Martingale driven BSDEs, PDEs and other related determinisic problems

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July 20th 2017

Abstract. We focus on a class of BSDEs driven by a cadlag martingale and corresponding Markov type BSDE which arise when the randomness of the driver appears through a Markov process. To those BSDEs we associate a deterministic problem which, when the Markov process is a Brownian diffusion, is nothing else but a parabolic type PDE. The solution of the deterministic problem is intended as *decoupled mild solution*, and it is formulated with the help of a time-inhomogeneous semigroup.

MSC 2010 Classification. 60H30; 60H10; 35S05; 60J35; 60J60; 60J75.

KEY WORDS AND PHRASES. Decoupled mild solutions; Martingale problem; cadlag martingale; pseudo-PDE; Markov processes; backward stochastic differential equation.

1 Introduction

Markovian backward stochastic differential equations (BSDEs) are BSDEs in the sense of [28] involving a forward dynamics described by a Markov (often a diffusion) process X. Those are naturally linked to a parabolic PDE, which constitute a particular deterministic problem. In particular, under reasonable conditions, which among others ensure well-posedness, the solutions of BSDEs produce *viscosity* type solutions for the mentioned PDE. In this paper we focus on *Pseudo-PDEs* which are the corresponding deterministic problems associated to the case of a Markovian BSDE when this is driven by a cadlag martingale and when the underlying forward process is a general Markov process. In that case the concept of viscosity solution (based on comparison theorems) is not

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completely appropriated. For this we propose a new type of solution called *decoupled mild* which extends the usual notion of mild solution which is very familiar to the experts of PDEs. We establish an existence and uniqueness theorem in the class of Borel functions having a certain growth condition.

In the Brownian framework, BSDEs were introduced first by E. Pardoux and S. Peng in [28]. An interesting particular case appears when the random dependence of the driver generally denoted by f comes through a diffusion process X and the terminal condition only depends on X_T. The solution, when it exists, is usually indexed by the starting time s and starting point x of the forward diffusion $X = X^{s,x}$, and it is expressed by

$$\left\{\begin{array}{lcl} X_t^{s,x} &=& x + \int_s^t \mu(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dB_r, \quad t \in [0,T] \\ Y_t^{s,x} &=& g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \quad t \in [0,T], \end{array}\right. \tag{1.1}$$

where B is a Brownian motion. In [30] and in [29] previous Markovian BSDE was linked to the semilinear PDE

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}\sum\limits_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_i x_j} u + \sum\limits_{i\leq d}\mu_i \partial_{x_i} u + f((\cdot,\cdot), u, \sigma\nabla u) = 0 \quad \text{ on } [0,T[\times\mathbb{R}^d \\ u(T,\cdot) = g. \end{array}\right. \tag{1.2}$$

In particular, if (1.2) has a classical smooth solution u then $(Y^{s,x}, Z^{s,x}) := (u(\cdot, X_\cdot^{s,x}), \sigma\nabla u(\cdot, X_\cdot^{s,x}))$ solves the second line of (1.1). Conversely, only under the Lipschitz type conditions on μ, σ, f, g, the solution of the BSDE can be expressed as a function (u, v) of the forward process, i.e. $(Y^{s,x}, Z^{s,x}) = (u(\cdot, X_\cdot^{s,x}), v(\cdot, X_\cdot^{s,x}))$, see [19]. When f and g are continuous, u is a viscosity solution of (1.2). In chapter 13 of [5], under some specific conditions on the coefficients of a Brownian BSDE, one produces a solution in the sense of distributions of the parabolic PDE. Later, a first notion of mild solution of the PDE was used in [2]. In [23] v was associated with a generalized form of $\sigma\nabla u$. Excepted in the case when previous u has some minimal differentiability properties, it is difficult to say something more on v. To express v in the general case, for instance when u is only a viscosity solution of the PDE, is not an easy task. Some authors call this the *identification problem*.

In [4] the authors introduced a new kind of Markovian BSDE including a term with jumps generated by a Poisson measure, where an underlying forward process X solves a jump diffusion equation with Lipschitz type conditions. They associated with it an Integro-Partial Differential Equation (in short IPDE) in which some non-local operators are added to the classical partial differential maps, and proved that, under some continuity and monotonicity conditions on the coefficients, the BSDE provides a viscosity solution of the IPDE. Concerning the study of BSDEs driven by more general martingales than Brownian motion, we have already mentioned BSDEs driven by Poisson measures. In this respect, more recently, BSDEs driven by marked point processes were introduced in [13], see also [3]; in that case the underlying process does not contain any diffusion term. Brownian BSDEs involving a supplementary orthogonal term were studied in [19]. A notion of BSDE driven by a martingale also involving a supplementary orthogonal martingale has appeared, see for instance [10], [12] and references therein.

In this paper, we consider a BSDE whose given data are a continuous increasing process $\hat{V}$, a square integrable martingale $\hat{M}$, a terminal condition ξ and a driver $\hat{f}$. A solution will be a couple (Y, M) verifying

$$Y = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot), \tag{1.3}$$

where Y is cadlag adapted and M is a square integrable martingale. We show existence and uniqueness of a solution for (1.3).

We will then be interested in a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with time interval $[0,T]$ and state space E being a Polish space. This will be supposed to be a solution of a martingale problem related to an operator $(\mathcal{D}(a), a)$ and a non-decreasing function V, meaning that for any $\phi \in \mathcal{D}(a)$, and $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x} := \mathbb{1}_{[s,T]}\left(\phi(\cdot, X_\cdot) - \phi(s,x) - \int_s^{\cdot} a(\phi)(r, X_r)dV_r\right)$ is a $\mathbb{P}^{s,x}$-square integrable martingale. We will fix some function $\psi := (\psi_1, \cdots, \psi_d) \in \mathcal{D}(a)^d$ and at Notation 5.7 we will introduce some special BSDEs driven by a martingale which we will call Markovian type BSDEs.

Those will be indexed by some $(s,x) \in [0,T] \times E$, defined in some stochastic basis $(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x})$ and will have the form

$$Y^{s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}(r)\right) dV_r - (M_T^{s,x} - M_\cdot^{s,x}), \tag{1.4}$$

where X is the canonical process, g is a Borel function with some growth condition and f is a Borel function with some growth condition with respect to the second variable, and Lipschitz conditions with respect to the third and fourth variables. Those Markovian BSDEs will be linked to the Pseudo-PDE

$$\left\{\begin{array}{rcll} a(u) + f\left(\cdot, \cdot, u, \Gamma^{\psi}(u)\right) &=& 0 & \text{on } [0,T] \times E \\ u(T, \cdot) &=& g, & \end{array}\right. \tag{1.5}$$

where $\Gamma^{\psi}(u) := (a(u\psi_i) - ua(\psi_i) - \psi_i a(u))_{i\in[\![1;d]\!]}$, see Definition 5.3. We introduce the notion of *classical* solution has to be an element of $\mathcal{D}(a)$ fulfilling (1.5). We call Γ^{ψ} the ψ-*generalized gradient*, due to the fact that when $E = \mathbb{R}^d$, $a = \partial_t + \frac{1}{2}\Delta$ and $\psi_i : (t,x) \longmapsto x_i$ for all $i \in [\![1,d]\!]$ then $\Gamma^{\psi}(u) = \nabla u$. In this particular setup, the forward Markov process is of course the Brownian motion. In that case the space $\mathcal{D}(a)$ where classical solutions are defined is $C^{1,2}([0,T] \times \mathbb{R}^d)$.

We show the existence of a Borel function u in some extended domain $\mathcal{D}(\mathfrak{a})$ such that for every $(s,x) \in [0,T] \times E$, $Y^{s,x}$ is, on $[s,T]$, a $\mathbb{P}^{s,x}$-modification of $u(\cdot, X_\cdot)$. At Definition 5.9 we will introduce the notion of *martingale solution* for the Pseudo-PDE (1.5). We then show that previous u is the unique martingale solution of (1.5), which means that it solves (1.5) where the maps a and Γ^{ψ} are respectively replaced with some extended operators $\mathfrak{a}$ and $\mathfrak{G}^{\psi}$. We also show that previous u is the unique *decoupled mild solution* of the same equation. We explain below that notion of solution which is introduced at Definition 5.14.

A Borel function u will be called decoupled mild solution if there exists an $\mathbb{R}^d$-valued Borel function $v := (v_1, \cdots, v_d)$ such that for every (s,x),

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_1 + ua(\psi_1) - \psi_1 f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dV_r \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_d + ua(\psi_d) - \psi_d f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dV_r, \end{array}\right. \tag{1.6}$$

where P is the time-dependent transition kernel associated to the Markov class and to the operator a, see Notation 5.12. v coincides with $\mathfrak{G}^{\psi}(u)$ and the couple (u,v) will be called solution to the *identification problem*, see Definition 5.14. The intuition behind this notion of solution relies to the fact that the equation $a(u) = -f(\cdot,\cdot,u,\Gamma^{\psi}(u))$ can be decoupled into the system

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ v_i & = & \Gamma^{\psi_i}(u), \quad i \in [\![1;d]\!], \end{array}\right. \tag{1.7}$$

which may be rewritten

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u\psi_i) & = & v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v), \quad i \in [\![1;d]\!]. \end{array}\right. \tag{1.8}$$

Martingale solutions were introduced in [6] and decoupled mild solutions in [7], in relation to a specific type of Pseudo-PDE, for which v was one-dimensional and which did not include the usual parabolic PDE related to classical BSDEs. A first approach to classical solutions to a general deterministic problem, associated with forward BSDEs with applications to the so called *Föllmer-Schweizer decomposition* was performed by [27].

The paper is organized as follows. In Section 3 we introduce an alternative formulation (1.3) for BSDEs driven by cadlag martingales discussed in [12]: we formulate in Theorem 3.3 existence and uniqueness for such equations. In Section 4, we introduce a canonical Markov class and the martingale problem which it is assumed to solve. We also define the extended domain $\mathcal{D}(\mathfrak{a})$ in Definition 4.11 and the extended operator $\mathfrak{a}$ (resp. $\mathfrak{G}^{\psi}$) in Definition 4.13 (resp. Notation 4.16). In Section 5, we introduce the Pseudo-PDE (1.5) (see Definition 5.3), the associated Markovian BSDEs (1.4), see Notation 5.7. We introduce the notion of martingale solution of the Pseudo-PDE in (5.9) and of decoupled mild solution in Definition 5.14. Propositions 5.16 and 5.17 show the equivalence between martingale solutions and decoupled mild solutions. Proposition 5.18 states that any classical solution is a decoupled mild solution and conversely that any decoupled mild solution belonging to $\mathcal{D}(\Gamma^{\psi})$ is a classical solution up to what we call a zero potential set. Let $(Y^{s,x}, M^{s,x})$ denote the unique solution of the associated BSDE (1.4), denoted by $BSDE^{s,x}(f,g)$. In Theorem 5.19 we show the existence of some $u \in \mathcal{D}(\mathfrak{a})$ such that for every $(s,x) \in [0,T] \times E$,

$Y^{s,x}$ is a $\mathbb{P}^{s,x}$-modification of $u(\cdot, X_\cdot)$ on $[s,T]$. Theorem 5.21 states that the function $(s,x) \longmapsto Y^{s,x}_s$ is the unique decoupled mild solution of (1.5). Proposition 5.24 states that if the functions (u,v) verify (1.6), then for any (s,x), the processes $\left(u(t,X_t), \quad u(t,X_t) - u(s,x) + \int_s^t f(\cdot,\cdot,u,v)(r,X_r)dV_r\right)_{t\in[s,T]}$ has a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f,g)$ on $[s,T]$. Finally in Section 6 we study some examples of applications. In section 6.1 we deal with parabolic semi-linear PDEs and in Section 6.2 with parabolic semi-linear PDEs with distributional drift.

2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary. For any integers $k \leq n$, $[\![k;n]\!]$ will denote the set of integers i verifying $k \leq i \leq n$. A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$. If (F, d_F) is a metric space, $\mathcal{C}(E,F)$ (respectively $\mathcal{C}_b(E,F)$, $\mathcal{B}(E,F)$, $\mathcal{B}_b(E,F)$) will denote the set of functions from E to F which are continuous (respectively bounded continuous, Borel, bounded Borel). $\mathbb{T}$ will stand for a real interval, of type $[0,T]$ with $T \in \mathbb{R}_+^*$ or $\mathbb{R}_+$.

On a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $p > 0$, L^p will denote the set of random variables with finite p-th moment. A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ will be called called a **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. We introduce now some notations and vocabulary about spaces of stochastic processes, on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$. Most of them are taken or adapted from [25] or [26]. We will denote $\mathcal{V}$ (resp $\mathcal{V}^+$) the set of adapted, bounded variation (resp non-decreasing) processes vanishing at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous; $\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1,\infty]$, $\mathcal{H}^p$ will denote the Banach space of elements of $\mathcal{M}$ for which $\|M\|_{\mathcal{H}^p} := \mathbb{E}[|\sup_{t\in\mathbb{T}} M_t|^p]^{\frac{1}{p}} < \infty$ and in this set we identify indistinguishable elements. $\mathcal{H}^p_0$ will denote the Banach subspace of $\mathcal{H}^p$ of elements vanishing at zero.

If $\mathbb{T} = [0,T]$ for some $T \in \mathbb{R}_+^*$, a stopping time will take values in $[0,T] \cup \{+\infty\}$. Let Y be a process and τ a stopping time, we denote by Y^τ the **stopped process** $t \mapsto Y_{t\wedge\tau}$. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exist a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$. By **localizing sequence of stopping times** we mean an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$.

For any $M, N \in \mathcal{M}_{loc}$, we denote $[M,N]$ their **quadratic covariation** and simply $[M]$ if $M = N$ and if moreover $M, N \in \mathcal{H}^2_{loc}$, $\langle M, N\rangle$ will denote their (predictable) **angular bracket**, or simply $\langle M\rangle$ if $M = N$.

$\mathcal{P}ro$ will denote the σ-field generated by progressively measurable processes defined on $[0,T]\times\Omega$.

From now on, we are given $T\in\mathbb{R}_+^*$. Until the end of Section 3, we also fix a stochastic basis $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}\right)$ fulfilling the usual conditions.

Definition 2.1. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA\ll dB$) if for almost all ω, $dA(\omega)\ll dB(\omega)$ as Borel measures on $[0,T]$.*

We will say that dB and dA are mutually singular **in the sense of stochastic measures** *(written $dA\perp dB$) if for almost all ω, the Borel measures $dA(\omega)$ and $dB(\omega)$ are mutually singular.*

Let $B\in\mathcal{V}^+$. $dB\otimes d\mathbb{P}$ will denote the positive measure on $(\Omega\times[0,T],\mathcal{F}\otimes\mathcal{B}([0,T]))$ defined for any $F\in\mathcal{F}\otimes\mathcal{B}([0,T])$ by $dB\otimes d\mathbb{P}(F)=\mathbb{E}\left[\int_0^T \mathbb{1}_F(\omega,r)dB_r(\omega)\right]$. A property which holds true everywhere except on a null set for this measure will be said to be true $dB\otimes d\mathbb{P}$ almost everywhere (a.e.).

The proposition below was the object of Proposition 3.2 in [6].

Proposition 2.2. *For any A and B in $\mathcal{V}^{p,+}$, there exists a (non-negative $dB\otimes d\mathbb{P}$ a.e.) predictable process $\frac{dA}{dB}$ and a process in $\mathcal{V}^{p,+}$ $A^{\perp B}$ such that*

$$dA^{\perp B}\perp dB \text{ and } A=A^B+A^{\perp B} \text{ a.s.},$$

where $A^B=\int_0^\cdot \frac{dA}{dB}(r)dB_r$. The process $A^{\perp B}$ is unique up to indistinguishability and the process $\frac{dA}{dB}$ is unique $dB\otimes d\mathbb{P}$ a.e.

The predictable process $\frac{dA}{dB}$ appearing in the statement of Proposition 2.2 will be called the **Radon-Nikodym derivative** of A by B.

If A belongs to $\mathcal{V}$, we will denote by $Var(A)$ (resp. $Pos(A)$, resp $Neg(A)$) the total (resp. positive, resp. negative) variation of A, meaning the unique pair of elements $\mathcal{V}^+$ such that $A=Pos(A)-Neg(A)$, see Proposition I.3.3 in [26] for their existence. If A is in $\mathcal{V}^p$, and $B\in\mathcal{V}^{p,+}$. We set $\frac{dA}{dB}:=\frac{dPos(A)}{dB}-\frac{dNeg(A)}{dB}$ and $A^{\perp B}:=Pos(A)^{\perp B}-Neg(A)^{\perp B}$.
Below we restate Proposition 3.4 in [6].

Proposition 2.3. *Let A_1 and A_2 be in $\mathcal{V}^p$, and $B\in\mathcal{V}^{p,+}$. Then, $\frac{d(A_1+A_2)}{dB}=\frac{dA_1}{dB}+\frac{dA_2}{dB}$ $dB\otimes d\mathbb{P}$ a.e. and $(A_1+A_2)^{\perp B}=A_1^{\perp B}+A_2^{\perp B}$.*

The following lemma was the object of Lemma 5.12 in [6].

Lemma 2.4. *Let V be a non-decreasing function. If two measurable processes are $\mathbb{P}$-modifications of each other, then they are also equal $dV\otimes d\mathbb{P}$ a.e.*

3 An alternative formulation of BSDEs driven by a cadlag martingale

We are now going to introduce here an alternative formulation for Backward Stochastic Differential Equations driven by a general cadlag martingale investigated for instance by [12].

Given some $\hat{V} \in \mathcal{V}^{c,+}$, we will indicate by $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ the set of (up to indistinguishability) progressively measurable processes ϕ such that $\mathbb{E}[\int_0^T \phi_r^2 d\hat{V}_r] < \infty$. $\mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ will denote the subspace of cadlag elements of $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$.

We will now fix a bounded process $\hat{V} \in \mathcal{V}^{c,+}$, an $\mathcal{F}_T$-measurable random variable ξ called the **final condition**, a square integrable **reference martingale** $\hat{M} := (\hat{M}^1, \cdots, \hat{M}^d)$ taking values in $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$, and a **driver** $\hat{f} : ([0,T] \times \Omega) \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, measurable with respect to $\mathcal{P}ro \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. We will assume that $(\xi, \hat{f}, \hat{M})$ verify the following hypothesis.

Hypothesis 3.1.

1. $\xi \in L^2$;
2. $\hat{f}(\cdot, \cdot, 0, 0) \in \mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$;
3. *There exist positive constants* K^Y, K^Z *such that,* $\mathbb{P}$ *a.s. for all* t, y, y', z, z', *we have*
$$|\hat{f}(t, \cdot, y, z) - \hat{f}(t, \cdot, y', z')| \leq K^Y |y - y'| + K^Z \|z - z'\|; \tag{3.1}$$
4. $\frac{d\langle \hat{M} \rangle}{d\hat{V}}$ *is bounded.*

We will now formulate precisely our BSDE.

Definition 3.2. *We say that a couple* $(Y, M) \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ *is a solution of* $BSDE(\xi, \hat{f}, V, \hat{M})$ *if it verifies*

$$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M} \rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot) \tag{3.2}$$

in the sense of indistinguishability.

The proof of the theorem below is very similar to the one of Theorem 3.22 in [6]. For the convenience of the reader, it is therefore postponed to Appendix A.

Theorem 3.3. *If* $(\xi, \hat{f}, \hat{V}, \hat{M})$ *verifies Hypothesis 3.1 then* $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ *has a unique solution.*

Remark 3.4. *Let $(\xi, \hat{f}, \hat{V}, \hat{M})$ satisfying Hypothesis 3.1. We can consider a BSDE on a restricted interval $[s,T]$ for some $s \in [0,T[$. Previous discussion and Theorem 3.3 extend easily to this case. In particular there exists a unique couple of processes (Y^s, M^s), indexed by $[s,T]$ such that Y^s is adapted, cadlag and verifies $\mathbb{E}[\int_s^T (Y_r^s)^2 d\hat{V}_r] < \infty$, such that M^s is a martingale vanishing in s and such that $Y^s = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r^s, \frac{d\langle M^s, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T^s - M_\cdot^s)$ in the sense of indistinguishability on $[s,T]$.*
Moreover, if (Y, M) denotes the solution of $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ then $(Y, M_\cdot - M_s)$ and (Y^s, M^s) coincide on $[s,T]$. This follows by an uniqueness argument resulting by Theorem 3.3 on time interval $[s,T]$.

4 Martingale Problem and Markov classes

In this section, we introduce the Markov process which will later explain the random dependence of the driver $\hat{f}$ of our BSDE driven by a cadlag martingale. For that reason that BSDE will be called Markovian.

For details about the exact mathematical necessary background for our Markov process, one can consult Section C of the Appendix. That process will be supposed to solve a martingale problem described below.

Let E be a Polish space. From now on, $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ denotes the canonical space defined in Definition C.1. We consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition kernel measurable in time as defined in Definitions C.5 and C.4, and for any $(s,x) \in [0,T] \times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ will denote the stochastic basis introduced in Definition C.7 and which fulfills the usual conditions.

Our Martingale problem will be associated to an operator, in a close formalism to the one of D.W. Stroock and S.R.S Varadhan in [35].

Definition 4.1. *Let $V : [0,T] \to \mathbb{R}_+$ be a non-decreasing continuous function vanishing at 0.*
Let us consider a linear operator $a : \mathcal{D}(a) \subset \mathcal{B}([0,T]\times E, \mathbb{R}) \longrightarrow \mathcal{B}([0,T]\times E, \mathbb{R})$, where the domain $\mathcal{D}(a)$ is a linear space.

We say that a family of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega, \mathcal{F})$ solves the **martingale problem associated to** $(\mathcal{D}(a), a, V)$ *if, for any $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ verifies the following.*

(a) $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;

(b) for every $\phi \in \mathcal{D}(a)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r) dV_r$, $t \in [s,T]$, is a cadlag $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[s,T]})$ square integrable martingale.

We say that the Martingale Problem is ***well-posed*** *if for any $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ is the only probability measure satisfying those two properties.*

We anticipate that well-posedness for the martingale problem will not be an hypothesis in the sequel.

Notation 4.2. *For every $(s,x) \in [0,T] \times E$ and $\phi \in \mathcal{D}(a)$, the process $t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dV_r\right)$ will be denoted $M[\phi]^{s,x}$.*

$M[\phi]^{s,x}$ is a cadlag $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[0,T]})$ square integrable martingale equal to 0 on $[0,s]$, and by Proposition C.8, it is also a $(\mathbb{P}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]})$ square integrable martingale.

Notation 4.3. *Let $\phi \in \mathcal{D}(a)$. We set, for $0 \le t \le u \le T$*

$$M[\phi]_u^t := \begin{cases} \phi(u,X_u) - \phi(t,X_t) - \int_t^u a(\phi)(r,X_r)dV_r \ \text{if } \int_t^u |a(\phi)|(r,X_r)dV_r < \infty, \\ 0 \ \text{otherwise}. \end{cases} \tag{4.1}$$

$M[\phi]$ is a square integrable Martingale Additive Functional (in short MAF), see Definition C.9, whose cadlag version under $\mathbb{P}^{s,x}$ for every $(s,x) \in [0,T] \times E$, is $M[\phi]^{s,x}$.

From now on we fix some $d \in \mathbb{N}^*$ and a vector $\psi = (\psi_1, \dots, \psi_d) \in \mathcal{D}(a)^d$.

For any $(s,x) \in [0,T]\times E$, the $\mathbb{R}^d$-valued martingale $(M[\psi_1]^{s,x}, \cdots, M[\psi_d]^{s,x})$ will be denoted $M[\psi]^{s,x}$.

Definition 4.4. *For any $\phi_1, \phi_2 \in \mathcal{D}(a)$ such that $\phi_1\phi_2 \in \mathcal{D}(a)$ we set $\Gamma(\phi_1,\phi_2) := a(\phi_1\phi_2) - \phi_1 a(\phi_2) - \phi_2 a(\phi_1)$. Γ will be called the* **carré du champs operator**. *We set $\mathcal{D}(\Gamma^\psi) := \left\{\phi \in \mathcal{D}(a) : \forall i \in [\![1;d]\!], \phi\psi^i \in \mathcal{D}(a)\right\}$. We define the linear operator $\Gamma^\psi : \mathcal{D}(\Gamma^\psi) \longrightarrow \mathcal{B}([0,T] \times E, \mathbb{R}^d)$ by*

$$\Gamma^\psi(\phi) := \left(\Gamma^{\psi_i}(\phi)\right)_{i\in[\![1;d]\!]} := \left(a(\phi\psi_i) - \phi a(\psi_i) - \psi_i a(\phi)\right)_{i\in[\![1;d]\!]}. \tag{4.2}$$

*Γ^ψ will be called the ψ-**generalized gradient operator**.*

We emphasize that this terminology is justified by the considerations below (1.5). This operator appears in the expression of the angular bracket of the local martingales that we have defined.

Proposition 4.5. *If $\phi \in \mathcal{D}(\Gamma^\psi)$, then for any $(s,x) \in [0,T] \times E$ and $i \in [\![1;d]\!]$ we have*

$$\langle M[\phi]^{s,x}, M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \Gamma^{\psi_i}(\phi)(r,X_r)dV_r, \tag{4.3}$$

in the stochastic basis $(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x})$.

Proof. The result follows from a slight modification of the proof of Proposition 4.8 of [6] in which $\mathcal{D}(a)$ was assumed to be stable by multiplication and $M[\phi]^{s,x}$ could potentially be a local martingale which is not a martingale. □

We will later need the following assumption.

Hypothesis 4.6. *For every $i \in [\![1;d]\!]$, the Additive Functional $\langle M[\psi_i]\rangle$ (see Proposition C.10) is absolutely continuous with respect to dV, see Definition C.9.*

Taking $\phi = \psi_i$ for some $i \in [\![1;d]\!]$ in Proposition 4.5, yields the following.

Corollary 4.7. *If $\psi_i^2 \in \mathcal{D}(a)$ for all $i \in [\![1;d]\!]$, then Hypothesis 4.6 is fulfilled.*

We will now consider suitable extensions of the domain $\mathcal{D}(a)$.

For any $(s,x) \in [0,T]\times E$ we define the positive bounded **potential measure** $U(s,x,\cdot)$ on $([0,T]\times E, \mathcal{B}([0,T])\otimes\mathcal{B}(E))$ by

$$U(s,x,\cdot): \begin{array}{ccc} \mathcal{B}([0,T])\otimes\mathcal{B}(E) & \longrightarrow & [0,V_T] \\ A & \longmapsto & \mathbb{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}}dV_t\right]. \end{array}$$

Definition 4.8. *A Borel set $A\subset[0,T]\times E$ will be said to be* **of zero potential** *if, for any $(s,x)\in[0,T]\times E$ we have $U(s,x,A)=0$.*

Notation 4.9. *Let $p>0$. We introduce*
$\mathcal{L}^p_{s,x} := \mathcal{L}^p(U(s,x,\cdot)) = \left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}) : \mathbb{E}^{s,x}\left[\int_s^T |f|^p(r,X_r)dV_r\right]<\infty\right\}.$
For $p\geq 1$, that classical $\mathcal{L}^p$-space is equipped with the seminorm
$\|\cdot\|_{p,s,x} : f\mapsto \left(\mathbb{E}^{s,x}\left[\int_s^T |f(r,X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$*. We also introduce*
$\mathcal{L}^0_{s,x} := \mathcal{L}^0(U(s,x,\cdot)) = \left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}) : \int_s^T |f|(r,X_r)dV_r<\infty \quad \mathbb{P}^{s,x}\ a.s.\right\}.$
For any $p\geq 0$ we set

$$\mathcal{L}^p_X = \bigcap_{(s,x)\in[0,T]\times E}\mathcal{L}^p_{s,x}. \tag{4.4}$$

Let $\mathcal{N}$ be the linear sub-space of $\mathcal{B}([0,T]\times E,\mathbb{R})$ containing all functions which are equal to 0, $U(s,x,\cdot)$ a.e. for every (s,x).
For any $p\geq 0$, we define the quotient space $L^p_X = \mathcal{L}^p_X/\mathcal{N}$.
If $p\geq 1$, L^p_X can be equipped with the topology generated by the family of seminorms $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ which makes it a separate locally convex topological vector space, see Theorem 5.76 in [1].

We recall that Proposition 4.14 in [6] states the following.

Proposition 4.10. *Let f and g be in $\mathcal{L}^0_X$. Then f and g are equal up to a set of zero potential if and only if for any $(s,x)\in[0,T]\times E$, the processes $\int_s^\cdot f(r,X_r)dV_r$ and $\int_s^\cdot g(r,X_r)dV_r$ are indistinguishable under $\mathbb{P}^{s,x}$. Of course in this case f and g correspond to the same element of L^0_X.*

We introduce now our notion of **extended generator** starting from its domain.

Definition 4.11. *We first define the* ***extended domain*** *$\mathcal{D}(\mathfrak{a})$ as the set of functions $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ for which there exists*
$\chi\in\mathcal{L}^0_X$ such that under any $\mathbb{P}^{s,x}$ the process

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \chi(r,X_r)dV_r\right) \tag{4.5}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}^2_0$.

A direct consequence of Proposition 4.16 in [6] is the following.

Proposition 4.12. *Let $\phi \in \mathcal{B}([0,T]\times E,\mathbb{R})$. There is at most one (up to zero potential sets) $\chi \in \mathcal{L}^0_X$ such that under any $\mathbb{P}^{s,x}$, the process defined in (4.5) has a modification which belongs to $\mathcal{H}^2$.*
If moreover $\phi \in \mathcal{D}(a)$, then $a(\phi) = \chi$ up to zero potential sets. In this case, according to Notation 4.2, for every $(s,x) \in [0,T]\times E$, $M[\phi]^{s,x}$ is the $\mathbb{P}^{s,x}$ cadlag modification in $\mathcal{H}^2_0$ of $\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot) - \phi(s,x) - \int_s^\cdot \chi(r,X_r)dV_r\right)$.

Definition 4.13. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 4.11. We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process (4.5) in $\mathcal{H}^2_0$. Taking Proposition 4.10 into account, this will not generate any ambiguity with respect to Notation 4.2. Proposition 4.10, also permits to define without ambiguity the operator*

$$\mathfrak{a} : \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L^0_X \\ \phi & \longmapsto & \chi. \end{array}$$

$\mathfrak{a}$ will be called the ***extended generator****.*

Remark 4.14. *$\mathfrak{a}$ extends a in the sense that $\mathcal{D}(a) \subset \mathcal{D}(\mathfrak{a})$ (comparing Definitions 4.11 and 4.1) and if $\phi \in \mathcal{D}(a)$ then $a(\phi)$ is an element of the class $\mathfrak{a}(\phi)$, see Proposition 4.12.*

We also introduce an extended ψ-generalized gradient.

Proposition 4.15. *Assume the validity of Hypothesis 4.6. Let $\phi \in \mathcal{D}(\mathfrak{a})$ and $i \in [\![1;d]\!]$. There exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T]\times E,\mathbb{R})$ which we will denote $\mathfrak{G}^{\psi_i}(\phi)$ such that under any $\mathbb{P}^{s,x}$, $\langle M[\phi]^{s,x}, M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\phi)(r,X_r)dV_r$ up to indistinguishability.*

Proof. We fix $i \in [\![1;d]\!]$. Let $M[\psi_i]$ be the square integrable MAF (see Definition C.9) presented in Notation 4.3. We introduce the random field $M[\phi] = (M[\phi]^t_u)_{(0\leq t\leq u\leq T)}$ as follows. We fix some χ in the class $\mathfrak{a}(\phi)$ and set

$$M[\phi]^t_u := \left\{ \begin{array}{l} \phi(u,X_u) - \phi(t,X_t) - \int_t^u \chi(r,X_r)dV_r \text{ if } \int_t^u |\chi|(r,X_r)dV_r < \infty, t \leq u, \\ 0 \text{ elsewhere,} \end{array} \right. \tag{4.6}$$

We emphasize that, a priori, the function χ is only in $\mathcal{L}^0_X$ implying that at fixed $t \leq u$, $\int_t^u |\chi|(r,X_r(\omega))dV_r$ is not finite for every $\omega \in \Omega$, but only on a set which is $\mathbb{P}^{s,x}$-negligible for all $(s,x) \in [0,t]\times E$.
According to Definition C.9 $M[\phi]$ is an AF whose cadlag version under $\mathbb{P}^{s,x}$ is $M[\phi]^{s,x}$. Of course $M[\psi_i]^{s,x}$ is the cadlag version of $M[\psi_i]$ under $\mathbb{P}^{s,x}$.
By Definition 4.13, since $\phi \in \mathcal{D}(\mathfrak{a})$, $M[\phi]^{s,x}$ is a square integrable martingale for every (s,x), so $M[\phi]$ is a square integrable MAF. Then by Corollary 4.7, the AF $\langle M[\psi_i]\rangle$ is absolutely continuous with respect to dV. The existence of $\mathfrak{G}^{\psi_i}(\phi)$ now follows from Proposition C.11 and the uniqueness follows by Proposition 4.10. □

Notation 4.16. *If 4.6 holds, we can introduce the linear operator*

$$\mathfrak{G}^{\psi}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & (L^0_X)^d \\ \phi & \longmapsto & (\mathfrak{G}^{\psi_1}(\phi), \cdots, \mathfrak{G}^{\psi_d}(\phi)), \end{array} \tag{4.7}$$

which will be called the ***extended*** ψ***-generalized gradient****.*

Corollary 4.17. *Let* $\phi \in \mathcal{D}(\Gamma^{\psi})$*. Then* $\Gamma^{\psi}(\phi) = \mathfrak{G}^{\psi}(\phi)$ *up to zero potential sets.*

Proof. Comparing Propositions 4.5 and 4.15, for every $(s,x) \in [0,T]\times E$ and $i \in [\![1;d]\!]$, $\int_s^{\cdot\vee s} \Gamma^{\psi_i}(\phi)(r,X_r)dV_r$ and $\int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\phi)(r,X_r)dV_r$ are $\mathbb{P}^{s,x}$-indistinguishable. We can conclude by Proposition 4.10. $\square$

$\mathfrak{G}^{\psi}$ therefore extends Γ^{ψ} as well as $\mathfrak{a}$ extends a, see Remark 4.14.

5 Pseudo-PDEs and associated Markovian type BSDEs driven by a cadlag martingale

5.1 The concepts

In this section, we still consider $T \in \mathbb{R}_+^*$, a Polish space E and the associated canonical space $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$, see Definition C.1. We also consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition kernel measurable in time (see Definitions C.5 and C.4) which solves a martingale problem associated to a triplet $(\mathcal{D}(a), a, V)$, see Definition 4.1.

We will investigate here a specific type of BSDE driven by a cadlag martingale, denoted by $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ which we will call **of Markovian type**, or **Markovian** BSDE, in the following sense. The process $\hat{V}$ will be the (deterministic) function V introduced in Definition 4.1, the final condition ξ will only depend on the final value of the canonical process X_T and the randomness of the driver $\hat{f}$ at time t will only appear via the value at time t of the forward process X. Given a function $f : [0,T] \times E \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we will set $\hat{f}(t,\omega,y,z) = f(t, X_t(\omega), y, z)$ for $t \in [0,T], \omega \in \Omega, y \in \mathbb{R}, z \in \mathbb{R}^d$.
Given d functions $\psi_1, \cdots, \psi_d$ in $\mathcal{D}(a)$, we will set $\hat{M} := (M[\psi_1]^{s,x}, \cdots, M[\psi_d]^{s,x})$.

That BSDE will be connected with the deterministic problem in Definition 5.3.

We fix an integer $d \in \mathbb{N}^*$ and some functions $\psi_1, \cdots, \psi_d \in \mathcal{D}(a)$ which in the sequel, will verify the following hypothesis.

Hypothesis 5.1. *For any* $i \in [\![1;d]\!]$ *we have the following*

- *Hypothesis 4.6 holds;*
- $a(\psi_i) \in \mathcal{L}^2_X$*;*

- $\mathfrak{G}^{\psi_i}(\psi_i)$ *is bounded.*

In particular, for every $i \in [\![1;d]\!]$, previous hypothesis implies the following.

Proposition 5.2.

- *For any* $(s,x) \in [0,T] \times E$, $\hat{M} := M[\psi]^{s,x}$ *verifies item 4. of Hypothesis 3.1 with respect to* $\hat{V} := V$.
- *for every* $(s,x) \in [0,T] \times E$, $\underset{t\in[s,T]}{sup} |\psi_i(t,X_t)|^2$ *belongs to* L^1 *under* $\mathbb{P}^{s,x}$*;*
- $\psi_i \in \mathcal{L}^2_X$.

Proof. The first item follows form the fact that for any $(s,x) \in [0,T] \times E$, $\langle M[\psi]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$ (see Proposition 4.15), and the fact that $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded. Concerning the second item, for any $(s,x) \in [0,T]\times E$, the martingale problem gives $\psi(\cdot,X) = \psi_i(s,x) + \int_s^{\cdot} a(\psi_i)(r,X_r)dV_r + M[\psi_i]^{s,x}$, see Definition 4.1. By Jensen's inequality, we have $\underset{t\in[s,T]}{\sup} |\psi_i(t,X_t)|^2 \leq C(\psi_i^2(s,x) + \int_s^T a^2(\psi_i)(r,X_r)dV_r + \underset{t\in[s,T]}{\sup} (M[\psi_i]^{s,x}_t)^2)$ for some $C > 0$. It is therefore L^1 since $a(\psi_i) \in \mathcal{L}^2_X$ and $M[\psi_i]^{s,x} \in \mathcal{H}^2$. The last item is a direct consequence of the second one. □

Definition 5.3. *Let us consider some* $g \in \mathcal{B}(E,\mathbb{R})$ *and* $f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R}^d)$.
We will call ***Pseudo-Partial Differential Equation*** *related to* (f,g) *(in short* $Pseudo-PDE(f,g)$*) the following equation with final condition:*

$$\begin{cases} a(u) + f\left(\cdot,\cdot,u,\Gamma^{\psi}(u)\right) &= 0 \quad on\ [0,T]\times E \\ u(T,\cdot) &= g. \end{cases} \tag{5.1}$$

We will say that u is a ***classical solution*** *of* $Pseudo-PDE(f,g)$ *if* $u, u\psi_i, i \in [\![1;d]\!]$ *belong to* $\mathcal{D}(a)$ *and if u verifies* (5.1).

The connection between a Markovian BSDE and a $Pseudo-PDE(f,g)$, will be possible under a hypothesis on some generalized moments on X, and some growth conditions on the functions (f,g). Those will be related to two fixed functions $\zeta,\eta \in \mathcal{B}(E,\mathbb{R}_+)$.

Hypothesis 5.4. *The Markov class will be said* ***to verify*** $H^{mom}(\zeta,\eta)$ *if*

1. *for any* $(s,x) \in [0,T]\times E$, $\mathbb{E}^{s,x}[\zeta^2(X_T)]$ *is finite;*
2. *for any* $(s,x) \in [0,T]\times E$, $\mathbb{E}^{s,x}\left[\int_0^T \eta^2(X_r)dV_r\right]$ *is finite.*

Until the end of this section, we assume that some ζ,η are given and that the Markov class verifies $H^{mom}(\zeta,\eta)$.

Hypothesis 5.5. *A couple of functions* $f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ *and* $g \in \mathcal{B}(E, \mathbb{R})$ *will be said* ***to verify*** $H^{lip}(\zeta, \eta)$ *if there exist positive constants* K^Y, K^Z, C, C' *such that*

1. $\forall x: \quad |g(x)| \leq C(1 + \zeta(x))$,
2. $\forall (t,x): \quad |f(t,x,0,0)| \leq C'(1 + \eta(x))$,
3. $\forall (t,x,y,y',z,z'): \quad |f(t,x,y,z) - f(t,x,y',z')| \leq K^Y|y-y'| + K^Z\|z-z'\|$.

(f,g) *will be said* ***to verify*** $H^{growth}(\zeta,\eta)$ *if the following lighter Hypothesis hold. There exist positive constants* C, C' *such that*

1. $\forall x: \quad |g(x)| \leq C(1 + \zeta(x))$;
2. $\forall (t,x,y,z): \quad |f(t,x,y,z)| \leq C'(1 + \eta(x) + |y| + \|z\|)$.

Remark 5.6. *We fix for now a couple* (f,g) *verifying* $H^{lip}(\zeta,\eta)$. *For any* $(s,x) \in [0,T] \times E$, *in the stochastic basis* $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ *and setting* $\hat{V} := V$, *the triplet* $\xi := g(X_T)$, $\hat{f} : (t,\omega,y,z) \longmapsto f(t, X_t(\omega), y, z)$, $\hat{M} := M[\psi]^{s,x}$ *verifies Hypothesis 3.1.*

With the equation $Pseudo-PDE(f,g)$, we will associate the following family of BSDEs indexed by $(s,x) \in [0,T] \times E$, driven by a cadlag martingale.

Notation 5.7. *For any* $(s,x) \in [0,T] \times E$, *we consider in the stochastic basis* $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ *and on the interval* $[0,T]$ *the* $BSDE(\xi, \hat{f}, V, \hat{M})$ *where* $\xi = g(X_T)$, $\hat{f} : (t,\omega,y,z) \longmapsto f(t, X_t(\omega), y, z)$, $\hat{M} = M[\psi]^{s,x})$.
This BSDE will from now on be denoted $BSDE^{s,x}(f,g)$ *and its unique solution (see Theorem 3.3 and Remark 5.6) will be denoted* $(Y^{s,x}, M^{s,x})$.

If $H^{lip}(\zeta,\eta)$ is fulfilled by (f,g), then $(Y^{s,x}, M^{s,x})$ is therefore the unique couple in $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ verifying

$$Y^{s,x}_{\cdot} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, Y^{s,x}_r, \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}(r)\right) dV_r - (M^{s,x}_T - M^{s,x}_{\cdot}). \tag{5.2}$$

Remark 5.8. *Even if the underlying process* X *admits no generalized moments, given a couple* (f,g) *such that* $f(\cdot,\cdot,0,0)$ *and* g *are bounded, the considerations of this section still apply. In particular the connections that we will establish between the* $BSDE^{s,x}(f,g)$ *and the corresponding* $Pseudo-PDE(f,g)$ *still take place.*

The goal of our work is to emphasize the precise link under general enough conditions between the solutions of equations $BSDE^{s,x}(f,g)$ and of $Pseudo-PDE(f,g)$. In particular we will emphasize that a solution of $BSDE^{s,x}(f,g)$ produces a solution of $Pseudo-PDE(f,g)$ and reciprocally.

We now introduce a probabilistic notion of solution for $Pseudo-PDE(f,g)$.

Definition 5.9. *A Borel function $u : [0,T] \times E \to \mathbb{R}$ will be said to be a* **martingale solution** *of $Pseudo - PDE(f,g)$ if $u \in \mathcal{D}(\mathfrak{a})$ and*

$$\left\{ \begin{array}{rcl} \mathfrak{a}(u) & = & -f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)) \\ u(T,\cdot) & = & g. \end{array} \right. \tag{5.3}$$

Remark 5.10. *The first equation of* (5.3) *holds in L^0_X, hence up to a zero potential set. The second one is a pointwise equality.*

Proposition 5.11. *Let (f,g) verify $H^{growth}(\zeta,\eta)$. Let u be a martingale solution of $Pseudo - PDE(f,g)$. Then for any $(s,x) \in [0,T] \times E$, the couple of processes*

$$\left(u(t,X_t), \quad u(t,X_t) - u(s,x) + \int_s^t f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u))(r,X_r)dV_r\right)_{t\in[s,T]} \tag{5.4}$$

has a $\mathbb{P}^{s,x}$-version which is a solution on $[s,T]$ of $BSDE^{s,x}(f,g)$, see Remark 3.4.

Moreover, $u \in \mathcal{L}^2_X$.

Proof. Let $u \in \mathcal{D}(\mathfrak{a})$ be a solution of (5.3) and let $(s,x) \in [0,T] \times E$ be fixed. By Definition 4.11 and Remark 3.4, the process $u(\cdot, X_\cdot)$ under $\mathbb{P}^{s,x}$ admits a cadlag modification $U^{s,x}$ on $[s,T]$, which is a special semi-martingale with decomposition

$$\begin{array}{rcl} U^{s,x} & = & u(s,x) + \int_s^\cdot \mathfrak{a}(u)(r,X_r)dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r,X_r,u(r,X_r),\mathfrak{G}^{\psi}(u)(r,X_r)\right)dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r,X_r,U^{s,x},\frac{d\langle M[u]^{s,x},M[\psi]^{s,x}\rangle}{dV}\right)dV_r + M[u]^{s,x}, \end{array} \tag{5.5}$$

where the third equality of (5.5) comes from Lemma 2.4 and Proposition 4.15. Moreover since $u(T,\cdot) = g$, then $U^{s,x}_T = u(T,X_T) = g(X_T)$ a.s. so the couple $(U^{s,x}, M[u]^{s,x})$ verifies the following equation on $[s,T]$ (with respect to $\mathbb{P}^{s,x}$):

$$U^{s,x}_\cdot = g(X_T) + \int_\cdot^T f\left(r,X_r,U^{s,x}_r,\frac{d\langle M[u]^{s,x},M[\psi]^{s,x}\rangle}{dV}(r)\right)dV_r - (M[u]^{s,x}_T - M[u]^{s,x}_\cdot). \tag{5.6}$$

$M[u]^{s,x}$ (introduced at Definition 4.13) belongs to $\mathcal{H}^2_0$ but we do not have a priori information on the square integrability of $U^{s,x}$. However we know that $M[u]^{s,x}$ is equal to zero at time s, and that $U^{s,x}_s$ is deterministic so square integrable. We can therefore apply Lemma A.12 which implies that $(U^{s,x}, M[u]^{s,x})$ solves $BSDE^{s,x}(f,g)$ on $[s,T]$. In particular, $U^{s,x}$ belongs to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$ for every (s,x), so by Lemma 2.4 and Definition 4.9, $u \in \mathcal{L}^2_X$. □

5.2 Decoupled mild solutions of Pseudo-PDEs

In this section we introduce an analytical notion of solution of our $Pseudo - PDE(f,g)$ that we will denominate *decoupled mild* since it inspired by the mild solution of partial differential equation. We will show that it is equivalent to the notion of martingale solution introduced in Definition 5.9.

Notation 5.12. *Let P denote the transition kernel of the canonical class (see Definition C.3). Let s,t in $[0,T]$ with $s \leq t$, $x \in E$ and $\phi \in \mathcal{B}(E,\mathbb{R})$, if ϕ is integrable with respect to $P_{s,t}(x,\cdot)$, then $P_{s,t}[\phi](x)$ will denote its integral.*

We recall two important measurability properties.

Remark 5.13.

- *Let $\phi \in \mathcal{B}(E,\mathbb{R})$ be such that for any (s,x,t), $\mathbb{E}^{s,x}[|\phi(X_t)|] < \infty$, then $(s,x,t) \longmapsto P_{s,t}[\phi](x)$ is Borel, see Proposition A.12 in [7].*
- *Let $\phi \in \mathcal{L}^1_X$, then $(s,x) \longmapsto \int_s^T P_{s,r}[\phi](x)dV_r$ is Borel, see Lemma A.11 in [7].*

Our notion of decoupled mild solution relies on the fact that the equation $a(u) + f\left(\cdot,\cdot,u,\Gamma^{\psi}(u)\right) = 0$ can be naturally decoupled into

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ v_i & = & \Gamma^{\psi_i}(u), \quad i \in [\![1;d]\!]. \end{array}\right. \tag{5.7}$$

Then, by definition of the carré du champ operator (see Definition 4.4), we formally have $i \in [\![1;d]\!]$, $a(u\psi_i) = \Gamma^{\psi_i}(u) + ua(\psi_i) + \psi_i a(u)$. So the system of equations (5.7) can be rewritten as

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u\psi_i) & = & v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v), \quad i \in [\![1;d]\!]. \end{array}\right. \tag{5.8}$$

Inspired by the usual notions of mild solution, this naturally leads us to the following definition of a mild solution.

Definition 5.14. *Let (f,g) be a couple verifying $H^{growth}(\zeta,\eta)$. Let $u \in \mathcal{B}([0,T]\times E,\mathbb{R})$ and $v \in \mathcal{B}([0,T]\times E,\mathbb{R}^d)$.*

1. *The couple (u,v) will be called* **solution of the identification problem determined by** *(f,g) or simply* **solution of** *$IP(f,g)$ if $u, v_1, \cdots, v_d$ belong to $\mathcal{L}^2_X$ and if for every $(s,x) \in [0,T]\times E$,*

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_1 + ua(\psi_1) - \psi_1 f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dV_r \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_d + ua(\psi_d) - \psi_d f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dV_r. \end{array}\right. \tag{5.9}$$

2. *The function u will be called* **decoupled mild solution** *of $Pseudo-PDE(f,g)$ if there exist a function v such that the couple (u,v) is a solution of $IP(f,g)$.*

Lemma 5.15. *Let $u, v_1, \cdots, v_d \in \mathcal{L}^2_X$, let (f,g) be a couple satisfying $H^{growth}(\zeta,\eta)$ and let $\psi_1, \cdots, \psi_d$ verify Hypothesis 5.1. Then $f\left(\cdot,\cdot,u,v\right)$ belongs to $\mathcal{L}^2_X$ and for every $i \in [\![1;d]\!]$, $\psi_i f\left(\cdot,\cdot,u,v\right)$, and $ua(\psi_i)$, belong to $\mathcal{L}^1_X$. For any $(s,x) \in [0,T]\times E$, $i \in [\![1;d]\!]$, $g(X_T)\psi_i(T,X_T)$ belongs to L^1 under $\mathbb{P}^{s,x}$. In particular, all terms in (5.9) make sense.*

Proof. Thanks to the growth condition on f in $H^{growth}(\zeta,\eta)$, there exists a constant $C>0$ such that for any $(s,x)\in[0,T]\times E$,

$$\begin{array}{cl} & \mathbb{E}^{s,x}\left[\int_t^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ \leq & C\mathbb{E}^{s,x}\left[\int_t^T (f^2(r,X_r,0,0)+u^2(r,X_r)+\|v\|^2(r,X_r))dV_r\right]. \end{array} \tag{5.10}$$

Previous quantity is finite since we have assumed that $u,v_1,\cdots,v_d$ belong to $\mathcal{L}^2_X$, taking into account Hypothesis 5.4 and $H^{growth}(\zeta,\eta)$. This means that $f^2(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^1_X$. All the other assertions are easily obtained taking into account Hypothesis 5.1, $H^{growth}(\zeta,\eta)$ and the classical inequality $2|ab|\leq a^2+b^2$. □

Proposition 5.16. *Let (f,g) verify $H^{growth}(\zeta,\eta)$. Let u be a martingale solution of $Pseudo-PDE(f,g)$, then $(u,\mathfrak{G}^\psi(u))$ is a solution of $IP(f,g)$ and in particular, u is a decoupled mild solution of $Pseudo-PDE(f,g)$.*

Proof. Let u be a martingale solution of $Pseudo-PDE(f,g)$. By Proposition 5.11, $u\in\mathcal{L}^2_X$. Taking into account Definition 4.13, for every (s,x), $M[u]^{s,x}\in\mathcal{H}^2_0$ under $\mathbb{P}^{s,x}$. So by Lemma A.2, for any $i\in[\![1;d]\!]$, $\frac{d\langle M[u]^{s,x},M[\psi^i]^{s,x}\rangle}{dV}$ belongs to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$. Taking Proposition 4.15 into account, this means that $\mathfrak{G}^{\psi_i}(u)\in\mathcal{L}^2_X$ for every i. By Lemma 5.15, it follows that $f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)$ belongs to $\mathcal{L}^2_X$ and so for any $i\in[\![1;d]\!]$, $\psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)$ and $ua(\psi_i)$, belong to $\mathcal{L}^1_X$.
We fix some $(s,x)\in[0,T]\times E$ and the correspondent probability $\mathbb{P}^{s,x}$, and we are going to show that

$$\left\{\begin{array}{rcl} u(s,x) &=& P_{s,T}[g](x)+\int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) &=& P_{s,T}[g\psi_1(T,\cdot)](x)-\int_s^T P_{s,r}\left[\left(\mathfrak{G}(u,\psi_1)+ua(\psi_1)-\psi_1 f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)\right)(r,\cdot)\right](x)dV_r \\ & \cdots & \\ u\psi_d(s,x) &=& P_{s,T}[g\psi_d(T,\cdot)](x)-\int_s^T P_{s,r}\left[\left(\mathfrak{G}(u,\psi_d)+ua(\psi_d)-\psi_d f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)\right)(r,\cdot)\right](x)dV_r. \end{array}\right. \tag{5.11}$$

Combining Definitions 4.11, 4.13, 5.9, we know that on $[s,T]$, the process $u(\cdot,X_\cdot)$ has a modification which we denote $U^{s,x}$ which is a special semimartingale with decomposition

$$U^{s,x}=u(s,x)-\int_s^\cdot f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,X_r)dV_r+M[u]^{s,x}, \tag{5.12}$$

and $M[u]^{s,x}\in\mathcal{H}^2_0$.
Definition 5.9 also states that $u(T,\cdot)=g$, implying that

$$u(s,x)=g(X_T)+\int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,X_r)dV_r-M[u]^{s,x}_T \text{ a.s.} \tag{5.13}$$

Taking the expectation, by Fubini's theorem we get

$$\begin{array}{rcl} u(s,x) &=& \mathbb{E}^{s,x}\left[g(X_T)+\int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,X_r)dV_r\right] \\ &=& P_{s,T}[g](x)+\int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}^\psi(u)(r,\cdot)\right)\right](x)dV_r. \end{array} \tag{5.14}$$

We now fix $i \in [\![1;d]\!]$. By integration by parts, taking (5.12) and Definition 4.1 into account, we obtain

$$
\begin{array}{rcl}
d(U^{s,x}_t \psi_i(t,X_t)) & = & -\psi_i(t,X_t) f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(t,X_t)dV_t + \psi_i(t^-,X_{t^-})dM[u]^{s,x}_t \\
& & +U^{s,x}_t a(\psi_i)(t,X_t)dV_t + U^{s,x}_{t^-} dM[\psi_i]^{s,x}_t + d[M[u]^{s,x}, M[\psi_i]^{s,x}]_t,
\end{array}
\tag{5.15}
$$

so integrating from s to T, we get

$$
\begin{array}{rl}
& u\psi_i(s,x) \\
= & g(X_T)\psi_i(T,X_T) + \int_s^T \psi_i(t,X_t) f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,X_r)dV_r - \int_s^T \psi_i(r^-,X_{r^-})dM[u]^{s,x}_r \\
& -\int_s^T U^{s,x}_t a(\psi_i)(r,X_r)dV_r - \int_s^T U^{s,x}_{r^-} dM[\psi_i]^{s,x}_r - [M[u]^{s,x}, M[\psi_i]^{s,x}]_T \\
= & g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r - \int_s^T \psi_i(r^-,X_{r^-})dM[u]^{s,x}_r \\
& -\int_s^T U^{s,x}_{r^-} dM[\psi_i]^{s,x}_r - [M[u]^{s,x}, M[\psi_i]^{s,x}]_T,
\end{array}
\tag{5.16}
$$

where the latter equality is a consequence of Lemma 2.4.
The next step will consist in taking the expectation in equation (5.16), but before, we will check that $\int_s^{\cdot} U^{s,x}_{r^-} dM[\psi_i]^{s,x}_r$ and $\int_s^{\cdot} \psi_i(r^-,X_{r^-})dM[u]^{s,x}_r$ are martingales.
By Proposition 4.15, $\langle M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$. So the angular bracket of $\int_s^{\cdot} U^{s,x}_{r^-} dM[\psi_i]^{s,x}_r$ at time T is equal to $\int_s^T u^2 \mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$ which is an integrable r.v. since $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded and $u \in \mathcal{L}^2_X$. Therefore $\int_s^{\cdot} U^{s,x}_{r^-} dM[\psi_i]^{s,x}_r$ is a square integrable martingale.
Then, by Hypothesis 5.1 and Proposition 5.2, $\sup\limits_{t\in[s,T]} |\psi_i(t,X_t)|^2 \in L^1$, and by Definition 4.13, $M[u]^{s,x} \in \mathcal{H}^2$ so by Lemma 3.15 in [6], $\int_s^{\cdot} \psi_i(r^-,X_{r^-})dM[u]^{s,x}_r$ is a martingale.

We can now take the expectation in (5.16), to get

$$
\begin{array}{rl}
& u\psi_i(s,x) \\
= & \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r - [M[u]^{s,x}, M[\psi_i]^{s,x}]_T\right] \\
= & \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) + \mathfrak{G}^{\psi_i}(u) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r\right],
\end{array}
\tag{5.17}
$$

since u and ψ_i belong to $\mathcal{D}(\mathfrak{a})$. Indeed the second equality follows from the fact $[M[u]^{s,x}, M[\psi_i]^{s,x}] - \langle M[u]^{s,x}, M[\psi_i]^{s,x}\rangle$ is a martingale and Proposition 4.15.

Since we have assumed that $u \in \mathcal{L}^2_X$, Lemma 5.15 says that $f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right) \in \mathcal{L}^2_X$, Hypothesis 5.1 implies that ψ_i and $a(\psi_i)$ are in $\mathcal{L}^2_X$, so all terms in the integral inside the expectation in the third line belong to $\mathcal{L}^1_X$. We can therefore apply Fubini's theorem to get

$$
u\psi_i(s,x) = P_{s,T}[g\psi_i(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(ua(\psi_i) + \mathfrak{G}^{\psi_i}(u) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,\cdot)\right](x)dV_r.
\tag{5.18}
$$

This concludes the proof. □

Proposition 5.16 admits a converse implication.

Proposition 5.17. *Let (f,g) verify $H^{growth}(\zeta,\eta)$. Every decoupled mild solution of $Pseudo-PDE(f,g)$ is a also a martingale solution. Moreover, if (u,v) solves $IP(f,g)$, then $v=\mathfrak{G}^{\psi}(u)$, up to zero potential sets.*

Proof. Let u and v_i, $i\in[\![1;d]\!]$ in $\mathcal{L}^2_X$ verify (5.9). We observe that the first line of (5.9) with $s=T$, gives $u(T,\cdot)=g$.

We fix $(s,x)\in[0,T]\times E$ and the associated probability $\mathbb{P}^{s,x}$. On $[s,T]$, we set $U:=u(\cdot,X)$ and $N:=u(\cdot,X)-u(s,x)+\int_s^{\cdot}f(r,X_r,u(r,X_r),v(r,X_r))dV_r$.

For some $t\in[s,T]$, we combine the first line of (5.9) applied in $(s,x)=(t,X_t)$ and the Markov property (C.3). Since $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ (see Lemma 5.15) we get the a.s. equalities

$$\begin{array}{rcl} U_t &=& u(t,X_t)\\ &=& P_{t,T}[g](X_t)+\int_t^T P_{t,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](X_t)dV_r\\ &=& \mathbb{E}^{t,X_t}\left[g(X_T)+\int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r\right]\\ &=& \mathbb{E}^{s,x}\left[g(X_T)+\int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right], \end{array} \tag{5.19}$$

from which we deduce that $N_t=\mathbb{E}^{s,x}\left[g(X_T)+\int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right]-u(s,x)$ a.s. and so N is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, the cadlag version $N^{s,x}$ of N. We extend now $N^{s,x}_t$, to $t\in[0,T]$, putting its value equal to zero on $[0,s]$, and consider the special semi-martingale

$$U^{s,x}:=u(s,x)-\int_s^{\cdot}f(r,X_r,u(r,X_r),v(r,X_r))dV_r+N^{s,x}, \tag{5.20}$$

indexed on $[s,T]$ which is obviously a cadlag version of U.

By Jensen's inequality, we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}[(N^{s,x}_T)^2] &=& \mathbb{E}^{s,x}\left[\left(g(X_T)+\int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r-u(s,x)\right)^2\right]\\ &\leq& 3u^2(s,x)+3\mathbb{E}^{s,x}[g^2(X_T)]+3\mathbb{E}^{s,x}\left[\int_s^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right]\\ &<& \infty, \end{array} \tag{5.21}$$

where the second term is finite because of $H^{mom}(\zeta,\eta)$ and $H^{growth}(\zeta,\eta)$, and the third one because $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$, see Lemma 5.15. So $N^{s,x}$ is square integrable. We have therefore shown that under any $\mathbb{P}^{s,x}$, the process $\left(u(\cdot,X_\cdot)-u(s,x)+\int_s^{\cdot}f(r,X_r,u(r,X_r),v(r,X_r))dV_r\right)\mathbb{1}_{[s,T]}$ has a modification in $\mathcal{H}^2_0$. According to Definitions 4.11 and 4.13 we have $u\in\mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u)=-f(\cdot,\cdot,u,v)$ and for any $(s,x)\in[0,T]\times E$, $M[u]^{s,x}=N^{s,x}$ in the sense of $P^{s,x}$-indistinguishability.

So to conclude that u is a martingale solution of $Pseudo-PDE(f,g)$, there is left to show $\mathfrak{G}^{\psi}(u)=v$, up to zero potential sets. By Proposition 4.15, this is equivalent to show that for every $(s,x)\in[0,T]\times E$ and $i\in[\![1;d]\!]$,

$$\langle M^{s,x}[u],M^{s,x}[\psi_i]\rangle=\int_s^{\cdot\vee s}v_i(r,X_r)dV_r, \tag{5.22}$$

in the sense of indistinguishability.

We fix again $(s,x) \in [0,T] \times E$, the associated probability, and some $i \in [\![1;d]\!]$ Combining the $(i+1)$th line of (5.9) applied in $(s,x) = (t,X_t)$ and the Markov property (C.3), taking into account the fact that all terms belong to $\mathcal{L}^1_X$ (see Lemma 5.15, Hypothesis 5.1) we get the a.s. equalities

$$\begin{array}{rcl} u\psi_i(t,X_t) & = & P_{t,T}[g\psi_i(T,\cdot)](X_t) - \int_t^T P_{t,r}\left[(v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,\cdot)\right](X_t)dV_r \\ & = & \mathbb{E}^{t,X_t}\left[g(X_T)\psi_i(T,X_T) - \int_t^T (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_t^T (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r|\mathcal{F}_t\right]. \end{array} \tag{5.23}$$

Setting, for $t \in [s,T]$, $N^i_t := u\psi_i(t,X_t) - \int_s^t (v_i + ua(\psi)^i - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r$, from (5.23) we deduce that, for any $t \in [s,T]$,

$$N^i_t = \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r \middle| \mathcal{F}_t\right]$$

a.s. So N^i is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, the cadlag version $N^{i,s,x}$ of N^i.

The process

$$\int_s^{\cdot} (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r + N^{i,s,x}, \tag{5.24}$$

is therefore a cadlag special semi-martingale which is a $\mathbb{P}^{s,x}$-version of $u\psi_i(\cdot,X)$ on $[s,T]$. But we also had shown, see (5.20), that
$U^{s,x} = u(s,x) - \int_s^{\cdot} f(r,X_r,u(r,X_r),v(r,X_r))dV_r + N^{s,x}$ is a version of $u(\cdot,X)$ which by integration by parts on the process $U^{s,x}\psi_i(\cdot,X.)$ implies that

$$\begin{array}{l} u\psi_i(s,x) + \int_s^{\cdot} U^{s,x}_r a(\psi_i)(r,X_r)dV_r + \int_s^{\cdot} U^{s,x}_{r^-} dM^{s,x}[\psi_i]_r \\ - \int_s^{\cdot} \psi_i f(\cdot,\cdot,u,v)(r,X_r)dV_r + \int_s^{\cdot} \psi_i(r^-,X_{r^-})dM^{s,x}[u]_r + [M^{s,x}[u], M^{s,x}[\psi_i]] \end{array} \tag{5.25}$$

is another cadlag semi-martingale which is a $\mathbb{P}^{s,x}$-version of $u\psi_i(\cdot,X)$ on $[s,T]$. Now (5.25) equals

$$\mathcal{M}^i + \mathcal{V}^i, \tag{5.26}$$

where

$$\begin{array}{rcl} \mathcal{M}^i_t & = & u\psi_i(s,x) + \int_s^t U^{s,x}_{r^-} dM^{s,x}[\psi_i]_r + \int_s^t \psi_i(r^-,X_{r^-})dM^{s,x}[u]_r \\ & + & ([M^{s,x}[u], M^{s,x}[\psi_i]]_t - \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle_t, \end{array}$$

is a local martingale and

$$\mathcal{V}^i_t = \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle_t + \int_s^t U^{s,x}_r a(\psi_i)(r,X_r)dV_r - \int_s^t \psi_i f(\cdot,\cdot,u,v)(r,X_r)dV_r,$$

is a predictable with bounded variation vanishing at zero process. Now (5.26) and (5.24) are two cadlag version of $u\psi_i(\cdot, X)$ on $[s,T]$.

By the uniqueness of the decomposition of a special semi-martingale, identifying the bounded variation predictable components and using Lemma 2.4 we get

$$\begin{aligned}\int_s^{\cdot}(v_i &+ ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r \\ &= \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle + \int_s^{\cdot} ua(\psi_i)(r,X_r)dV_r - \int_s^{\cdot} \psi_i f(\cdot,\cdot,u,v)(r,X_r)dV_r,\end{aligned}$$

This yields $\langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle = \int_s^{\cdot\vee s} v_i(r,X_r)dV_r$ as desired, which implies (5.22).

□

Proposition 5.18. *Let (f,g) verify $H^{growth}(\zeta,\eta)$. A classical solution of $Pseudo-PDE(f,g)$ is also a decoupled mild solution.*

Conversely, a decoupled mild solution of $Pseudo-PDE(f,g)$ belonging to $\mathcal{D}(\Gamma^{\psi})$ is a classical solution of $Pseudo-PDE(f,g)$ up to a zero-potential set, meaning that it verifies the first equality of (5.1) *up to a set of zero potential.*

Proof. Let u be a classical solution of $Pseudo-PDE(f,g)$. Definition 5.3 and Corollary 4.17 imply that $u(T,\cdot) = g$, and the equalities up to zero potential sets

$$\mathfrak{a}(u) = a(u) = -f(\cdot,\cdot,u,\Gamma^{\psi}(u)) = -f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)), \tag{5.27}$$

which shows that u is a martingale solution and by Proposition 5.16 it is also a decoupled mild solution.

Similarly, the second statement follows by Proposition 5.17, Definition 5.9, and again Corollary 4.17. □

5.3 Existence and uniqueness of a decoupled mild solution

In this subsection, the positive functions ζ,η and the functions (f,g) appearing in $Pseudo-PDE(f,g)$ are fixed. We still assume that the Markov class verifies $H^{mom}(\zeta,\eta)$.

Using arguments which are very close to those developed in [6], one can show the following theorem. For the convenience of the reader, we postpone the adapted proof to Appendix B.

Let $(Y^{s,x}, M^{s,x})$ be for any $(s,x)\in[0,T]\times E$ the unique solution of (5.2), see Notation 5.7.

Theorem 5.19. *Let (f,g) verify $H^{lip}(\zeta,\eta)$. There exists $u\in\mathcal{D}(\mathfrak{a})$ such that for any $(s,x)\in[0,T]\times E$*

$$\begin{cases}\forall t\in[s,T]: Y^{s,x}_t &= u(t,X_t) \quad \mathbb{P}^{s,x} a.s. \\ M^{s,x} &= M[u]^{s,x},\end{cases}$$

and in particular $\frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(u)(\cdot, X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ *a.e. on* $[s,T]$*. Moreover, for every* (s,x)*,* $Y_s^{s,x}$ *is* $\mathbb{P}^{s,x}$ *a.s. equal to a constant (which we shall still denote* $Y_s^{s,x}$*) and* $u(s,x) = Y_s^{s,x}$ *for every* $(s,x) \in [0,T] \times E$*.*

Corollary 5.20. *Let* (f,g) *verify* $H^{lip}(\zeta,\eta)$*. For any* $(s,x) \in [0,T] \times E$*, the functions* u *obtained in Theorem 5.19 verifies* $\mathbb{P}^{s,x}$ *a.s. on* $[s,T]$

$$u(t,X_t) = g(X_T) + \int_t^T f\left(r, X_r, u(r,X_r), \mathfrak{G}^\psi(u)(r,X_r)\right) dV_r - (M[u]_T^{s,x} - M[u]_t^{s,x}),$$

and in particular, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,\mathfrak{G}^\psi(u))$*.*

Proof. The corollary follows from Theorem 5.19 and Lemma 2.4. □

Theorem 5.21. *Let* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *be a Markov class associated to a transition kernel measurable in time (see Definitions C.5 and C.4) which solves a martingale problem associated with the triplet* $(\mathcal{D}(a), a, V)$*. Moreover we suppose Hypothesis* $H^{mom}(\zeta,\eta)$ *for some positive* ζ,η*. Let* (f,g) *be a couple verifying* $H^{lip}(\zeta,\eta)$*.*

Then $Pseudo-PDE(f,g)$ *has a unique decoupled mild solution given by*

$$u : \begin{array}{ccc} [0,T]\times E & \longrightarrow & \mathbb{R} \\ (s,x) & \longmapsto & Y_s^{s,x}, \end{array} \tag{5.28}$$

where $(Y^{s,x}, M^{s,x})$ *denotes the (unique) solution of* $BSDE^{s,x}(f,g)$ *for fixed* (s,x)*.*

Proof. Let u be the function exhibited in Theorem 5.19. In order to show that u i.e. a decoupled mild solution of $Pseudo-PDE(f,g)$, it is enough by Proposition 5.16 to show that it is a martingale solution.
In Corollary 5.20, we have already seen that $\mathfrak{a}(u) = -f(\cdot,\cdot,u,\mathfrak{G}^\psi(u))$.
Concerning the second line of (5.3), for any $x \in E$, we have
$u(T,x) = u(T,X_T) = g(X_T) = g(x)$ $\mathbb{P}^{T,x}$ a.s., so $u(T,\cdot) = g$, in the deterministic pointwise sense.
We now show uniqueness. By Proposition 5.17, it is enough to show that $Pseudo-PDE(f,g)$ admits at most one martingale solution. Let u, u' be two martingale solutions of $Pseudo-PDE(f,g)$. We fix $(s,x) \in [0,T]\times E$. By Proposition 5.11, both couples, indexed by $[s,T]$,
$\left(u(\cdot,X_\cdot),\quad u(\cdot,X_\cdot) - u(s,x) + \int_s^\cdot f(\cdot,\cdot,u,\mathfrak{G}^\psi(u))(r,X_r)dV_r\right)$ and
$\left(u'(\cdot,X_\cdot),\quad u'(\cdot,X_\cdot) - u'(s,x) + \int_s^\cdot f(\cdot,\cdot,u',\mathfrak{G}^\psi(u))(r,X_r)dV_r\right)$ admit a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f,g)$ on $[s,T]$. By Theorem 3.3 and Remark 3.4, $BSDE^{s,x}(f,g)$ admits a unique solution, so $u(\cdot,X_\cdot)$ and $u'(\cdot,X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications one of the other on $[s,T]$. In particular, considering their values at time s, we have $u(s,x) = u'(s,x)$. We therefore have $u' = u$. □

Corollary 5.22. *Let* (f,g) *verify* $H^{lip}(\zeta,\eta)$*. There is at most one classical solution of* $Pseudo-PDE(f,g)$ *and this only possible classical solution is the unique decoupled mild solution* $(s,x) \longmapsto Y_s^{s,x}$*, where* $(Y^{s,x}, M^{s,x})$ *denotes the (unique) solution of* $BSDE^{s,x}(f,g)$ *for fixed* (s,x)*.*

Proof. The proof follows from Proposition 5.18 and Theorem 5.21. □

Remark 5.23. *The function v such that (u, v) is the unique solution of the identification problem $IP(f, g)$ also has a stochastic representation since it verifies for every $(s, x) \in [0, T] \times E$, on the interval $[s, T]$, $\frac{d\langle M^{s,x}, M^{s,x}[\psi]\rangle}{dV} = v(\cdot, X_\cdot)\ dV \otimes d\mathbb{P}^{s,x}$ a.e. where $M^{s,x}$ is the martingale part of the solution of $BSDE^{s,x}(f, g)$.*

The existence of a decoupled mild solution of $Pseudo - PDE(f, g)$ provides in fact an existence theorem for $BSDE^{s,x}(f, g)$ for any (s, x). The following constitutes the converse of Theorem 5.21.

Proposition 5.24. *Assume (f, g) verifies $H^{mom}(\zeta, \eta)$. Let (u, v) be a solution of $IP(f, g)$, let $(s, x) \in [0, T] \times E$ and the associated probability $\mathbb{P}^{s,x}$ be fixed. The couple*

$$\left(u(t, X_t), \quad u(t, X_t) - u(s, x) + \int_s^t f(\cdot, \cdot, u, v)(r, X_r)dV_r\right)_{t\in[s,T]} \tag{5.29}$$

has a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f, g)$ on $[s, T]$.

In particular if (f, g) verifies the stronger hypothesis $H^{lip}(\zeta, \eta)$ and (u, v) is the unique solution of $IP(f, g)$, then for any $(s, x) \in [0, T] \times E$, $\left(u(t, X_t), \quad u(t, X_t) - u(s, x) + \int_s^t f(\cdot, \cdot, u, v)(r, X_r)dV_r\right)_{t\in[s,T]}$ is a $\mathbb{P}^{s,x}$ modification of the unique solution of $BSDE^{s,x}(f, g)$ on $[s, T]$.

Proof. It follows from Propositions 5.17, and 5.11.

□

6 Examples of applications

We now develop some examples. In all the items below there will be a canonical Markov class with transition kernel being measurable in time which is solution of a Martingale Problem associated to some triplet $(\mathcal{D}(a), a, V)$ as introduced in Definition 4.1. Therefore all the results of this paper will apply to all the examples below. In particular, Propositions 5.17, 5.18, Theorem 5.21, Corollary 5.22 and Proposition 5.24 will apply but we will mainly emphasize Theorem 5.21 and Corollary 5.22.

In all the examples $T > 0$ will be fixed.

6.1 A new approach to Brownian BSDEs and associate semilinear PDEs

In this subsection, the state space will be $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$.

Notation 6.1. *A function $\phi \in \mathcal{B}([0,T]\times\mathbb{R}^d,\mathbb{R})$ will be said to have* ***polynomial growth*** *if there exists $p \in \mathbb{N}$ and $C > 0$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $|\phi(t,x)| \leq C(1 + \|x\|^p)$. For any $k,p \in \mathbb{N}$, $\mathcal{C}^{k,p}([0,T] \times \mathbb{R}^d)$ (resp. $\mathcal{C}_b^{k,p}([0,T] \times \mathbb{R}^d)$, resp. $\mathcal{C}_{pol}^{k,p}([0,T]\times\mathbb{R}^d)$) will denote the sublinear algebra of $\mathcal{C}([0,T]\times\mathbb{R}^d,\mathbb{R})$ of functions admitting continuous (resp. bounded continuous, resp. continuous with polynomial growth) derivatives up to order k in the first variable and order p in the second.*

We consider bounded Borel functions $\mu \in \mathcal{B}_b([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and $\alpha \in \mathcal{B}_b([0,T] \times \mathbb{R}^d, S_d^+(\mathbb{R}))$ where $S_d^+(\mathbb{R})$ is the space of symmetric non-negative $d \times d$ real matrices. We define for $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$ the operator a by

$$a(\phi) = \partial_t\phi + \frac{1}{2}\sum_{i,j\leq d}\alpha_{i,j}\partial^2_{x_i x_j}\phi + \sum_{i\leq d}\mu_i\partial_{x_i}\phi. \tag{6.1}$$

We will assume the following.

Hypothesis 6.2. *There exists a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ which solves the Martingale Problem associated to $(\mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d), a, V_t \equiv t)$ in the sense of Definition 4.1.*

We now recall a non-exhaustive list of sets of conditions on μ, α under which Hypothesis 6.2 is satisfied.

1. α is continuous non-degenerate, in the sense that for any t, x, $\alpha(t,x)$ is invertible, see Theorem 4.2 in [34];

2. μ and α are continuous in the second variable, see Exercise 12.4.1 in [35];

3. $d = 1$ and α is uniformly positive on compact sets, see Exercise 7.3.3 in [35].

Remark 6.3. *When the first or third item above is verified, the mentioned Markov class is unique, but if the second one holds, uniqueness may not hold. We therefore fix a Markov class which solves the martingale problem. We wish to emphasize that given a fixed Markov class, we will obtain some uniqueness results concerning the martingale solution and the decoupled mild solution of an associated PDE, but that for every Markov class solving the martingale problem may correspond a different solution.*

In this context, for ϕ, ψ in $\mathcal{D}(a)$, the carré du champs operator (see Definition 4.4) is given by $\Gamma(\phi,\psi) = \sum\limits_{i,j\leq d}\alpha_{i,j}\partial_{x_i}\phi\partial_{x_j}\psi$.

Remark 6.4. *By a localization procedure, it is also clear that for every $(s,x) \in [0,T]\times\mathbb{R}^d$, $\mathbb{P}^{s,x}$ verifies that for any $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr \in \mathcal{H}^2_{loc}$ and that Proposition 4.5 extends to all $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$.*

We set now $\mathcal{D}(a) = \mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d)$.

For any $i \in [\![1;d]\!]$, the function Id_i denotes $(t,x) \longmapsto x_i$ which belongs to $\mathcal{D}(a)$ and $Id := (Id_1,\cdots,Id_d)$.
It is clear that for any i, $a(Id_i) = \mu_i$, and for any i,j, $Id_iId_j \in \mathcal{D}(a)$ and $\Gamma(Id_i,Id_j) = \alpha_{i,j}$. In particular, by Corollary 4.7, $(Id_1,\cdots,Id_d)$ verify Hypothesis 4.6 and since μ,α are bounded, they verify Hypothesis 5.1.

For any i we can therefore consider the MAF $M[Id_i] : (t,u) \mapsto X^i_u - X^i_t - \int_t^u \mu_i(r,X_r)dr$ whose cadlag version under $\mathbb{P}^{s,x}$ for every $(s,x)\in[0,T]\times\mathbb{R}^d$ is $M[Id_i]^{s,x} = \mathbb{1}_{[s,T]}\left(X^i - x_i - \int_s^\cdot \mu_i(r,X_r)dr\right)$ and we have for any i,j $\langle M[Id_i]^{s,x}, M[Id_j]^{s,x}\rangle = \int_s^{\cdot\vee s}\alpha_{i,j}(r,X_r)dr$.

Lemma 6.5. *Let $(s,x)\in[0,T]\times\mathbb{R}^d$ and associated probability $\mathbb{P}^{s,x}$, $i\in[\![1;d]\!]$ and $p\in[1,+\infty[$ be fixed. Then $\underset{t\in[s,T]}{\sup}|X^i_t|^p \in L^1$.*

Proof. We have $X^i = x_i + \int_s^\cdot \mu_i(r,X_r)dr + M[Id_i]^{s,x}$ where μ_i is bounded so it is enough to show that $\underset{t\in[s,T]}{\sup}|M[Id_i]^{s,x}_t|^p \in L^1$. Since $\langle M[Id_i]^{s,x}\rangle = \int_s^{\cdot\vee s}\alpha_{i,i}(r,X_r)dr$, which is bounded, the result holds by Burkholder-Davis-Gundy inequality. □

Corollary 6.6. *$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves the Martingale Problem associated to $(\mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d), a, V_t\equiv t)$ in the sense of Definition 4.1.*

Proof. By Remark 6.4, for any $\phi\in\mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d)$ and $(s,x)\in[0,T]\times\mathbb{R}^d$, $\phi(\cdot,X_\cdot) - \int_s^\cdot a(\phi)(r,X_r)dr$ is a $\mathbb{P}^{s,x}$-local martingale. Since ϕ and $a(\phi)$ have polynomial growth, Lemma 6.5 and Jensen's inequality imply that it is also a square integrable martingale. □

We now consider a couple (f,g) verifying $H^{lip}(\|\cdot\|^p,\|\cdot\|^p)$ for some $p\geq 1$. In this case Hypothesis 5.5 can be retranslated into what follows.

- g is Borel with polynomial growth;
- f is Borel with polynomial growth in x (uniformly in t), and Lipschitz in y,z.

We consider the PDE

$$\left\{\begin{array}{l}\partial_t u + \frac{1}{2}\sum\limits_{i,j\leq d}\alpha_{i,j}\partial^2_{x_ix_j}u + \sum\limits_{i\leq d}\mu_i\partial_{x_i}u + f(\cdot,\cdot,u,\alpha\nabla u) = 0\\ u(T,\cdot) = g.\end{array}\right. \tag{6.2}$$

We emphasize that for $u\in\mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d)$, $\alpha\nabla u = \Gamma^{Id}(u)$. The associated decoupled mild equation is given by

$$\begin{cases} u(s,x) &= P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dr \\ u(s,x)x_i &= P_{s,T}[gId_i](x) - \int_s^T P_{s,r}\left[\left(v_i + u\mu_i - Id_i f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dr, i \in [\![1;d]\!], \end{cases} \tag{6.3}$$

$(s,x) \in [0,T] \times \mathbb{R}^d$, where P is the transition kernel of the Markov class.

Proposition 6.7. *Assume that Hypothesis 6.2 is verified and that (f,g) verifies $H^{lip}(\|\cdot\|^p, \|\cdot\|^p)$ for some $p \geq 1$. Then equation (6.2) has a unique decoupled mild solution u.*

Moreover it has at most one classical solution which (when it exists) equals this function u.

Proof. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ verifies a martingale problem in the sense of Definition 4.1 and has a transition kernel which is measurable in time. Moreover $(Id_1, \cdots, Id_d)$ verify Hypothesis 5.1, $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ verifies (by Lemma 6.5) $H^{mom}(\|\cdot\|^p, \|\cdot\|^p)$ for some $p \geq 1$ and (f,g) verifies $H^{lip}(\|\cdot\|^p, \|\cdot\|^p)$. So Theorem 5.21 and Corollary 5.22 apply. □

Remark 6.8. *The unique decoupled mild solution mentioned in the previous proposition admits the probabilistic representation given in Theorem 5.21.*

Remark 6.9. *In the classical literature, a Brownian BSDE is linked to a slightly different type of parabolic PDE, see the introduction of the present paper, or [29] for more details.*

The PDE which is generally considered is of the type

$$\begin{cases} \partial_t u + \frac{1}{2}\sum_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_i x_j} u + \sum_{i\leq d}\mu_i \partial_{x_i} u + f(\cdot,\cdot,u,\sigma\nabla u) = 0 \\ u(T,\cdot) = g, \end{cases} \tag{6.4}$$

(where $\sigma = \sqrt{\alpha}$ in the sense of non-negative symmetric matrices) rather than (6.2). In fact, the only difference is that the term $\sigma\nabla u$ replaces $\alpha\nabla u$ in the fourth argument of the driver f.

We recall that the Markovian BSDE was given in (1.1).

Under the probability $\mathbb{P}^{s,x}$ (for some fixed (s,x)), one can introduce the square integrable martingale $\tilde{M}[Id]^{s,x} := \int_s^\cdot (\sigma^\intercal)^+(r,X_r)dM[Id]^{s,x}_r$ where $A \mapsto A^+$ denotes the Moore-Penrose pseudo-inverse operator, see [9] chapter 1. The Brownian BSDE (1.1) can then be reexpressed here as

$$Y^{s,x}_t = g(X_T) + \int_t^T f\left(r, X_r, Y^{s,x}_r, \frac{d\langle M^{s,x}, \tilde{M}[Id]^{s,x}\rangle_r}{dr}\right)dr - (M^{s,x}_T - M^{s,x}_t). \tag{6.5}$$

Under the assumptions of Proposition 6.7 where $\alpha = \sigma\sigma^\intercal$), it is possible to show that (6.5) constitutes the probabilistic representation of (6.4) performing similar arguments as in our approach for (6.2). In particular we can show existence and *uniqueness of a function $u \in \mathcal{L}^2_X$ for which there exists $v_1, \cdots, v_d \in \mathcal{L}^2_X$ such that for all $(s,x) \in [0,T] \times \mathbb{R}^d$,*

$$\left\{ \begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,(\sigma^\intercal)^+ v\right)(r,\cdot)\right](x)dr \\ u(s,x)x_i & = & P_{s,T}[gId_i](x) - \int_s^T P_{s,r}\left[\left(v_i + u\mu_i - Id_i f\left(\cdot,\cdot,u,(\sigma^\intercal)^+ v\right)\right)(r,\cdot)\right](x)dr, i \in [\![1;d]\!], \end{array} \right. \tag{6.6}$$

and that this function u is the only possible classical solution of (6.4) *in* $\mathcal{C}^{1,2}_{pol}([0,T] \times \mathbb{R}^d)$. (6.6) *constitutes the "good" version of decoupled mild solution for the* (6.4). *This technique is however technically more complicated and for purpose of illustration we prefer to remain in our set up (which is by the way close to* (6.4)*) to keep our notion of decoupled-mild solution more comprehensible.*

Remark 6.10. *It is also possible to treat jump diffusions instead of continuous diffusions (see [34]), and under suitable conditions on the coefficients, it is also possible to prove existence and uniqueness of a decoupled mild solution for equations of type*

$$\left\{ \begin{array}{l} \partial_t u + \frac{1}{2}Tr(\alpha\nabla^2 u) + (\mu, \nabla u) + \int \left(u(\cdot,\cdot+y) - u - \frac{(y,\nabla u)}{1+\|y\|^2}\right) K(\cdot,\cdot,dy) + f(\cdot,\cdot,u,\Gamma^{Id}(u)) = 0 \\ u(T,\cdot) = g, \end{array} \right. \tag{6.7}$$

where K is a Lévy kernel: this means that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $K(t,x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d \backslash \{0\}$, $\underset{t,x}{sup} \int \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy)$ is Borel. In that framework we have

$$\Gamma^{Id} : \phi \longmapsto \alpha\nabla\phi + \left(\int y_i(\phi(\cdot,\cdot+y) - \phi(\cdot,\cdot))K(\cdot,\cdot,dy)\right)_{i\in[\![1;d]\!]}. \tag{6.8}$$

6.2 Parabolic semi-linear PDEs with distributional drift

In this subsection we will use the formalism and results obtained by in [21] and [22], see also [32] and [14] for more recent developments in dimension 1. Those authors make reference to stochastic differential equations with distributional drift whose solution are possibly non-semimartingales. Those papers introduced a suitable framework of Martingale Problem related to a PDE operator involving a distributional drift b' which is the derivative of a continuous function. [20] established a first work in the n-dimensional setting, later developments were discussed by [11]. Other Markov processes associated to diffusion operators which are not semimartingales were produced when the diffusion operator is in divergence form, see e.g. [31] or Markov processes associated to singular PDEs involving paracontrolled distributions introduced in [11].

Let $b, \sigma \in \mathcal{C}^0(\mathbb{R})$ such that $\sigma > 0$. By a mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)dx = 1$.
We denote $\Phi_n(x) = n\Phi(nx)$, $\sigma_n^2 = \sigma^2 * \Phi_n$, $b_n = b * \Phi_n$.
We then define $L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'$. $f \in \mathcal{C}^1(\mathbb{R})$ is said to be a solution to $Lf = \dot{l}$

where $\dot{l} \in \mathcal{C}^0(\mathbb{R})$, if for any mollifier Φ, there are sequences (f_n) in $\mathcal{C}^2(\mathbb{R})$, $(\dot{l}_n)$ in $\mathcal{C}^0(\mathbb{R})$ such that

$$L_n f_n = (\dot{l}_n),\ f_n \overset{\mathcal{C}^1(\mathbb{R})}{\longrightarrow} f,\ \dot{l}_n \overset{\mathcal{C}^0(\mathbb{R})}{\longrightarrow} \dot{l}. \tag{6.9}$$

We will assume that $\Sigma(x) = \lim_{n\to\infty} 2\int_0^x \frac{b'_n}{\sigma_n^2}(y)dy$ exists in $\mathcal{C}^0(\mathbb{R})$ independently from the mollifier.

By Proposition 2.3 in [21] there exists a solution $h \in \mathcal{C}^1(\mathbb{R})$ to $Lh = 0$, $h(0) = 0$, $h'(0) = 1$. Moreover it verifies $h' = e^{-\Sigma}$.

$\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1(\mathbb{R})$ such that there exists some $\dot{l} \in \mathcal{C}^0(\mathbb{R})$ with $Lf = \dot{l}$ and it is a linear algebra.

Let v be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$ (see Remark 2.4 in [21]), we will assume that

$$v(-\infty) = v(+\infty) = +\infty, \tag{6.10}$$

which represents a non-explosion condition. In this case, Proposition 3.13 in [21] states that a certain martingale problem associated to $(\mathcal{D}_L, L)$ is well-posed. Its solution will be denoted $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$.

X is a $\mathbb{P}^{s,x}$-Dirichlet process for every (s,x) and $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ defines a Markov class and Proposition B.2 in [7] implies that its transition kernel is measurable in time.

We introduce the domain that we will indeed use.

Definition 6.11. *We set*

$$\mathcal{D}^{max}(a) = \left\{\phi \in \mathcal{C}^{1,1}([0,T]\times\mathbb{R}) : \frac{\partial_x \phi}{h'} \in \mathcal{C}^{1,1}([0,T]\times\mathbb{R})\right\} \tag{6.11}$$

On $\mathcal{D}^{max}(a)$, we set $L\phi := \frac{\sigma^2 h'}{2}\partial_x(\frac{\partial_x\phi}{h'})$ and $a(\phi) := \partial_t\phi + L\phi$. We then define the smaller domain

$$\mathcal{D}(a) = \left\{\phi \in \mathcal{D}^{max}(a) : \sigma\partial_x\phi \in \mathcal{C}^{0,0}_{pol}([0,T]\times\mathbb{R})\right\}. \tag{6.12}$$

We formulate here some supplementary assumptions that we will make, the first one being called (TA) in [21].

Hypothesis 6.12.

- *There exists $c_1, C_1 > 0$ such that $c_1 \le \sigma h' \le C_1$;*
- *σ has linear growth.*

The first item states in particular that $\sigma h'$ is bounded so $h \in \mathcal{D}(a)$. Proposition 3.2 in [21] states that for every (s,x), $\langle M[h]^{s,x}\rangle = \int_s^{\cdot\vee s}(\sigma h')^2(X_r)dr$. Moreover the AF $\langle M[h]\rangle_u^t = \int_t^u(\sigma h')^2(X_r)dr$ is absolutely continuous with respect to $\hat{V}_t \equiv t$. Therefore Hypothesis 4.6 is verified (for $\psi = h$) and $\mathfrak{G}^h(h) = (\sigma h')^2$.

Since this function is bounded and clearly $a(h) = 0$ then h verifies Hypothesis 5.1.

We will therefore consider Γ^h the h-generalized gradient associated to a, and Proposition 4.23 in [7] implies the following.

Proposition 6.13. *Let $\phi \in \mathcal{D}(\Gamma^h)$, then $\Gamma^h(\phi) = \sigma^2 h' \partial_x \phi$.*

Remarking that b' is a distribution, the equation that we will study in this section is the following.

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}\sigma^2 \partial_x^2 u + b' \partial_x u + f(\cdot,\cdot,u,\sigma^2 h' \partial_x u) = 0 \quad \text{on } [0,T]\times\mathbb{R} \\ u(T,\cdot) = g. \end{array}\right. \tag{6.13}$$

The associated PDE in the decoupled mild sense is given by

$$\left\{\begin{array}{rcl} u(s,x) &=& P_{T-s}[g](x) + \int_s^T P_{r-s}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dr \\ u(s,x)h(x) &=& P_{T-s}[gh](x) - \int_s^T P_{r-s}\left[\left(v - hf\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dr, \end{array}\right. \tag{6.14}$$

$(s,x) \in [0,T]\times\mathbb{R}$, where P is the (homogeneous) transition kernel of the Markov class.

In order to consider the $BSDE^{s,x}(f,g)$ for functions (f,g) having polynomial growth in x, we had shown in [7] the following result, stated as Proposition 4.26.

Proposition 6.14. *We suppose that Hypothesis 6.12 is fulfilled. Then, for any $p \in \mathbb{N}$ and $(s,x) \in [0,T]\times\mathbb{R}$, $\mathbb{E}^{s,x}[|X_T|^p] < \infty$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. In other words, for any $p \geq 1$, the Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ verifies $H^{mom}(|\cdot|^p, |\cdot|^p)$, see Hypothesis 5.4.*

Proposition 6.15. *We suppose that Hypothesis 6.12 is fulfilled. Then $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves the Martingale Problem associated to $(a, \mathcal{D}(a), V_t \equiv t)$ in the sense of Definition 4.1.*

Proof. Let $(s,x) \in [0,T]\times\mathbb{R}$ be fixed. Proposition 4.24 in [7] implies that for any $\phi \in \mathcal{D}(a)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r,X_r)dr$ is a (continuous) $\mathbb{P}^{s,x}$-local martingale, so taking Definition 4.1 into account, it is enough to show that this local martingale is a square integrable martingale. Considering Definition 4.22, Proposition 4.23 and Proposition 2.7 in [7], we know that the angular bracket of this local martingale is given by $\int_s^\cdot (\sigma\partial_x\phi)^2(r,X_r)dr$. Since $\phi \in \mathcal{D}(a)$ then $\sigma\partial_x\phi$ has polynomial growth, so by Proposition 6.14, $\int_s^T (\sigma\partial_x\phi)^2(r,X_r)dr \in L^1$ and this implies that the overmentioned local martingale is a square integrable martingale. ☐

We can now state the main result of this section.

Proposition 6.16. *Assume that the non-explosion condition* (6.10) *is verified, that Hypothesis 6.12 is fulfilled and (f,g) verifies $H^{lip}(|\cdot|^p, |\cdot|^p)$ for some $p \geq 1$, see Hypothesis 5.5. Then equation* (6.13) *has a unique decoupled mild solution u. It has at most one classical solution which can only be equal to u.*

Remark 6.17. *The unique decoupled mild solution u can be of course represented by* (5.28)*, Theorem 5.21.*

Proof. The assertions come from Theorem 5.21 and Corollary 5.22 which applies thanks to Propositions 6.15, 6.14, and the fact that h verifies Hypothesis 5.1. □

Remark 6.18.

1. *[33] has made a first analysis linking elliptic PDEs (in fact second order ODEs) with distributional drift and BSDEs. In those BSDEs the final horizon was a stopping time.*
2. *[24] have considered a class of BSDEs involving distributions in their setting.*

Appendices

A Proof of Theorem 3.3 and related technicalities

We adopt here the same notations as at the beginning of Section 3. We will denote $L^2(d\hat{V} \otimes d\mathbb{P})$ the quotient space of $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ with respect to the subspace of processes equal to zero $d\hat{V} \otimes d\mathbb{P}$ a.e.
$L^2(d\hat{V} \otimes d\mathbb{P})$ is a Hilbert space equipped with its usual norm.
$L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P}))$ will denote the subspace of $L^2(d\hat{V} \otimes d\mathbb{P}))$ of elements having a cadlag representative. We emphasize that $L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ is not a closed subspace of $L^2(d\hat{V} \otimes d\mathbb{P})$.
The application which to a process associate its class will be denoted $\phi \mapsto \dot{\phi}$.

Proposition A.1. *If (Y, M) solves $BSDE(\xi, \hat{f}, V, \hat{M})$, and if we denote $\hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right)$ by $\hat{f}_r$, then for any $t \in [0, T]$, a.s. we have*

$$\left\{ \begin{array}{rcl} Y_t &=& \mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] \\ M_t &=& \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right]. \end{array} \right. \tag{A.1}$$

Proof. Since $Y_t = \xi + \int_t^T \hat{f}_r d\hat{V}_r - (M_T - M_t)$ a.s., Y being an adapted process and M a martingale, taking the expectation in (3.2) at time t, we directly get $Y_t = \mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right]$ and in particular that $Y_0 = \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right]$.
Since $M_0 = 0$, looking at the BSDE at time 0 we get

$$M_T = \xi + \int_0^T \hat{f}_r d\hat{V}_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right].$$

Taking the expectation with respect to $\mathcal{F}_t$ in the above inequality gives the second line of (A.1). □

Lemma A.2. *Let $M \in \mathcal{H}^2$ and ϕ be a bounded positive process. Then there exists a constant $C > 0$ such that for any $i \in [\![1; d]\!]$,*
$\int_0^T \phi_r \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r \leq C \int_0^T \phi_r d\langle M\rangle_r$. In particular, $\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}$ belongs to $L^2(d\hat{V} \otimes d\mathbb{P})$.

Proof. We fix $i \in [\![1; d]\!]$. By Hypothesis 3.1 $\frac{d\langle \hat{M}^i\rangle}{d\hat{V}}$ is bounded; using Proposition B.1 and Remark 3.3 in [6], we show the existence of $C > 0$ such that

$$\begin{array}{rcl} \int_0^T \phi_r \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r & \leq & \int_0^T \phi_r \frac{d\langle \hat{M}^i\rangle}{d\hat{V}}(r) \frac{d\langle M\rangle}{d\hat{V}}(r) d\hat{V}_r \\ & \leq & C \int_0^T \phi_r \frac{d\langle M\rangle}{d\hat{V}}(r) d\hat{V}_r \\ & \leq & C \int_0^T \phi_r d\langle M\rangle_r. \end{array} \tag{A.2}$$

In particular, setting $\phi = 1$, we have $\int_0^T \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r \leq C\langle M\rangle_T$ which belongs to L^1 since $M \in \mathcal{H}^2_0$. ☐

We fix for now a couple $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ and we consider a representative U of $\dot{U}$. Until Proposition A.6 included, we will use the notation $\hat{f}_r := \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right)$.

Proposition A.3. *For any $t \in [0, T]$, $\int_t^T \hat{f}_r^2 d\hat{V}_r$ belongs to L^1 and $\left(\int_t^T \hat{f}_r d\hat{V}_r\right)$ is in L^2.*

Proof. By Jensen's inequality, thanks to the Lipschitz conditions on $\hat{f}$ in Hypothesis 3.1 and by Lemma A.2 there exist positive constants $C, C', C"$ such that, for any $t \in [0, T]$, we have

$$\begin{array}{rcl} \left(\int_t^T \hat{f}_r d\hat{V}_r\right)^2 & \leq & C \int_t^T \hat{f}_r^2 d\hat{V}_r \\ & \leq & C' \left(\int_t^T \hat{f}^2(r, \cdot, 0, 0) d\hat{V}_r + \int_t^T U_r^2 d\hat{V}_r + \sum_{i=1}^d \int_t^T \left(\frac{d\langle N, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\right) \\ & \leq & C" \left(\int_t^T \hat{f}^2(r, \cdot, 0, 0) d\hat{V}_r + \int_t^T U_r^2 d\hat{V}_r + (\langle N\rangle_T - \langle N\rangle_t)\right). \end{array} \tag{A.3}$$

All terms on the right-hand side are in L^1. Indeed, N is taken in $\mathcal{H}^2$, $\dot{U}$ in $L^2(d\hat{V} \otimes d\mathbb{P})$ and by Hypothesis 3.1, $f(\cdot, \cdot, 0, 0)$ is in $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$. This concludes the proof. ☐

We can therefore state the following definition.

Definition A.4. *Let M be the cadlag version of the martingale $t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right]$.*

M is square integrable by Proposition A.3. It admits a cadlag version taking into account Theorem 4 in Chapter IV of [16], since the stochastic basis fulfills the usual conditions. We denote by Y the cadlag process defined by

$Y_t = \xi + \int_t^T \hat{f}_r d\hat{V}_r - (M_T - M_t)$. *This will be called the* **cadlag reference process** *and we omit its dependence to* $(\dot{U}, N)$.

Proposition A.5. *Y and M take square integrable values.*

Proof. We already know that M is a square integrable martingale. As we have seen in Proposition A.3, $\int_t^T \hat{f}_r d\hat{V}_r$ belongs to L^2 for any $t \in [0,T]$ and by Hypothesis 3.1, $\xi \in L^2$. So by (A.1) and Jensen's inequality for conditional expectation we have

$$\begin{array}{rcl} \mathbb{E}\left[Y_t^2\right] & = & \mathbb{E}\left[\mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right]^2\right] \\ & \leq & \mathbb{E}\left[\mathbb{E}\left[\left(\xi + \int_t^T \hat{f}_r d\hat{V}_r\right)^2 \middle| \mathcal{F}_t\right]\right] \\ & \leq & \mathbb{E}\left[2\xi^2 + 2\int_t^T \hat{f}_r^2 d\hat{V}_r\right], \end{array}$$

which is finite. □

Proposition A.6. $\sup\limits_{t\in[0,T]} |Y_t| \in L^2$ *and in particular,* $Y \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes \mathbb{P})$.

Proof. Since $dY_r = -\hat{f}_r d\hat{V}_r + dM_r$, by integration by parts formula we get

$$d(Y_r^2 e^{-\hat{V}_r}) = -2e^{-\hat{V}_r} Y_r \hat{f}_r d\hat{V}_r + 2e^{-\hat{V}_r} Y_{r-} dM_r + e^{-\hat{V}_r} d[M]_r - e^{-\hat{V}_r} Y_r^2 d\hat{V}_r.$$

So integrating from 0 to some $t \in [0,T]$, we get

$$\begin{array}{rcl} Y_t^2 e^{-\hat{V}_t} & = & Y_0^2 - 2\int_0^t e^{-\hat{V}_r} Y_r \hat{f}_r d\hat{V}_r + 2\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r \\ & & + \int_0^t e^{-\hat{V}_r} d[M]_r - \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r \\ & \leq & Y_0^2 + \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r + \int_0^t e^{-\hat{V}_r} \hat{f}_r^2 d\hat{V}_r \\ & & +2\left|\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r\right| + \int_0^t e^{-\hat{V}_r} d[M]_r - \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r \\ & \leq & Z + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r\right|, \end{array}$$

where $Z = Y_0^2 + \int_0^T e^{-\hat{V}_r} \hat{f}_r^2 d\hat{V}_r + \int_0^T e^{-\hat{V}_r} d[M]_r$. Therefore, for any $t \in [0,T]$ we have $(Y_t e^{-\hat{V}_t})^2 \leq Y_t^2 e^{-\hat{V}_t} \leq Z + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r\right|$. Thanks to Propositions A.3 and A.5, Z is integrable, so we can conclude by Lemma 3.16 in [6] applied to the process $Ye^{-\hat{V}}$, and the fact that $\hat{V}$ is bounded.

Since Y is cadlag progressively measurable, $\sup\limits_{t\in[0,T]} |Y_t| \in L^2$ and since $\hat{V}$ is bounded, it is clear that $Y \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ and the corresponding class $\dot{Y}$ belongs to $L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$. □

Thanks to Propositions A.5 and A.6, we are allowed to introduce the following operator.

Notation A.7. *We denote by Φ the operator which associates to a couple $(\dot{U},N)$ the couple $(\dot{Y},M)$.*

$$\Phi : \begin{array}{ccc} L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}^2_0 & \longrightarrow & L^{2,cadlag}(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}^2_0 \\ (\dot{U},N) & \longmapsto & (\dot{Y},M). \end{array}$$

Proposition A.8. *The mapping $(Y,M)\longmapsto(\dot{Y},M)$ induces a bijection between the set of solutions of $BSDE(\xi,\hat{f},\hat{V},\hat{M})$ and the set of fixed points of Φ.*

Proof. First, let (U,N) be a solution of $BSDE(\xi,\hat{f},V,\hat{M})$, let $(\dot{Y},M):=\Phi(\dot{U},N)$ and let Y be the reference cadlag process associated to U as in Definition A.4. By this same definition, M is the cadlag version of
$t\mapsto\mathbb{E}\left[\xi+\int_0^T\hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r\middle|\mathcal{F}_t\right]-\mathbb{E}\left[\xi+\int_0^T\hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r\middle|\mathcal{F}_0\right]$,
but by Proposition A.1, so is N, meaning $M=N$. Again by Definition A.4, $Y=\xi+\int_\cdot^T\hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r-(N_T-N_\cdot)$ which is equal to U thanks to (3.2), so $Y=U$ in the sense of indistinguishability. In particular, $\dot{U}=\dot{Y}$, implying $(\dot{U},N)=(\dot{Y},M)=\Phi(\dot{U},N)$. Therefore, the mapping $(Y,M)\longmapsto(\dot{Y},M)$ does indeed map the set of solutions of $BSDE(\xi,\hat{f},V,\hat{M})$ into the set of fixed points of Φ.

The map Φ is surjective. Indeed let $(\dot{U},N)$ be a fixed point of Φ, the couple (Y,M) of Definition A.4 verifies $Y=\xi+\int_\cdot^T\hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r-(M_T-M_\cdot)$ in the sense of indistinguishability, and $(\dot{Y},M)=\Phi(\dot{U},N)=(\dot{U},N)$, so by Lemma 3.9 in [6], $\int_\cdot^T\hat{f}\left(r,\cdot,Y_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r$ and $\int_\cdot^T\hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r$ are indistinguishable and $Y=\xi+\int_\cdot^T\hat{f}\left(r,\cdot,Y_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r-(M_T-M_\cdot)$, meaning that (Y,M) (which is a preimage of $(\dot{U},N)$) solves $BSDE(\xi,\hat{f},V,\hat{M})$.

We finally show that it is injective. Let us consider two solutions (Y,M) and (Y',M) of $BSDE(\xi,\hat{f},V,\hat{M})$ with $\dot{Y}=\dot{Y}'$. By Lemma 3.9 in [6] the processes $\int_\cdot^T\hat{f}\left(r,\cdot,Y_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r$ and $\int_\cdot^T\hat{f}\left(r,\cdot,Y'_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r$ are indistinguishable, so taking (3.2) into account, we have $Y=Y'$. □

Proposition A.9. *Let $\lambda\in\mathbb{R}$, let $(\dot{U},N)$, $(\dot{U}',N')\in L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}^2_0$, let $(\dot{Y},M)$, $(\dot{Y}',M')$ be their images through Φ and let Y,Y' be the cadlag representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition A.4. Then $\int_0^\cdot e^{\lambda\hat{V}_r}Y_{r-}dM_r$, $\int_0^\cdot e^{\lambda\hat{V}_r}Y'_{r-}dM'_r$, $\int_0^\cdot e^{\lambda\hat{V}_r}Y_{r-}dM'_r$ and $\int_0^\cdot e^{\lambda\hat{V}_r}Y'_{r-}dM_r$ are martingales.*

Proof. $\hat{V}$ is bounded and thanks to Proposition A.6 we know that $\sup\limits_{t\in[0,T]}|Y_t|$ and $\sup\limits_{t\in[0,T]}|Y'_t|$ are L^2. Moreover, since M and M' are square integrable, the statement yields therefore as a consequence Lemma 3.15 in [6]. □

Starting from now, if $(\dot{Y}, M)$ is the image by Φ of some $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, by default, we will always refer to the cadlag reference process Y of $\dot{Y}$ defined in Definition A.4.

For any $\lambda \geq 0$, on $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ we define the norm $\|(\dot{Y}, M)\|^2_\lambda := \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} Y_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle M\rangle_r\right]$. Since $\hat{V}$ is bounded, these norms are all equivalent.

Proposition A.10. *There exists $\lambda > 0$ such that for any $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, $\left\|\Phi(\dot{U}, N)\right\|^2_\lambda \leq \frac{1}{2}\left\|(\dot{U}, N)\right\|^2_\lambda$. In particular, Φ is a contraction in $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ for the norm $\|\cdot\|_\lambda$.*

Proof. Let $(\dot{U}, N)$ and $(\dot{U}', N')$ be two couples belonging to $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$ and $(\dot{Y}', M')$ be their images via Φ and let Y, Y' be the cadlag reference process of $\dot{Y}$, $\dot{Y}'$ introduced in Definition A.4. We will write $\bar{Y}$ for $Y - Y'$ and we adopt a similar notation for other processes. We will also write $\bar{f}_t := \hat{f}\left(t, \cdot, U_t, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(t)\right) - \hat{f}\left(t, \cdot, U'_t, \frac{d\langle N', \hat{M}\rangle}{d\hat{V}}(t)\right)$.

By additivity, we have $d\bar{Y}_t = -\bar{f}_t d\hat{V}_t + d\bar{M}_t$. Since $\bar{Y}_T = \xi - \xi = 0$, applying the integration by parts formula to $\bar{Y}_t^2 e^{\lambda \hat{V}_t}$ between 0 and T we get

$$\bar{Y}_0^2 - 2\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r \bar{f}_r d\hat{V}_r + 2\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_{r^-} d\bar{M}_r + \int_0^T e^{\lambda \hat{V}_r} d[\bar{M}]_r + \lambda \int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r = 0.$$

Since, by Proposition A.9, the stochastic integral with respect to $\bar{M}$ is a real martingale, by taking the expectations we get

$$\mathbb{E}\left[\bar{Y}_0^2\right] - 2\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r \bar{f}_r d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{M}\rangle_r\right] + \lambda \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] = 0.$$

So by re-arranging previous expression, by the Lipschitz condition on $\hat{f}$ stated in Hypothesis 3.1, by the linearity of the Radon-Nikodym derivative stated in Proposition 2.3 and by Lemma A.2, we get

$$\begin{array}{ll}
 & \lambda \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{M}\rangle_r\right] \\
\leq & 2\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} |\bar{Y}_r||\bar{f}_r| d\hat{V}_r\right] \\
\leq & 2K^Y \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} |\bar{Y}_r||\bar{U}_r| d\hat{V}_r\right] + 2K^Z \sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} |\bar{Y}_r| \left|\frac{d\langle \bar{N}, \hat{M}^i\rangle}{d\hat{V}}(r)\right| d\hat{V}_r\right] \\
\leq & (K^Y \alpha + dK^Z \beta)\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] + \frac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{U}_r^2 d\hat{V}_r\right] \\
 & + \frac{K^Z}{\beta} \sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \left(\frac{d\langle \bar{N}, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\right] \\
\leq & (K^Y \alpha + dK^Z \beta)\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] + \frac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{U}_r^2 d\hat{V}_r\right] \\
 & + \frac{CdK^Z}{\beta}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{N}\rangle_r\right],
\end{array}$$

for some positive C and any positive α and β. The latter equality holds by Hypothesis 3.1 4. Then we pick $\alpha = 2K^Y$ and $\beta = 2CdK^Z$, which gives us

$$\begin{aligned} & \lambda \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{M}\rangle_r\right] \\ \leq \quad & 2((K^Y)^2 + C(dK^Z)^2)\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] \\ + \quad & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{U}_r^2 d\hat{V}_r\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{N}\rangle_r\right]. \end{aligned}$$

We choose now $\lambda = 1 + 2((K^Y)^2 + C(dK^Z)^2)$ and we get

$$\begin{aligned} & \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{M}\rangle_r\right] \\ \leq \quad & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{U}_r^2 d\hat{V}_r\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{N}\rangle_r\right], \end{aligned} \tag{A.4}$$

which proves the contraction for the norm $\|\cdot\|_\lambda$. □

Proof of Theorem 3.3.

The space $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ is complete and Φ defines on it a contraction for the norm $\|(\cdot,\cdot)\|_\lambda$ for some $\lambda > 0$, so Φ has a unique fixed point in $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$. Then by Proposition A.8, $BSDE(\xi, \hat{f}, V, \hat{M})$ has a unique solution. □

Remark A.11. *Let (Y, M) be the solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ and $\dot{Y}$ the class of Y in $L^2(d\hat{V} \otimes d\mathbb{P})$. Thanks to Proposition A.8, we know that $(\dot{Y}, M) = \Phi(\dot{Y}, M)$ and therefore by Propositions A.6 and A.9 that $\underset{t\in[0,T]}{sup} |Y_t|$ is L^2 and that $\int_0^\cdot Y_{r-} dM_r$ is a real martingale.*

The lemma below shows that, in order to verify that a couple (Y, M) is the solution of $BSDE(\xi, \hat{f}, V, \hat{M})$, it is not necessary to verify the square integrability of Y since it will be automatically fulfilled.

Lemma A.12. *In this lemma we consider $(\xi, \hat{f}, V, \hat{M})$ such that $\xi, \hat{M}$ verify items 1., 2. of Hypothesis 3.1 but where item 3. is replaced by the weaker following hypothesis on $\hat{f}$. There exists $C > 0$ such that $\mathbb{P}$ a.s., for all t, y, z,*

$$|\hat{f}(t, \omega, y, z)| \leq C(1 + |y| + \|z\|). \tag{A.5}$$

Assume that there exists a cadlag adapted process Y with $Y_0 \in L^2$, and $M \in \mathcal{H}_0^2$ such that

$$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot), \tag{A.6}$$

in the sense of indistinguishability. Then $\underset{t\in[0,T]}{sup} |Y_t|$ is L^2. In particular, $Y \in \mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ and if $(\xi, \hat{f}, V, \hat{M})$ verify Hypothesis 3.1 (Y, M) is the unique solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ in the sense of Definition 3.2.

On the other hand if (Y, M) verifies (A.6) *on* $[s, T]$ *with* $s < T$, $Y_s \in L^2$ *and* $M_s = 0$ *then* $\sup_{t \in [s,T]} |Y_t|$ *is* L^2.

In particular if $(\xi, \hat{f}, V, \hat{M})$ *verify Hypothesis 3.1 and if we denote* (U, N) *the unique solution of* $BSDE(\xi, \hat{f}, V, \hat{M})$*, then* (Y, M) *and* $(U, N. - N_s)$ *are indistinguishable on* $[s, T]$.

Proof. Let $\lambda > 0$ and $t \in [0, T]$. By integration by parts formula applied to $Y^2 e^{-\lambda \hat{V}}$ between 0 and t we get

$$\begin{array}{rcl} Y_t^2 e^{-\lambda \hat{V}_t} - Y_0^2 & = & -2 \int_0^t e^{-\lambda \hat{V}_r} Y_r \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r + 2 \int_0^t e^{-\lambda \hat{V}_r} Y_{r-} dM_r \\ & & + \int_0^t e^{-\lambda \hat{V}_r} d[M]_r - \lambda \int_0^t e^{-\lambda \hat{V}_r} Y_r^2 d\hat{V}_r. \end{array}$$

By re-arranging the terms and using the Lipschitz conditions item 3. of in Hypothesis 3.1, we get

$$\begin{array}{rl} & Y_t^2 e^{-\lambda \hat{V}_t} + \lambda \int_0^t e^{-\lambda \hat{V}_r} Y_r^2 d\hat{V}_r \\ \leq & Y_0^2 + 2 \int_0^t e^{-\lambda \hat{V}_r} |Y_r| |\hat{f}| \left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r + 2 \left| \int_0^t e^{-\lambda \hat{V}_r} Y_{r-} dM_r \right| \\ & + \int_0^t e^{-\lambda \hat{V}_r} d[M]_r \\ \leq & Y_0^2 + \int_0^t e^{-\lambda \hat{V}_r} \hat{f}^2(r, \cdot, 0, 0) d\hat{V}_r + (2K^Y + 1 + K^Z) \int_0^t e^{-\lambda \hat{V}_r} Y_r^2 d\hat{V}_r \\ & + K^Z \sum_{i=1}^d \int_0^t e^{-\lambda \hat{V}_r} \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r) \right)^2 d\hat{V}_r + 2 \left| \int_0^t e^{-\lambda \hat{V}_r} Y_{r-} dM_r \right| + \int_0^t e^{-\lambda \hat{V}_r} d[M]_r. \end{array}$$

Picking $\lambda = 2K^Y + 1 + K^Z$ and using Lemma A.2, this gives

$$\begin{array}{rcl} Y_t^2 e^{-\lambda \hat{V}_t} & \leq & Y_0^2 + \int_0^t e^{-\lambda \hat{V}_r} |\hat{f}|^2(r, \cdot, 0, 0) d\hat{V}_r + K^Z C \int_0^t e^{-\lambda \hat{V}_r} d\langle M\rangle_r \\ & & +2 \left| \int_0^t e^{-\lambda \hat{V}_r} Y_{r-} dM_r \right| + \int_0^t e^{-\lambda \hat{V}_r} d[M]_r, \end{array}$$

for some $C > 0$. Since $\hat{V}$ is bounded, there is a constant $C' > 0$, such that for any $t \in [0, T]$

$$Y_t^2 \leq C' \left(Y_0^2 + \int_0^T |\hat{f}|^2(r, \cdot, 0, 0) d\hat{V}_r + \langle M\rangle_T + [M]_T + \left| \int_0^t Y_{r-} dM_r \right| \right).$$

By Hypothesis 3.1 and since we assumed $Y_0 \in L^2$ and $M \in \mathcal{H}^2$, the first four terms on the right-hand side are integrable and we can conclude by Lemma 3.16 in [6].

An analogous proof also holds on the interval $[s, T]$ taking into account Remark 3.4. In particular, if (U, N) is the unique solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ then $(U, N - N_s)$ is a solution on $[s, T]$. The final statement result follows by the uniqueness argument of Remark 3.4. □

Notation A.13. *Let* $\Phi : L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ *be the operator introduced in Notation A.7.*

In the sequel we will not distinguish between a couple $(\dot{Y}, M)$ in $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ and (Y, M), where Y is the reference cadlag process of $\dot{Y}$, according to Definition A.4. We then convene the following.

1. $(Y^0, M^0) := (0, 0)$;
2. $\forall k \in \mathbb{N}^* : (Y^k, M^k) := \Phi(Y^{k-1}, M^{k-1})$,

meaning that for $k \in \mathbb{N}^*$, (Y^k, M^k) is the solution of the BSDE

$$Y^k = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y^{k-1}, \frac{d\langle M^{k-1}, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M^k_T - M^k_\cdot). \tag{A.7}$$

Definition A.14. *The processes $(Y^k, M^k)_{k\in\mathbb{N}}$ will be called the Picard iterations associated to $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$.*

We know that Φ is a contraction in $L^2(d\hat{V} \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ for a certain norm, so that (Y^k, M^k) tends to (Y, M) in this topology. The proposition below also shows an a.e. corresponding convergence, adapting the techniques of Corollary 2.1 in [19].

Proposition A.15. $Y^k \underset{k\to\infty}{\longrightarrow} Y \quad d\hat{V} \otimes d\mathbb{P}$ *a.e. and for any* $i \in [\![1; d]\!]$, $\frac{d\langle M^k, \hat{M}^i\rangle}{d\hat{V}} \underset{k\to\infty}{\longrightarrow} \frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}} \quad d\hat{V} \otimes d\mathbb{P}$ *a.e.*

Proof of Proposition A.15.

For any $i \in [\![1; d]\!]$ and $k \in \mathbb{N}$ we set $Z^{i,k} := \frac{d\langle M^k, \hat{M}^i\rangle}{d\hat{V}}$ and $Z^i := \frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}$. By Proposition A.10, there exists $\lambda > 0$ such that for any $k \in \mathbb{N}^*$

$$\begin{array}{ll} & \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^{k+1}_r - Y^k_r|^2 d\hat{V}_r + \int_0^T e^{-\lambda\hat{V}_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^k_r - Y^{k-1}_r|^2 d\hat{V}_r + \int_0^T e^{-\lambda\hat{V}_r} d\langle M^k - M^{k-1}\rangle_r\right], \end{array}$$

therefore

$$\begin{array}{ll} & \sum\limits_{k\geq 0}\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^{k+1}_r - Y^k_r|^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq & \sum\limits_{k\geq 0}\frac{1}{2^k}\left(\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^1_r|^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^1\rangle_r\right]\right) \\ < & \infty. \end{array} \tag{A.8}$$

For every fixed (i, k), the linearity property stated in Proposition 2.3) says that $Z^{i,k+1}_r - Z^{i,k}_r = \frac{d\langle M^{k+1} - M^k, \hat{M}^i\rangle}{d\hat{V}}$. Therefore combining equation (A.8) and Lemma A.2, we get

$$\sum_{k\geq 0}\left(\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^{k+1}_r - Y^k_r|^2 d\hat{V}_r\right] + \sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Z^{i,k+1}_r - Z^{i,k}_r|^2 d\hat{V}_r\right]\right) < \infty.$$

So by Fubini's theorem we have

$$\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}\left(\sum_{k\geq 0}\left(|Y^{k+1}_r - Y^k_r|^2 + \sum_{i=1}^d |Z^{i,k+1}_r - Z^{i,k}_r|^2\right)\right) d\hat{V}_r\right] < \infty.$$

Consequently the sum $\sum_{k\geq 0}\left(|Y_r^{k+1}(\omega)-Y_r^k(\omega)|^2+\sum_{i=1}^d |Z_r^{i,k+1}(\omega)-Z_r^{i,k}(\omega)|^2\right)$ is finite on a set of full $d\hat{V}\otimes d\mathbb{P}$ measure. So on this set of full measure, the sequence $(Y_t^k(\omega),(Z_t^{i,k}(\omega))_{i\in[\![1;d]\!]})$ converges, and the limit is necessarily equal to $(Y_t(\omega),(Z_t^i(\omega))_{i\in[\![1;d]\!]})$ $d\hat{V}\otimes d\mathbb{P}$ a.e. Indeed, as we have mentioned in the lines before the statement of the present Proposition A.15, we already know that Y^k converges to Y in $L^2(d\hat{V}\otimes d\mathbb{P})$. Since by Lemma A.2, $\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Z_r^{i,k}-Z_r^i|^2 d\hat{V}_r\right]\leq C\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}d\langle M^k-M\rangle_r\right]$, for every (i,k), where C is a positive constant which does not depend on (i,k), the convergence of M^k to M in $\mathcal{H}_0^2$ also implies the convergence of $Z^{i,k}$ to Z^i in $L^2(d\hat{V}\otimes d\mathbb{P})$.

□

B Proof of Theorem 5.19

Lemma B.1. *Let $\tilde{f}\in\mathcal{L}^2_X$. For every $(s,x)\in[0,T]\times E$, let $(\tilde{Y}^{s,x},\tilde{M}^{s,x})$ be the unique (by Theorem 3.3 and Remark 3.4) solution of*

$$\tilde{Y}_\cdot^{s,x}=g(X_T)+\int_\cdot^T \mathbb{1}_{[s,T]}(r)\tilde{f}(r,X_r)\,dV_r-(\tilde{M}_T^{s,x}-\tilde{M}_\cdot^{s,x}) \tag{B.1}$$

in $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}_t^{s,x})_{t\in[0,T]},\mathbb{P}^{s,x}\right)$. Then there exist $\tilde{u}\in\mathcal{D}(\mathfrak{a})$ such that for any $(s,x)\in[0,T]\times E$

$$\begin{cases}\forall t\in[s,T]:\tilde{Y}_t^{s,x} &= \tilde{u}(t,X_t)\quad \mathbb{P}^{s,x}\,a.s.\\ \tilde{M}^{s,x} &= M[\tilde{u}]^{s,x},\end{cases}$$

and in particular $\frac{d\langle\tilde{M}^{s,x},M[\psi]^{s,x}\rangle}{dV}=\mathfrak{G}^\psi(\tilde{u})(\cdot,X_\cdot)\ dV\otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$.

Proof. We set $\tilde{u}:(s,x)\mapsto\mathbb{E}^{s,x}\left[g(X_T)+\int_s^T\tilde{f}(r,X_r)\,dV_r\right]$ which is Borel by Proposition A.10 and Lemma A.11 in [7]. Therefore by (C.3) in Remark C.6, for every fixed $t\in[s,T]$ we have $\mathbb{P}^{s,x}$- a.s.

$$\begin{aligned}\tilde{u}(t,X_t) &= \mathbb{E}^{t,X_t}\left[g(X_T)+\textstyle\int_t^T\tilde{f}(r,X_r)\,dV_r\right]\\ &= \mathbb{E}^{s,x}\left[g(X_T)+\textstyle\int_t^T\tilde{f}(r,X_r)\,dV_r\middle|\mathcal{F}_t\right]\\ &= \mathbb{E}^{s,x}\left[\tilde{Y}_t^{s,x}+(\tilde{M}_T^{s,x}-\tilde{M}_t^{s,x})|\mathcal{F}_t\right]\\ &= \tilde{Y}_t^{s,x}.\end{aligned}$$

By (B.1) we have $d\tilde{Y}_t^{s,x}=-\tilde{f}(t,X_t)dV_t+d\tilde{M}_t^{s,x}$, so for every fixed $t\in[s,T]$, $\tilde{u}(t,X_t)=\tilde{u}(s,x)-\int_s^t\tilde{f}(r,X_r)dV_r-\tilde{M}_t^{s,x}$ $\mathbb{P}^{s,x}$- a.s. Since $\tilde{M}^{s,x}$ is square integrable and since previous relation holds for any (s,x) and t, Definition 4.13 implies that $\tilde{u}\in\mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(\tilde{u})=-\tilde{f}$ and $\tilde{M}^{s,x}=M[\tilde{u}]^{s,x}$ for every (s,x), hence the announced results. □

Notation B.2. *For a fixed* $(s,x) \in [0,T]\times E$, *we will denote by* $(Y^{k,s,x}, M^{k,s,x})_{k\in\mathbb{N}}$ *the Picard iterations associated to* $BSDE^{s,x}(f,g)$.

Proposition B.3. *For each* $k \in \mathbb{N}$, *there exists* $u_k \in \mathcal{D}(\mathfrak{a})$, *such that for every* $(s,x) \in [0,T] \times E$

$$\left\{\begin{array}{rcll} \forall t \in [s,T] : Y^{k,s,x}_t &=& u_k(t,X_t) & \mathbb{P}^{s,x} a.s. \\ M^{k,s,x} &=& M[u_k]^{s,x} & \end{array}\right. \tag{B.2}$$

Remark B.4. *In particular,* (B.2) *implies that* $\frac{d\langle M^{k,s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(u_k)(\cdot, X_\cdot)$ $dV \otimes d\mathbb{P}^{s,x}$ *a.e. on* $[s,T]$.

Proof. We proceed by induction on k. It is clear that $u_0 = 0$ verifies the assertion for $k = 0$.
Now let us assume that the function u_{k-1} exists, for some integer $k \geq 1$, verifying (B.2) and in particular Remark B.4, for k replaced with $k-1$.
We fix $(s,x) \in [0,T] \times E$. By Lemma 2.4, $(Y^{k-1,s,x}, \frac{d\langle M^{k-1,s,x}, M[\psi]^{s,x}\rangle}{dV}) = (u_{k-1}, \mathfrak{G}^\psi(u_{k-1}))(\cdot, X_\cdot)$ $dV \otimes \mathbb{P}^{s,x}$ a.e. on $[s,T]$. Therefore by (A.7), on $[s,T]$ $Y^{k,s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, u_{k-1}(r,X_r), \mathfrak{G}^\psi(u_{k-1})(r,X_r)\right) dV_r - (M^{k,s,x}_T - M^{k,s,x}_\cdot)$.
Since $\Phi^{s,x}$ maps $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ into itself (see Definition A.7), obviously all the Picard iterations belong to $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$. In particular, by Lemma A.2 $Y^{k-1,s,x}$ and for every $i \in [\![1;d]\!]$, $\frac{d\langle M^{k-1,s,x}, M[\psi_i]^{s,x}\rangle}{dV}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. So, by recurrence assumption on u_{k-1}, it follows that u_{k-1} and for any $i \in [\![1;d]\!]$, $\mathfrak{G}^{\psi_i}(u_{k-1})$ belong to $\mathcal{L}^2_X$.

Combining $H^{mom}(\zeta,\eta)$ and the growth condition of f (item 3.) in $H^{lip}(\zeta,\eta)$ (see Hypotheses 5.4 and 5.5), $f(\cdot,\cdot,0,0)$ also belongs to $\mathcal{L}^2_X$. Therefore thanks to the Lipschitz conditions on f assumed in $H^{lip}(\zeta,\eta)$, $f(\cdot,\cdot,u_{k-1},\mathfrak{G}^\psi(u_{k-1})) \in \mathcal{L}^2_X$.
The existence of u_k now comes from Lemma B.1 applied to $\tilde{f} := f(\cdot,\cdot,u_{k-1},\mathfrak{G}^\psi(u_{k-1}))$, which establishes the induction step for a general k and allows to conclude the proof.

□

Proof of Theorem 5.19. We set $\bar{u} := \limsup\limits_{k\in\mathbb{N}} u_k$, in the sense that for any $(s,x) \in [0,T] \times E$, $\bar{u}(s,x) = \limsup\limits_{k\in\mathbb{N}} u_k(s,x)$ and $v := \limsup\limits_{k\in\mathbb{N}} v_k$. $\bar{u}$ and v are Borel functions. Let us fix now $(s,x) \in [0,T] \times E$. We know by Propositions B.3, A.15 and Lemma 2.4 that

$$\left\{\begin{array}{rcll} u_k(\cdot,X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Y^{s,x} & dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T] \\ \mathfrak{G}^\psi(u_k)(\cdot,X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Z^{s,x} & dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T], \end{array}\right.$$

where $Z^{s,x} := \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}$. Therefore, and on the subset of $[s,T] \times E$ of full

$dV\otimes d\mathbb{P}^{s,x}$ measure on which these convergences hold we have

$$\begin{cases} \bar{u}(t,X_t(\omega)) &= \limsup_{k\in\mathbb{N}} u_k(t,X_t(\omega)) &= \lim_{k\in\mathbb{N}} u_k(t,X_t(\omega)) &= Y_t^{s,x}(\omega) \\ v(t,X_t(\omega)) &= \limsup_{k\in\mathbb{N}} \mathfrak{G}^{\psi}(u_k)(t,X_t(\omega)) &= \lim_{k\in\mathbb{N}} \mathfrak{G}^{\psi}(u_k)(t,X_t(\omega)) &= Z_t^{s,x}(\omega). \end{cases} \tag{B.3}$$

Thanks to the $dV\otimes d\mathbb{P}^{s,x}$ equalities concerning v and $\bar{u}$ stated in (B.3), under $\mathbb{P}^{s,x}$ we actually have

$$Y^{s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, \bar{u}(r,X_r), v(r,X_r)\right) dV_r - (M_T^{s,x} - M_{\cdot}^{s,x}). \tag{B.4}$$

Now (B.4) can be considered as a BSDE where the driver does not depend on y and z. $Y^{s,x}$ and $Z^{s,x}$ belong to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$ (see Lemma A.2), then by (B.3), so do $\bar{u}(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$ and $v(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$, meaning that $\bar{u}$ and v belong to $\mathcal{L}^2_X$. Combining $H^{mom}(\zeta,\eta)$ and the Lipschitz condition on f assumed in $H^{lip}(\zeta,\eta)$, $f(\cdot,\cdot,\bar{u},v)$ also belongs to $\mathcal{L}^2_X$. We can therefore apply Lemma B.1 to $\tilde{f} := f(\cdot,\cdot,\bar{u},v)$, and conclude.
Concerning the last statement of the theorem, for any $(s,x)\in[0,T]\times E$, we have $Y_s^{s,x} = u(s,X_s) = u(s,x)$ $\mathbb{P}^{s,x}$ a.s. so $Y_s^{s,x}$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant and u is the mapping $(s,x)\longmapsto Y_s^{s,x}$. □

C Markov classes and Martingale Additive Functionals

We recall in this Appendix section some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [17], concerning non-homogeneous Markov classes, our reference was chapter VI of [18]. Some results are only stated, but the interested reader can consult [8] in which all announced results are carefully proved in our exact setup.

The first definition refers to the canonical space that one can find in [25], see paragraph 12.63.

Notation C.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space). It will be called the* ***state space****.*

We consider $T\in\mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}(E)$ the Skorokhod space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T (for which we also use the french acronym cadlag). For any $t\in[0,T]$ we denote the coordinate mapping $X_t : \omega\mapsto\omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r\in[0,T])$.

On the measurable space $(\Omega,\mathcal{F})$, we introduce the ***canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ *where* $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r|r \leq s)$ *if* $t < T$, *and* $\mathcal{F}_T := \sigma(X_r|r \in [0,T]) = \mathcal{F}$.

$\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ *will be called the **canonical space** (associated to* T *and* E*).*

For any $t \in [0,T]$ *we denote* $\mathcal{F}_{t,T} := \sigma(X_r|r \geq t)$, *and for any* $0 \leq t \leq u < T$ *we will denote* $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r|r \in [t, u + \frac{1}{n}])$.

Remark C.2. *Previous definitions and all the notions of this Appendix, extend to a time interval equal to* $\mathbb{R}_+$ *or replacing the Skorokhod space with the space of continuous functions from* $[0,T]$ *(or* $\mathbb{R}_+$*) to* E. *but since our goal is to work on a finite time interval, we will not consider this situation.*

Definition C.3. *The function*

$$P : \begin{array}{rcl} (s,t,x,A) & \longmapsto & P_{s,t}(x,A) \\ [0,T]^2 \times E \times \mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

*will be called **transition kernel** if, for any* s,t *in* $[0,T]$, $x \in E$, $A \in \mathcal{B}(E)$, *it verifies the following.*

1. $P_{s,t}(\cdot, A)$ *is Borel,*
2. $P_{s,t}(x, \cdot)$ *is a probability measure on* $(E, \mathcal{B}(E))$,
3. *if* $t \leq s$ *then* $P_{s,t}(x,A) = \mathbb{1}_A(x)$,
4. *if* $s < t$, *for any* $u > t$, $\int_E P_{s,t}(x,dy)P_{t,u}(y,A) = P_{s,u}(x,A)$.

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition C.4. *A transition kernel* P *for which the first item is reinforced supposing that* $(s,x) \longmapsto P_{s,t}(x,A)$ *is Borel for any* t, A, *will be said **measurable in time**.*

Definition C.5. *A **canonical Markov class** associated to a transition kernel* P *is a set of probability measures* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *defined on the measurable space* $(\Omega, \mathcal{F})$ *and verifying for any* $t \in [0,T]$ *and* $A \in \mathcal{B}(E)$

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x,A), \tag{C.1}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A|\mathcal{F}_t) = P_{t,u}(X_t, A) \quad \mathbb{P}^{s,x} \ a.s. \tag{C.2}$$

Remark C.6. *Formula 1.7 in Chapter 6 of [18] states that for any* $F \in \mathcal{F}_{t,T}$ *yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \ \ \mathbb{P}^{s,x} a.s. \tag{C.3}$$

Property (C.3) *will be called **Markov property**.*

For the rest of this section, we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ which transition kernel is measurable in time.

Definition C.7. *For any $(s,x)\in[0,T]\times E$ we will consider the (s,x)-**completion** $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x}\right)$ of the stochastic basis $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}^{s,x}\right)$ by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}^{s,x}_t$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$ for every $t\in[0,T]$.*

We remark that, for any $(s,x)\in[0,T]\times E$, $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x}\right)$ is a stochastic basis fulfilling the usual conditions.

We recall the following simple consequence of Remark 32 in [15] Chapter II.

Proposition C.8. *Let $(s,x)\in[0,T]\times E$ be fixed, Z be a random variable and $t\in[s,T]$, then $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]=\mathbb{E}^{s,x}[Z|\mathcal{F}^{s,x}_t]$ $\mathbb{P}^{s,x}$ a.s.*

We now introduce the notion of non-homogeneous Additive Functional that we use in the paper.

Definition C.9. *We denote $\Delta:=\{(t,u)\in[0,T]^2|t\leq u\}$. On $(\Omega,\mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened AF) as a random-field $A:=(A^t_u)_{(t,u)\in\Delta}$ with values in $\mathbb{R}$ verifying the two following conditions.*

1. *For any $(t,u)\in\Delta$, A^t_u is $\mathcal{F}_{t,u}$-measurable;*
2. *for any $(s,x)\in[0,T]\times E$, there exists a real cadlag $\mathcal{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x\in E$ and $s\leq t\leq u$, $A^t_u=A^{s,x}_u-A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s.*

*$A^{s,x}$ will be called the **cadlag version of A under** $\mathbb{P}^{s,x}$.*

*An AF will be called a **non-homogeneous square integrable Martingale Additive Functional** (shortened square integrable MAF) if under any $\mathbb{P}^{s,x}$ its cadlag version is a square integrable martingale. More generally an AF will be said to verify a certain property (being non-negative, increasing, of bounded variation, square integrable, having L^1 terminal value) if under any $\mathbb{P}^{s,x}$ its cadlag version verifies it.*

*Finally, given an increasing AF A and an increasing function V, A will be said to be **absolutely continuous with respect to** V if for any $(s,x)\in[0,T]\times E$, $dA^{s,x}\ll dV$ in the sense of stochastic measures.*

The two following results are proven in [8].

Proposition C.10. *Let M, M' be two square integrable MAFs, let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under $\mathbb{P}^{s,x}$. Then there exists a bounded variation AF with L^1 terminal condition denoted $\langle M,M'\rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M,M'\rangle$ is $\langle M^{s,x},M'^{s,x}\rangle$. If $M=M'$ the AF $\langle M,M'\rangle$ will be denoted $\langle M\rangle$ and is increasing.*

Proposition C.11. *Let V be a continuous non-decreasing function. Let M, N be two square integrable MAFs, and assume that the AF $\langle N \rangle$ is absolutely continuous with respect to V. There exists a function $v \in \mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any (s,x), $\langle M^{s,x}, N^{s,x} \rangle = \int_s^{\cdot \vee s} v(r, X_r) V_r$.*

ACKNOWLEDGMENTS. The authors are grateful to Andrea Cosso for stimulating discussions. The research of the first named author was provided by a PhD fellowship (AMX) of the Ecole Polytechnique.

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