$

Since ε is arbitrarily small, we get $\lim_{n \rightarrow \infty} |J_1(n, M) + J_2(n, M) + J_3(n, M)| = 0$ and the result follows. \square

A Technicalities

Proposition A.1. *Let Y_0 be distributed according to x_0 . Let $a : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function such there are $0 < c < C$ with $c \leq a(s, \xi) \leq C$, $\forall (s, \xi) \in [0, T] \times \mathbb{R}$. We fix $0 \leq r \leq t \leq T$. We set $a_n(t, x) = \int_{\mathbb{R}} \rho_n(x - y) a(t, y) dy$ where (ρ_n) is the usual sequence of mollifiers converging to the Dirac delta. The unique solutions S^n to $S_t^n = Y_0 + \int_r^t a_n(s, S_s^n) dB_s$, B being a classical Wiener process, converges in law to the (weak unique solution) of $S_t = Y_0 + \int_r^t a(s, S_s) dB_s$.*

Proof. The proof follows by standard arguments, see Stroock-Varadhan ([27], Problem 7.3.3), tightness and Kolmogorov-Centsov type arguments. For a detailed proof, the reader may consult [7]. \square

Proof (of Corollary 7.7).

By (7.8), the family $(Q^{\kappa_\ell}, \ell \in \mathbb{N}, \omega \in \Omega)$ is tight. So, for every positive integer n there exists a compact subset K_n of $\tilde{\Omega}_1$ such that

$$Q^{\kappa_\ell}(K_n^c, \omega) < \frac{1}{n}, \forall \ell \in \mathbb{N}, \omega \in \Omega. \quad (1.1)$$

Since each $C(K_n) := C(K_n; \mathbb{R})$ is separable with respect to the sup-norm $\|\cdot\|_\infty$ then $C(K_n), \|\cdot\|_\infty$ is a separable Banach space. So we apply Appendix 1, Lemma A.1.4 in [21], to the map $\Omega \ni \omega \mapsto F(\cdot|_{K_n}, \omega) \in C(K_n)$, where $F(\cdot|_{K_n}, \omega)$ denotes the map $K_n \ni \eta \mapsto F(\eta, \omega)$. Therefore we can find a sequence $\tilde{F}_{n,k} : \Omega \rightarrow C(K_n)$, $\omega \mapsto \tilde{F}_{n,k}(\cdot, \omega) \in K_n$ such that for $\|F\|_\infty := \sup_{\eta \in \tilde{\Omega}_1, \omega \in \Omega} |F(\eta, \omega)|$, we have

$$\|\tilde{F}_{n,k}\|_\infty \leq 1 + \|F\|_\infty, \quad \tilde{F}_{n,k}(\Omega) \subset \{\tilde{g}_{n,k}^{(1)}, \dots, \tilde{g}_{n,k}^{(N_{n,k})}\} \subset C(K_n),$$

where $\tilde{g}_{n,k}^{(i)} \neq \tilde{g}_{n,k}^{(j)}$ if $i \neq j$, and for all $\omega \in \Omega$

$$\sup_{\eta \in K_n} |F(\eta, \omega) - F_{n,k}(\eta, \omega)| \rightarrow 0, \quad (1.2)$$

as $k \rightarrow \infty$. Clearly, for all $\omega \in \Omega$, $\tilde{F}_{n,k}(\cdot, \omega) = \sum_{j=1}^{N_{n,k}} \tilde{g}_{n,k}^{(j)} 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\cdot, \omega)$.

By Tietze's extension theorem there exist extensions $g_{n,k}^{(1)}, \dots, g_{n,k}^{(N_{n,k})} \in C(\tilde{\Omega}_1)$ of $\tilde{g}_{n,k}^{(1)}, \dots, \tilde{g}_{n,k}^{(N_{n,k})}$ such that for all $1 \leq j \leq N_{n,k}$, $\sup_{\eta \in \tilde{\Omega}_1} |g_{n,k}^{(j)}(\eta)| \leq \sup_{\eta \in \tilde{K}_n} |\tilde{g}_{n,k}^{(j)}(\eta)|$. Now we define $F_{n,k} : \Omega \rightarrow C(\tilde{\Omega}_1)$, $\omega \mapsto F_{n,k}(\cdot, \omega)$ by

$$F_{n,k}(\cdot, \omega) = \sum_{j=1}^{N_{n,k}} g_{n,k}^{(j)} 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\cdot, \omega).$$

Clearly, still

$$\|F_{n,k}\|_\infty \leq 1 + \|F\|_\infty. \quad (1.3)$$

Note that for all $\eta \in \tilde{\Omega}_1$

$$\tilde{F}_{n,k}(\eta, \omega) = \sum_{j=1}^{N_{n,k}} g_{n,k}^{(j)}(\eta) 1(\eta)_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\eta, \omega),$$

hence of the form that Lemma 7.6 applies. Therefore using the standard notation $\mu(f) := \int f d\mu$, for a measure μ and a function f , we can argue as follows. Fix $n \in \mathbb{N}$. Then for all $\ell, k \in \mathbb{N}$

$$\begin{aligned} & \left| \int Q^{\kappa_\ell}(F(\cdot, \omega), \omega) P(d\omega) - \int Q(F(\cdot, \omega), \omega) P(d\omega) \right| \\ & \leq \left| \int Q^{\kappa_\ell}(F(\cdot, \omega) 1_{K_n}, \omega) P(d\omega) - \int Q(F(\cdot, \omega) 1_{K_n}, \omega) P(d\omega) \right| + \frac{2}{n} \|F\|_\infty \\ & \leq \int \underbrace{Q^{\kappa_\ell}(|F(\cdot, \omega) - F_{n,k}(\cdot, \omega)| 1_{K_n}, \omega)}_{\leq \sup_{\eta \in K_n} |F(\eta, \omega) - F_{n,k}(\eta, \omega)|} P(d\omega) \\ & + \left| \int Q^{\kappa_\ell}(F_{n,k}(\cdot, \omega), \omega) P(d\omega) - \int Q(F_{n,k}(\cdot, \omega), \omega) P(d\omega) \right| \\ & + \frac{2}{n} (1 + \|F\|_\infty) + \int Q(|F(\cdot, \omega) - F_{n,k}(\cdot, \omega)|, \omega) 1_{K_n} P(d\omega) + \frac{2}{n} \|F\|_\infty. \end{aligned}$$

The first inequality is a consequence of (1.1), the second one of (1.1) and (1.3). Now, letting first $\ell \rightarrow \infty$ (using Lemma 7.6), then $k \rightarrow \infty$ (using (1.2)) and finally $n \rightarrow \infty$, the assertion follows. \square

B Uniqueness for the porous media equation with noise

We state here a general uniqueness lemma which only holds under even weaker hypotheses than Assumption 1.1 i.e. $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz and that the functions belong to $W^{1,\infty}$.

Theorem B.1. *Let $x_0 \in \mathcal{S}'(\mathbb{R}^d)$ and suppose $\psi : \mathbb{R} \rightarrow \mathbb{R}$ to be Lipschitz. Then equation (1.1) admits at most one solution among the random fields $X :]0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ such that*

$$\int_{[0,T] \times \mathbb{R}} X^2(s, \xi) ds d\xi < \infty \quad a.s. \quad (\text{B.1})$$

Remark B.2. 1. *Suppose moreover that $e^i, 0 \leq i \leq N$, belong to H^1 . If $x_0 \in L^2$ or ψ is non-degenerate then Theorem 3.4 of [8] provides an existence theorem for (1.1). It states the existence of a random field X such that*

$$E \left(\int_{[0,T] \times \mathbb{R}} X^2(s, \xi) ds d\xi \right) < \infty,$$

such that $t \mapsto X(t, \cdot)$ belongs to $C([0, T]; H^{-1}(\mathbb{R}))$ and $t \mapsto \int_0^t \psi(X(s, \cdot)) ds \in C([0, T]; H^1(\mathbb{R}))$ a.s.

2. So, under the assumption of item 1., the solution X is unique among those fulfilling (B.1).

3. X of point ii) fulfills the equation, for almost all ω , in H^{-1}

$$X(t, \cdot) = x_0 + \int_0^t \Delta(\psi(X(s, \cdot))) ds + \int_0^t \mu(ds, \cdot) X(s, \cdot), \quad t \in [0, T]. \quad (2.1)$$

The proof of Theorem B.1 is a consequence of the result stated in Theorem B.1 of [24], see also [7].

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