# About classical solutions of the path-dependent heat equation

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**Abstract**

This paper investigates two existence theorems for the path-dependent heat equation, which is the Kolmogorov equation related to the window Brownian motion, considered as a $C([-T,0])$-valued process. We concentrate on two general existence results of its classical solutions related to different classes of final conditions: the first one is given by a cylindrical non necessarily smooth r.v., the second one is a smooth generic functional.

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## 1 Introduction

The path-dependent heat equation is a natural extension of the classical heat equation to the path-dependent world. If the heat equation constitutes the Kolmogorov equation associated with Brownian motion viewed as a real valued process, then the path-dependent heat equation is the Kolmogorov equation related to the Wiener process as $C([-T,0)$-valued process, that we will denominate as *window Brownian motion*. One particularity of $C([-T,0)$ is that it is a (even non-reflexive) Banach space, which does not make it an easy task for stochastic calculus.

An important recent contribution in the study of path-dependent heat equation was carried on by [5], which considered a (not necessarily smooth in time) mild type solutions, involving at the same time a path-dependent drift, see also references therein for related contributions. In this paper we focus on classical solutions of the path-dependent heat equation with two types of final condition. Let $H : C([-T,0]) \longrightarrow \mathbb{R}$

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be continuous and let $\sigma$ be a real constant. Our path dependent heat equation can be expressed as

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^-\eta(x) + \frac{1}{2}\sigma^2 \langle D^2 u(t,\eta), \mathbb{1}_{\{0\}} \otimes \mathbb{1}_{\{0\}} \rangle = 0, & \text{for } (t,\eta) \in [0,T[\times C([-T,0]) \\ u(T,\eta) = H(\eta) & \text{for } \eta \in C([-T,0]). \end{cases} \tag{1.1}$$

Even though some results can be extended to a more general context we have preferred for clarity to work with $\sigma$ being a constant. A function $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ will be a classical solution of (1.1) if it belongs to $u \in C^{1,2}\left([0,T[\times C([-T,0])\right) \cap C^0\left([0,T] \times C([-T,0])\right)$ in the Fréchet sense and if it verifies (1.1). For any given $(t,\eta) \in [0,T] \times C([-T,0])$, we denote by $D^{\delta_0} F(t,\eta)$ the component of $DF(t,\eta)$ concentrated on the Dirac zero defined by $D^{\delta_0} F(t,\eta) := DF(t,\eta)(\{0\})$ and we denote by $D^{\perp} F(t,\eta)$ the component of $DF(t,\eta)$ *singular* to the Dirac zero component, i.e. the measure defined by $D^{\perp} F(t,\eta) := DF(t,\eta) - DF(t,\eta)(\{0\})\delta_0$. For every $\eta \in C([-\tau,0])$, we observe that $t \mapsto D^{\delta_0} F(t,\eta)$ is a real valued function. If for each $(t,\eta)$, $D^{\perp} F(t,\eta)$ is absolutely continuous, we denote by $D^{ac} F(t,\eta)$ its density and in particular it holds that $D^{\perp}_{dx} F(t,\eta) = D^{ac}_x F(t,\eta)\, dx$.

A central object appearing in the path-dependent heat equation PDE (1.1) is the deterministic integrals denoted by

$$\int_{]-t,0]} D^{\perp}_{dx} u(t,\eta) d^-\eta(x),$$

where $D^{\perp} u(t,\eta)$ is a measure on $[-T,0]$ and $\eta \in C([-T,0])$. We will give a sense, for $-T \leq a \leq b \leq 0$, to the term $\int_{]a,b]} D^{\perp} u(t,x) d^-\eta(x)$ as essentially the *deterministic forward integral* $\lim_{\epsilon \to 0} \int_{]a,b]} D^{\perp}_{dx} u(t,x) \frac{\eta(x+\epsilon)-\eta(x)}{\epsilon} dx$, see Definition 2.2. More generally, let $\mu$ be a finite Borel measure on $[-T,0]$ and $f$ a cadlag function, we will give a sense to the integral $\int_{]a,b]} \mu(dx) d^- f(x)$. Whenever $f$ has bounded variation and $\mu$ is absolutely continuous, it will coincide with the classical expected Riemann-Stieltjes integral, see Proposition 2.3.

As we mentioned, we state two existence theorems of the classical solution of (1.1) under two different types of final condition $H$. In Proposition 3.4, we consider as final condition a possibly not smooth function $H$ of a finite numbers of integrals of the type $\int_{-T}^0 \varphi d^-\eta$. The reason of validity of that result (when $\sigma \neq 0$) can be understood through the non-degeneracy feature of Brownian motion.

In Theorem 4.10 we suppose the final condition function $H$ to be $C^3(C([-T,0]))$. This result generalizes an existence results already established in the unpublished monograph [3] Sections 9.8 and 9.9, where we assumed a Fréchet smooth dependence with respect to $L^2([-T,0])$.

In this paper we have only concentrated our efforts on the problem of existence of solution of (1.1), the uniqueness constituting a simpler task which can obtained as an application of a Banach space valued Itô formula established in [4]. Let $W$ be a Brownian motion, we denote by $(W_t)_{0 \leq t \leq T}$ the real valued Brownian motion and by $(W_t(\cdot))$ or simply $W(\cdot)$ its window Brownian process with values in $C([-T,0])$ defined by $W_t(x) := W_{t+x}$, see Definition 2.1.

An application of our two existence results consists in obtaining a Clark-Ocone type formula for a path-dependent random variable $h := H(X_T(\cdot))$, where $X$ is a finite quadratic variation process with quadratic variation given by $[X]_t = \sigma^2 t$, but $X$ non necessarily a semimartingale. A possible example of such process is given by $X = W + B^H$, i.e. a Brownian motion plus a fractional Brownian motion of parameter $H > 1/2$ or the weak $k$-order Brownian motion of [6].

Let $u$ be the solution of (1.1) provided by Proposition 3.4 or Theorem 4.10. By Itô formula, see e.g. Theorem 5.2 in [4], if $u$ verifies some more technical conditions then

$$h = u(0, X_0(\cdot)) + \int_0^T \mathcal{L}u(t, X_t(\cdot))\, dt + \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t, \tag{1.2}$$

where $\mathcal{L}$ denotes differential operator for $u \in C^{1,2}\left([0,T[\times C([-T,0])\right)$ defined by

$$\mathcal{L}u(t,\eta) := \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^-\eta(x) + +\frac{1}{2}\sigma^2 \langle D^2 u(t,\eta), \mathbb{1}_{\{0\}} \otimes \mathbb{1}_{\{0\}} \rangle,$$

for $(t,\eta) \in [0,T] \times C([-T,0])$. Now by (1.2)

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t, \tag{1.3}$$

where we remind that $\int_0^t Y d^- X$ is the forward integral via regularization defined first in [10] and [9] for $X$ (respectively $Y$) a continuous (resp. locally integrable) real process, see also [11] for a survey. Whenever $X = W$, the forward real valued integral equals the classical Itô integral, see Proposition 1.1 in [10]. In particular, if $h \in \mathbb{D}^{1,2}$, it holds that the representation stated in (1.3) coincides with the classical Clark-Ocone formula $h = \mathbb{E}[h] + \int_0^T \mathbb{E}\left[D^m_t h | \mathcal{F}_t\right] dW_t$ , i.e. $u(0, W_0(\cdot)) = \mathbb{E}[h]$ and $D^{\delta_0} u(t, W_t(\cdot)) = D^m_t(h|\mathcal{F}_t)$, $D^m$ denoting the Malliavin derivative. This follows by the uniqueness of decomposition of square integrable random variables with respect to the Brownian filtration. We remark that our representation (1.3) can be proved in some cases, where $h \notin \mathbb{D}^{1,2}$, see e.g. Section 3.

The paper is organized as follows. After this introduction, in Section 2 we recall some preliminaries: basic notions of calculus via regularization in finite and infinite dimension, Fréchet derivatives of a functionals and the important subsection 2.2 about deterministic calculus via regularization. In Section 3 we show the existence of a classical solution of the Kolmogorov PDE for a cylindrical $H$. Finally in Section 4 we show that existence for $H$ being general but smooth.

# 2 Preliminaries

## 2.1 General notations

Let $A$ and $B$ be two general sets such that $A \subset B$; $\mathbb{1}_A : B \to \{0,1\}$ will denote the indicator function of the set $A$, so $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. Let $k \in \mathbb{N} \cup \{+\infty\}$, we denote by $C^k(\mathbb{R}^n)$

the set of all function $g : \mathbb{R}^n \to \mathbb{R}$ which admits all partial derivatives of order $0 \le p \le k$ and they are continuous. If $I$ is a real interval and $g$ is a function from $I \times \mathbb{R}^n$ to $\mathbb{R}$ which belongs to $C^{1,2}(I \times \mathbb{R}^n)$, the symbols $\partial_t g(t,x)$, $\partial_i g(t,x)$ and $\partial^2_{ij} g(t,x)$ will denote respectively the partial derivative with respect to variable $I$, the partial derivative with respect to the $i$-th component and the second order mixed derivative with respect to $j$-th and $i$-th component evaluated in $(t,x) \in I \times \mathbb{R}^n$.

Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed probability space, equipped with a given filtration $\mathcal{F} = (\mathcal{F}_t)_{t \ge 0}$ fulfilling the usual conditions. Let $a < b$ be two real numbers, $C([a,b])$ will denote the Banach linear space of real continuous functions equipped with the uniform norm denoted by $\| \cdot \|_\infty$. Let $B$ be a Banach space over the scalar field $\mathbb{R}$. The space of bounded linear mappings from $B$ to $E$ will be denoted by $L(B;E)$ and we will write $L(B)$ instead of $L(B;B)$. The topological dual space of $B$, i.e. when $L(B;\mathbb{R})$, will be denoted by $B^*$. If $\phi$ is a linear functional on $B$, we shall denote the value of $\phi$ at an element $b \in B$ either by $\phi(b)$ or $\langle \phi, b \rangle$ or even ${}_{B^*}\langle \phi, b \rangle_B$. Let $K$ be a compact space, $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the so-called set of finite signed measures on $K$.

Let $E, F, G$ be Banach spaces; we shall denote the space of $G$-valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}(E \times F; G)$ with the norm given by $\|\phi\|_{\mathcal{B}} = \sup\{\|\phi(e,f)\|_G \, : \, \|e\|_E \le 1; \|f\|_F \le 1\}$. If $G = \mathbb{R}$ we simply denote it by $\mathcal{B}(E \times F)$. We recall that $\mathcal{B}(B \times B)$ is identified with $(B \hat{\otimes}_\pi B)^*$, see [12, 7] for more details.

We recall some notions about differential calculus in Banach spaces; for more details reader can refer to [1]. Let $B$ be a Banach space. A function $F : [0,T] \times B \longrightarrow \mathbb{R}$, is said to be $C^{1,2}([0,T] \times B)$ (Fréchet), or $C^{1,2}$ (Fréchet), if the following three properties are fulfilled. 1. $F$ is once continuously differentiable; the partial derivative with respect to $t$ will be denoted by $\partial_t F : [0,T] \times B \longrightarrow \mathbb{R}$; 2. for any $t \in [0,T]$, $x \mapsto DF(t,x)$ is of class $C^1$ where $DF : [0,T] \times B \longrightarrow B^*$ denotes the derivative with respect to the second argument and 3. the second order derivative with respect to the second argument $D^2 F : [0,T] \times B \to \mathcal{B}(B \times B)$ is continuous.

If $B = C([-T,0])$, we remark that $DF$ defined on $[0,T] \times B$ takes values in $B^* \cong \mathcal{M}([-T,0])$. For all $(t,\eta) \in [0,T] \times C([-T,0])$, we will denote by $D_{dx} F(t,\eta)$ the measure such that

$$
{}_{\mathcal{M}([-T,0])}\langle DF(t,\eta), h \rangle_{C([-T,0])} = DF(t,\eta)(h) = \int_{[-T,0]} h(x) D_{dx} F(t,\eta) \qquad \forall\, h \in C([-T,0]).
$$

Whenever $B = E = F = C([-T,0])$, then the space of finite signed Borel measures on $[-T,0]^2$ is included in the space $\mathcal{B}(B \times B)$ in the following way:

$$
{}_{\mathcal{M}([-T,0]^2)}\langle \mu, \eta \rangle_{C([-T,0]^2)} = \int_{[-T,0]^2} \eta(x,y) \mu(dx,dy) = \int_{[-T,0]^2} \eta_1(x) \eta_2(y) \mu(dx,dy) \; .
$$

We convene that the continuous functions (and real processes) defined on $[0,T]$ or $[-T,0]$, are extended by continuity to the real line.

**Definition 2.1.** Given a real continuous process $X = (X_t)_{t\in[0,T]}$, we will call **window process**, and denoted by $X(\cdot)$, the $C([-T,0])$-valued process

$$X(\cdot) = \left(X_t(\cdot)\right)_{t\in[0,T]} = \{X_t(x) := X_{t+x}; x \in [-T,0], t \in [0,T]\} .$$

$X(\cdot)$ will be understood, sometimes without explicit mention, as $C([-T,0])$-valued. In this paper $B$ will be often $C([-T,0])$.

We recall now the integration by parts in Wiener space. Let $\delta$ be the it Skorohod integral or the adjoint operator of Malliavin derivative $D^m$ as defined in Definition 1.3.1 in [8]. If $u$ belongs to $Dom\ \delta$, then $\delta(u)$ is an element of $L^2(\Omega)$ characterized by

$$\mathbb{E}\left[F\,\delta(u)\right] = \mathbb{E}\left[\int_0^T D_t^m F\ u_t\ dt\right], \tag{2.1}$$

for any $F \in \mathbb{D}^{1,2}$.

## 2.2 Deterministic calculus via regularization

Let $-T \le a \le b \le 0$, we will essentially concentrate in the definite integral on an interval $J = ]a,b]$ and $\bar{J} = [a,b]$,where $a < b$ are two real numbers. Typically, in our applications we will consider $a = -T$ or $a = -t$ and $b = 0$. That integral will be a real number.

We start with a convention. If $f : [a,b] \to \mathbb{R}$ is a cadlag function, we extend it naturally to two possible cadlag functions $f_J$ and $f_{\bar{J}}$ on real line setting

$$f_J(x) = \begin{cases} f(b) & : \quad x > b, \\ f(x) & : \quad x \in [a,b], \\ f(a) & : \quad x < a. \end{cases} \qquad \text{and} \qquad f_{\bar{J}}(x) = \begin{cases} f(b) & : \quad x > b, \\ f(x) & : \quad x \in [a,b], \\ 0 & : \quad x < a. \end{cases}$$

**Definition 2.2.** Let $\mu$ be a finite Borel measure on $[0,T]$, $\mu \in \mathcal{M}([-T,0])$ and $f : [a,b] \to \mathbb{R}$ be a cadlag function. We define the **deterministic forward integral** on $J = ]a,b]$ and on $\bar{J} = [a,b]$ denoted by

$$\int_{]a,b]} \mu(dx) d^- f(x) \quad \text{or simply} \int_{]a,b]} \mu d^- f \qquad \text{and} \qquad \int_{[a,b]} \mu(dx) d^- f(x) \quad \text{or simply} \int_{[a,b]} \mu d^- f$$

as the limit of

$$I^-(]a,b],f,\epsilon) = \int_{]a,b]} \frac{f_J(x+\epsilon) - f_J(x)}{\epsilon} \mu(dx) \qquad \text{and of} \qquad I^-([a,b],f,\epsilon) = \int_{[a,b]} \frac{f_{\bar{J}}(x+\epsilon) - f_{\bar{J}}(x)}{\epsilon} \mu(dx),$$

when $\epsilon \downarrow 0$, provided it exists.

If $\mu$ is absolutely continuous we denote by $\mu^{ac}$ the density with respect to Lebesgue measure. In this case we set

$$\int_{]a,b]} \mu d^- f := \int_{]a,b]} \mu^{ac} d^- f, \quad \int_{[a,b]} \mu d^- f := \int_{[a,b]} \mu^{ac} d^- f. \tag{2.2}$$

The first type integral on $]a, b]$ appears in the path-dependent PDE (1.1), the second one on the closed interval $[a, b]$ is fundamental in Section 3. Now Proposition 2.3 The proposition below discusses the case when $f$ or $\mu$ is absolutely continuous.

**Proposition 2.3.** Let $\mu(dx) = \mu^{ac}(x)dx$, i.e. $\mu$ be absolutely continuous with Radon-Nikodym derivative density denoted by $\mu^{ac}$. By default, the bounded variation functions will be considered as cadlag.

1. If $f$ has bounded variation then

$$\int_{]a,b]} \mu^{ac}(x) d^- f(x) = \int_{]a,b]} \mu^{ac}(x-) df(x) \qquad \text{(classical Lebesgue-Stieltjes integral).}$$

   In particular, whenever $\mu^{ac} \equiv 1$, $\int_{]a,b]} \mu^{ac}(x) d^- f(x) = f(b) - f(a)$.

2. If the function $\mu^{ac} : [a, b] \longrightarrow \mathbb{R}$ is cadlag with bounded variation, then

   (a)

$$\int_{[a,b]} \mu^{ac}(x) d^- f(x) = \mu^{ac}(b) f(b) - \int_{]a,b]} f(x) d\mu^{ac}(x) \tag{2.3}$$

   (b)

$$\int_{]a,b]} \mu^{ac}(x) d^- f(x) = \mu^{ac}(b) f(b) - \mu^{ac}(a) f(a) - \int_a^b f(x) d\mu^{ac}(x). \tag{2.4}$$

*Proof.* The statements follow directly from the definition. Concerning the case when the integration interval is $[a, b]$ we remark that our definition is compatible with Definitions 4, 18, see also Proposition 18 of [2]. By Proposition 4 ibidem, we get item 2.(a). The other items can be established by similar considerations and are left to the reader. □

# 3 The existence result for cylindrical final condition

The central object of this section is Proposition 3.4 which gives an existence result of the solution of the path-dependent heat equation (1.1) when the terminal condition $H$ depends on a finite number of integrals, but it is not necessarily smooth. As we mentioned, the idea is here to exploit here the non-degeneracy aspect of the Brownian motion in the sense that the covariance matrix of every finite dimensional distribution is invertible. This in opposition to Section 4 where $H$ is Fréchet smooth, but not necessarily cylindrical. In this section the standard deviation parameter $\sigma$ will never vanish. We introduce now the functional $H$. For all $i = 1, \dots, n$, let $\varphi_i : [0,T] \longrightarrow \mathbb{R}$ be $C^2([0,T];\mathbb{R})$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and with linear growth. We consider the functional

$$H : C([-T,0]) \to \mathbb{R}$$

defined by

$$H(\eta) = f\left(\int_{[-T,0]} \varphi_1(u+T)d^-\eta(u), \dots, \int_{[-T,0]} \varphi_n(u+T)d^-\eta(u)\right). \tag{3.1}$$

We recall that for smooth $\varphi_i$, $i \in \{1, \dots, n\}$, the deterministic integral $\int_{[-T,0]} \varphi_i(u+T)d^-\eta(u)$ exists pointwise, according to Definition 2.2. That integral exists since, by (2.3) in Proposition 2.3, we have

$$\int_{[-T,0]} \varphi_i(u+T)d^-\eta(u) = \varphi_i(T)\eta(0) - \int_0^T \eta(s-T)d\varphi_i(s). \tag{3.2}$$

So, replacing $\eta$ with the random path $\sigma W_T(\cdot)$ in (3.1) we get

$$\begin{aligned} h = H(W_T(\cdot)) &= f\left(\sigma \int_{[-T,0]} \varphi_1(u+T)d^-W_T(u), \dots, \sigma \int_{[-T,0]} \varphi_n(u+T)d^-W_T(u)\right) \\ &= f\left(\sigma \int_0^T \varphi_1(s)d^-W_s, \dots, \sigma \int_0^T \varphi_n(s)d^-W_s\right) \\ &= f\left(\sigma \int_0^T \varphi_1(s)dW_s, \dots, \sigma \int_0^T \varphi_n(s)dW_s\right). \end{aligned} \tag{3.3}$$

We stress that in the first line of (3.3) the integrands are deterministic forward integrals; those integrals exist pathwise, however in the second line of (3.3) appear stochastic forward integrals. The second equality is justified because the convergence for every realization $\omega$ implies of course the convergence in probability, which characterizes the stochastic forward integral. The latter equality holds because Itô integrals with Brownian motion are also forward integrals, see Proposition 1.1 in [10]. On the other hand, for every $i \in \{1, \dots, n\}$, since $\varphi_i$ are of class $C^2$ then Proposition 2.3 and in particular (2.3) gives

$$\int_0^t \varphi_i(s)d^-W_s = \int_{[-t,0]} \varphi_i(u+t)d^-W_t(u) = \varphi_i(t)W_t - \int_0^t W_s d\varphi_i(s), \tag{3.4}$$

where the first equality holds by similar reasons as for the first equality in (3.3). The second equality holds by (2.3).

We formulate the following *non-degeneracy* assumption.

**Assumption 1.** For $t \in [0,T]$, we denote $\Sigma_t$ the matrix in $\mathbb{M}_{n\times n}(\mathbb{R})$ defined by

$$(\Sigma_t)_{1\le i,j\le n} = \left(\int_t^T \varphi_i(s)\varphi_j(s)ds\right)_{1\le i,j\le n}.$$

We suppose

$$\det(\Sigma_t) > 0 \qquad\qquad \forall\, t \in [0, T[\ .$$

**Remark 3.1.** 1. We observe that, by continuity of function $t \mapsto \det(\Sigma_t)$, there is always $\tau > 0$ such that $\det(\Sigma_t) \neq 0$ for all $t \in [0, \tau[$.

2. It is not restrictive to consider $\det(\Sigma_0) \neq 0$ since it is always possible to orthogonalize $(\varphi_i)_{i=1,\dots,n}$ in $L^2([0, T])$ via a Gram-Schmidt procedure.

Let $W$ be a classical Wiener process equipped with its canonical filtration $(\mathcal{F}_t)$. We set $h = H(W_T(\cdot))$ and we evaluate the conditional expectation $\mathbb{E}[h|\mathcal{F}_t]$. It gives

$$\begin{aligned}
\mathbb{E}[h|\mathcal{F}_t] &= \mathbb{E}\left[f\left(\sigma\int_0^T \varphi_i(s)dW_s, \dots, \sigma\int_0^T \varphi_n(s)dW_s\right)|\mathcal{F}_t\right] \\
&= \Psi\left(t, \sigma\int_0^t \varphi_1(s)dW_s, \dots, \sigma\int_0^t \varphi_n(s)dW_s\right) \\
&= \Psi\left(t, \int_{[-t,0]} \varphi_1(u+t)d^-\sigma W_t(u), \dots, \int_{[-t,0]} \varphi_n(u+t)d^-\sigma W_t(u)\right) \\
&= \Psi\left(t, \int_{[-T,0]} \varphi_1(u+t)d^-\sigma W_t(u), \dots, \int_{[-T,0]} \varphi_n(u+t)d^-\sigma W_t(u)\right)\ ,
\end{aligned} \tag{3.5}$$

where the function $\Psi : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$\Psi(t, y_1, \dots, y_n) = \mathbb{E}\left[f\left(y_1 + \sigma\int_t^T \varphi_1(s)dW_s, \dots\dots, y_n + \sigma\int_t^T \varphi_n(s)dW_s\right)\right], \tag{3.6}$$

for any $t \in [0, T], y_1, \dots, y_n \in \mathbb{R}$. In particular

$$\Psi(T, y_1, \dots, y_n) = f(y_1, \dots\dots, y_n)\ .$$

The second equality in (3.5) holds because for every $1 \le i \le n$

$$\int_0^t \varphi_n(s)\sigma dW_s = \int_{-t}^0 \varphi_n(u+t)d^-\sigma W_t(u),$$

for the same reasons as in (3.4). We evaluate expression (3.6) introducing the density function $p$ of the Gaussian vector

$$\left(\int_t^T \varphi_1(s)dW_s, \dots, \int_t^T \varphi_n(s)dW_s\right)\ ,$$

whose covariance matrix equals to $\Sigma_t$. The function $p : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is characterized by

$$p(t, z_1, \dots, z_n) = \sqrt{\frac{1}{(2\pi)^n \det(\Sigma_t)}} \exp\left\{-\frac{(z_1, \dots, z_n)\Sigma_t^{-1}(z_1, \dots, z_n)^*}{2}\right\}\ ,$$

and function $\Psi$ becomes

$$\Psi(t,y_1,\dots,y_n)=\begin{cases}\int_{\mathbb{R}^n} f\left(\tilde z_1,\dots,\tilde z_n\right) p\left(t,\frac{\tilde z_1-y_1}{\sigma},\dots,\frac{\tilde z_n-y_n}{\sigma}\right) d\tilde z_1\cdots d\tilde z_n & \text{if } t\in[0,T[\\ f\left(y_1,\dots\dots,y_n\right) & \text{if } t=T\ .\end{cases} \tag{3.7}$$

**Remark 3.2.** 1. If $f$ is not continuous, we remark that, at time $t=T$, $\Psi(T,\cdot)$ is a function which strictly depends on the representative of $f$ and not only on its Lebesgue a.e. representative. So $\Psi$, as a class does not admit a restriction to $t=T$ which can not be identified with its class in a.e. sense.

2. Function $p$ is a solution $C^{3,\infty}([0,T[\times\mathbb{R}^n)$ of

$$\partial_t p(t,z_1,\dots,z_n)=-\frac{1}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\partial^2_{ij}p(t,z_1,\dots,z_n)\ .$$

Therefore function $\Psi$ is $C^{1,2}([0,T[\times\mathbb{R}^n)$ and solves

$$\partial_t\Psi(t,z_1,\dots,z_n)=-\frac{\sigma^2}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\partial^2_{ij}\Psi(t,z_1,\dots,z_n). \tag{3.8}$$

We define now a function $u:[0,T]\times C([-T,0])\longrightarrow\mathbb{R}$ by

$$u(t,\eta)=\Psi\left(t,\int_{[-t,0]}\varphi_1(s+t)d^-\eta(s),\dots,\int_{[-t,0]}\varphi_n(s+t)d^-\eta(s)\right)\ , \tag{3.9}$$

where $\Psi(t,y_1,\dots,y_n)$ is defined by (3.7).
By the fact that, for every $i$, the functions $\varphi_i$ are $C^2$, so in particular with bounded variation, similarly to (3.2) we can write

$$\int_{[-t,0]}\varphi_i(s+t)d^-\eta(s)=\eta(0)\varphi_i(t)-\int_0^t\eta(s-t)\dot\varphi_i(s)ds,\ , \tag{3.10}$$

see (2.3).

**Remark 3.3.** By construction we have

$$u(t,\sigma W_t(\cdot))=\mathbb{E}[h|\mathcal{F}_t]$$

and in particular $u(0,W_0(\cdot))=\mathbb{E}[h]$.

We state now the main proposition of this section.

**Proposition 3.4.** Let $H:C([-T,0])\longrightarrow\mathbb{R}$ be defined by (3.1) and $u:[0,T]\times C([-T,0])\longrightarrow\mathbb{R}$ be defined by (3.9).

1. Function $u$ belongs to $C^{1,2}\left([0,T[\times C([-T,0])\right)$ and it is a classical solution of (1.1).

2. If $f$ is continuous then we have moreover $u \in C^0([0,T] \times C([-T,0]))$.

*Proof.* We will see that that $D^{\perp}u(t,\eta)$ is absolutely continuous with density that we will denote $x \mapsto D_x^{ac}$, so (1.1) simplifies in

$$\begin{cases} \mathcal{L}u\,(t,\eta) = \partial_t u(t,\eta) + \int_{]-t,0]} D_x^{ac} u(t,\eta)\, d^-\eta(x) + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = 0, \\ u(T,\eta) = H(\eta). \end{cases}$$

We first evaluate the derivative $\partial_t u\,(t,\eta)$, for a given $(t,\eta) \in [0,T] \times C([-T,0])$:

$$\begin{aligned} \partial_t u(t,\eta) &= \partial_t \Psi \left( t, \int_{[-t,0]} \varphi_1(s+t) d^-\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t) d^-\eta(s) \right) \\ &+ \sum_{i=1}^n \left( \partial_i \Psi \left( t, \int_{[-t,0]} \varphi_1(s+t) d^-\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t) d^-\eta(s) \right) \cdot \left( \partial_t \int_{[-t,0]} \varphi_i(s+t) d^-\eta(s) \right) \right) \\ &= \partial_t \Psi \left( t, \int_{[-t,0]} \varphi_1(s+t) d^-\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t) d^-\eta(s) \right) \\ &+ \sum_{i=1}^n \left( \partial_i \Psi \left( t, \int_{[-t,0]} \varphi_1(s+t) d^-\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t) d^-\eta(s) \right) \cdot I_i \right), \end{aligned} \tag{3.11}$$

where

$$I_i := \left( \int_{]-t,0]} \dot{\varphi}_i(s+t) d^-\eta(s) \right).$$

Indeed, by usual theorems of Lebesgue integration theory and by Proposition 2.3, (2.3) and (2.4), for every $1 \le i \le n$, we obtain

$$\begin{aligned} \partial_t \left( \int_{[-t,0]} \varphi_i(s+t) d^-\eta(s) \right) &= \partial_t \left( \eta(0)\varphi_i(t) - \int_{-t}^0 \eta(s) \dot{\varphi}_i(s+t) ds \right) \\ &= \eta(0)\dot{\varphi}_i(t) - \eta(-t)\dot{\varphi}_i(0^+) - \int_{-t}^0 \eta(s)\ddot{\varphi}_i(s+t) ds = I_i. \end{aligned}$$

In order to evaluate the derivatives of $u$ with respect to $\eta$, by (3.9) and (3.10), it yields

$$u(t,\eta) = \Psi \left( t, \eta(0)\varphi_1(t) - \int_0^t \eta(s-t)\dot{\varphi}_1(s) ds, \ldots, \eta(0)\varphi_n(t) - \int_0^t \eta(s-t)\dot{\varphi}_n(s) ds \right).$$

For every $t \in [0,T]$, $\eta \in C([-T,0])$, the first derivative $Du$ evaluated at $(t,\eta)$ is the measure on $[-T,0]$

defined by

$$D_{dx}u(t,\eta) = D^{ac}_x u(t,\eta)\,dx + D^{\delta_0}u(t,\eta)\delta_0(dx) \quad \text{with}$$
$$D^{ac}_x u(t,\eta) = -\sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)d^-\eta(s)\right)\right) \cdot \left(\mathbb{1}_{[-t,0]}(x)\dot{\varphi}_i(x+t)\right),$$
$$D^{\delta_0}u(t,\eta) = \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)d^-\eta(s)\right)\right) \cdot \varphi_i(t)\,.$$

As anticipated, we observe that $x \mapsto D^{ac}_x u\,(t,\eta)$ has bounded variation.

Deriving again in a similar way, for every $t \in [0,T]$, $\eta \in C([-T,0])$, the second order derivative $D^2 u$ evaluated at $(t,\eta)$ gives

$$\begin{aligned} D^2_{dx,dy}u(t,\eta) = \sum_{i,j=1}^n & \left(\partial^2_{i,j}\Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)d^-\eta(s)\right)\right) \cdot \\ & \cdot \Bigg(\varphi_i(t)\varphi_j(t)\delta_0(dx)\delta_0(dy) - \varphi_i(t)\mathbb{1}_{[-t,0]}(x)\ddot{\varphi}_j(x+t)\delta_0(dy) \\ & - \varphi_j(t)\mathbb{1}_{[-t,0]}(y)\ddot{\varphi}_i(y+t)\delta_0(dx) + \mathbb{1}_{[-t,0]}(x)\mathbb{1}_{[-t,0]}(y)\ddot{\varphi}_i(x+t)\ddot{\varphi}_j(y+t)\Bigg)\,. \end{aligned} \tag{3.12}$$

We get

$$\int_{]-t,0]} D^{ac}_x u(t,\eta)\,d^-\eta(x) = \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)d^-\eta(s)\right)\right) \cdot I_i. \tag{3.13}$$

Using (3.8), (3.11), (3.13) and (3.12) we obtain that

$$\mathcal{L}u\,(t,\eta) = 0.$$

Condition $u(T,\eta) = H(\eta)$ is trivially verified by definition. This concludes the proof of point 1.

Point 2. is immediate. □

**Remark 3.5.** In this section we have often used the concept of deterministic forward integral on a closed interval $[-t,0]$, given at Definition 2.2

$$\int_{[-t,0]} \varphi_i(s+t)d^-\eta(s), \tag{3.14}$$

instead of

$$\int_{]-t,0]} \varphi_i(s+t)d^-\eta(s).$$

Since $W_0 = 0$, when we inject $\eta = W_t(\cdot)$, the two integrals are the same:

$$\int_{[-t,0]} \varphi_i(s+t) d^- \eta(s)|_{\eta=W_t(\cdot)} = \int_{]-t,0]} \varphi_i(s+t) d^- \eta(s)|_{\eta=W_t(\cdot)}.$$

The choice of (3.14), which is compatible to the fact of considering $\int_{]-t,0]} D_x^{ac} u(t,\eta)\, d^-\eta(x)$ in (1.1) is justified since

$$t \mapsto \int_{]-t,0]} \varphi_i(s+t) d^- \eta(s)$$

is not differentiable.

# 4 The existence result for smooth Fréchet terminal condition

## 4.1 Preliminary considerations

In this section, we will show existence theorem for classical solutions of (1.1) under smooth Fréchet terminal condition. In order to define explicitly the solution of the PDE, we need to introduce two central objects for this section: the functional Brownian stochastic flow which is a $C([-T,0])$-valued stochastic flow denoted by $(Y_t^{s,\eta})_{0\le s\le t\le T, \eta\in C([-T,0])}$ and the Markovian stochastic flow which is a real valued stochastic flow denoted by $(X_t^{s,x})_{0\le s\le t\le T, x\in\mathbb{R}}$.

**Definition 4.1.** Let $\Delta := \{(s,t)|\ 0 \le s \le t \le T\}$ and $\eta \in C([-T,0])$. We define the flow that will appear in this section.

1. We call **Brownian stochastic flow** the real valued flow and denoted by $(X_t^{s,x})_{0\le s\le t\le T, x\in\mathbb{R}}$ is a real random field defined over $\Delta \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$X_t^{s,x} = x + \sigma(W_t - W_s)\ . \tag{4.1}$$

2. We call **functional Brownian stochastic flow** the $C([-T,0])$-valued random field defined over $\Delta \times C([-T,0]) \longrightarrow C([-T,0])$ by

$$(s,t,\eta) \mapsto Y_t^{s,\eta}(x) = \begin{cases} \eta(x+t-s) & x \in [-T, s-t[ \\ \eta(0) + \sigma\left(W_t(x) - W_s\right) & x \in [s-t, 0]. \end{cases} \tag{4.2}$$

Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ be the functional appearing in (1.1) and a path-dependent random variable $h := H(\sigma W_T(\cdot))$. We define the functional $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ by

$$u(t,\eta) = \mathbb{E}\left[H\big(Y_T^{t,\eta}\big)\right]. \tag{4.3}$$

Since $\sigma W_T(\cdot) = Y_T^{t,\sigma W_t(\cdot)}$ we have

$$\mathbb{E}\left(h|\mathcal{F}_t\right) = \mathbb{E}\left[H\big(\sigma W_T(\cdot)\big)|\mathcal{F}_t\right] = \mathbb{E}\left[H\big(Y_T^{t,\sigma W_t(\cdot)}\big)|\mathcal{F}_t\right] = u(t, \sigma W_t(\cdot)).$$

For this reason, $u$ defined in (4.3) is a natural candidate to be a solution of (1.1). In Theorem 4.10 we will show, under smooth regularity of $H$, that such a $u$ is sufficiently smooth to be a classical solution of the path-dependent heat equation (1.1).

We dedicate next Subsection 4.2. to investigate some properties of $Y_T^{t,\eta}$ that we will use in the proof of the main theorem.

## 4.2 About a Markovian stochastic flow

We recall that, given $X$ and $Y$ two random elements taking values in the same space, we say that $X \sim Y$ if they have the same law.

**Remark 4.2.**

1. $(X_t^{s,x})$ introduced in Definition 4.1 is a particular case of the Markovian flow coming from the solutions to an SDE of the type

$$X_t = x + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds, t \in [0, T],$$

and $\sigma, b : [0, T] \times \mathbb{R} \to \mathbb{R}$ being Lipschitz functions with linear growth. We could have formulated in this more general framework the present paper but for simplicity we have restricted us to the pure Brownian case, i.e. $\sigma(t, x) = \sigma$, $\sigma$ a constant.

2. The Markovian stochastic flow admits a continuous modification in $(s, t, x)$. It can be considered as an $\mathbb{R}$-valued continuous random field and it verifies the well-known flow property for $0 \leq s \leq t \leq r \leq T$,

$$X_r^{s,x} = X_r^{t, X_t^{s,x}} . \tag{4.4}$$

We set

$$Y_t^{s,\eta}(x) = \begin{cases} \eta(x + t - s) & x \in [-T, s - t] \\ X_{t+x}^{s,\eta(0)} & x \in [s - t, 0]. \end{cases} \tag{4.5}$$

The functional flow $(Y_t^{s,\eta})$ coincides of course with (4.2) when $(X_t^{s,x})$ is given by (4.1).

The following lemma shows a "flow property" for the functional Brownian stochastic flow.

**Lemma 4.3.** Let $\eta \in C([-T, 0])$, for $0 \leq s \leq t \leq r \leq T$, the following flow property holds

$$Y_r^{s,\eta} = Y_r^{t, Y_t^{s,\eta}}. \tag{4.6}$$

*Proof.* It follows from the flow property (4.4) for the Markovian stochastic flow.
For fixed $\omega \in \Omega$, we inject $\tilde{\eta} = Y_s^{t,\eta}$ into $Y_r^{t,\tilde{\eta}}$ obtaining

$$Y_r^{t, Y_t^{s,\eta}}(x) = \left\{ \begin{array}{ll} \eta(x + r - s) & x \in [-T, s - r] \\ X_{r+x}^{s,\eta(0)} & x \in [s - r, t - r] \\ X_{r+x}^{t,\tilde{\eta}(0)} = X_{r+x}^{t, X_t^{s,\eta(0)}} = X_{r+x}^{s,\eta(0)} & x \in [t - r, 0] \end{array} \right\} = Y_r^{s,\eta}(x) ,$$

which concludes the proof of the Lemma. $\square$

In the sequel, as already mentioned, for simplicity of illustration we will concentrate on the functional Brownian stochastic flow. First of all we observe that it is time-homogeneous in law.

**Proposition 4.4.** $Y_t^{s,\eta}$ and $Y_{t-s}^{0,\eta}$ have the same law as $C([-T,0])$-valued r.v.
In particular, for every $x \in [-T,0]$, $Y_{t-s}^{0,\eta}(x) \sim Y_t^{s,\eta}(x)$.

*Proof.* It follows from the two following arguments. For $x \in [-T, s-t]$, $Y_t^{s,\eta}(x)$ and $Y_{t-s}^{0,\eta}(x)$ deterministic and are equal to $\eta(x+t-s)$. For $x \in [s-t, 0]$, the real-valued processes $Y_t^{s,\eta}(x) = \eta(0) + \sigma\left(W_t(x) - W_s\right)$ and $Y_{t-s}^{0,\eta}(x) = \eta(0) + \sigma\left(W_{t-s}(x) - W_0\right)$, have the same law by well-known properties of Brownian motion. $\square$

From now on a realization $\omega \in \Omega$ will be often fixed. The next proposition concerns the continuity of the field $Y_t^{s,\eta}$ with respect to its three variables.

**Proposition 4.5.** $(Y_t^{s,\eta})_{0 \le s \le t \le T,\, \eta \in C([-T,0])}$ is a continuous random field.

*Proof.* As usual in this section $\omega \in \Omega$ is fixed and $\varpi_\eta$ (resp. $\varpi_{W(\omega)}$) is respectively the modulus of continuity of $\eta$ (resp. the Brownian path $W(\omega)$).
Let $(s,t,\eta)$ such that $0 \le s \le t \le T, \eta \in C([-T,0])$ and a sequence $(s_n, t_n, \eta_n)$ also such that $0 \le s_n \le t_n \le T, \eta_n \in C([-T,0])$ with

$$\lim_{n\to\infty} \left(|s - s_n| + |t - t_n| + \|\eta - \eta_n\|_\infty\right) = 0.$$

We have to show that $Y_{t_n}^{s_n,\eta_n} \longrightarrow Y_t^{s,\eta}$ in $C([0,T]$, when $n \to \infty$ i.e. uniformly. For $x \in [0,T]$, we evaluate

$$|Y_{t_n}^{s_n,\eta_n} - Y_t^{s,\eta}|(x) \le (I_1(n) + I_2(n) + I_3(n))(x),$$

where

$$\begin{aligned} I_1(n)(x) &= |Y_{t_n}^{s_n,\eta_n} - Y_{t_n}^{s_n,\eta}|(x) \\ I_2(n)(x) &= |Y_{t_n}^{s,\eta} - Y_t^{s,\eta}|(x) \\ I_3(n)(x) &= |Y_{t_n}^{s_n,\eta} - Y_{t_n}^{s,\eta}|(x). \end{aligned}$$

By Definition 4.1, it is easy to see that

$$\begin{aligned} \|I_1(n)\|_\infty &\le \|\eta - \eta_n\|_\infty + |\eta_n(0) - \eta(0)| \\ &\le 2\|\eta - \eta_n\|_\infty. \end{aligned}$$

Since $I_3(n)$ behaves similarly to $I_2(n)$, we only show that

$$\lim_{n\to\infty} I_2(n) = 0.$$

Without restriction to generality, we will suppose that $t_n \le t$ for any $n$, since the case when the sequence $(t_n)$ is greater or equal than $t$, could be treated analogously. We observe that following equality holds:

$$
\begin{aligned}
(Y^{s,\eta}_{t_n} - Y^{s,\eta}_{t})(x) = \eta(x + t_n - s)\mathbb{1}_{[-T,s-t_n]}(x) - \eta(x + t - s)\mathbb{1}_{[-T,s-t]}(x) + \\
+ (\eta(0) + \sigma W_{t_n}(x) - \sigma W_s)\, \mathbb{1}_{[s-t_n,0]}(x) - (\eta(0) + \sigma W_t(x) - \sigma W_s)\, \mathbb{1}_{[s-t,0]}(x) = \\
= (\eta(x + t_n - s) - \eta(x + t - s))\, \mathbb{1}_{[-T,s-t]}(x) + \\
+ (\eta(x + t_n - s) - \eta(0) - \sigma W_t(x) + \sigma W_s)\, \mathbb{1}_{[s-t,s-t_n]}(x) \\
+ (\sigma W_{t_n}(x) - \sigma W_t(x))\, \mathbb{1}_{[s-t_n,0]}(x)\ .
\end{aligned}
\tag{4.7}
$$

Using (4.7) to evaluate $\|I_2(n)\|_\infty$ we obtain

$$
\begin{aligned}
\sup_{x\in[-T,0]} |Y^{s,\eta}_{t_n}(x) - Y^{s,\eta}_{t}(x)| \le \sup_{x\in[-T,0]} |\eta(x + t_n - s) - \eta(x + t - s)| + \\
+ \sup_{x\in[s-t,s-t_n]} |\eta(x + t_n - s) - \eta(0)| + \sup_{x\in[s-t,s-t_n]} \sigma|W_t(x) - W_s| + \\
+ \sup_{x\in[-T,0]} \sigma|W_{t_n}(x) - W_t(x)| \le \\
\le 2\, \varpi_\eta(|t_n - t|) + 2\, \sigma\varpi_{W(\omega)}(|t_n - t|) \xrightarrow[n\longrightarrow+\infty]{} 0.
\end{aligned}
$$

Since $\eta$ and $W(\omega)$ are uniformly continuous on the compact set $[0, T]$ both modulus of continuity converge to zero when $t_n \to t_0$. □

At this point we make some simple observations that will be often used in the sequel.

**Remark 4.6.**

1. There are universal constants $C_1$, $C_2$, $C_3$ and $C_4$ such that for every $t \in [0, T], \epsilon$ with $t + \epsilon \in [0, T]$ such that

$$
\left\|Y^{t,\eta}_T\right\|_\infty \le C_1 \left(1 + \|\eta\|_\infty + \sup_{t\in[0,T]} \sigma|W_t|\right) \ ; \quad \left\|Y^{t+\epsilon,\eta}_T\right\|_\infty \le C_2 \left(1 + \|\eta\|_\infty + \sigma \sup_{t\in[0,T]} |W_t|\right)
\tag{4.8}
$$

   and

$$
\left\|Y^{T-t,\eta}_0\right\|_\infty \le C_3 \left(1 + \|\eta\|_\infty + \sigma \sup_{t\in[0,T]} |W_t|\right) \ .
\tag{4.9}
$$

   (4.8) implies that, for any $\alpha \in [0, 1], t \in [0, T], \epsilon$ with $t + \epsilon \in [0, T]$, it holds

$$
\left\|\alpha Y^{t+\epsilon,\eta}_T + (1-\alpha) Y^{t+\epsilon, Y^{t,\eta}_{t+\epsilon}}_T\right\|_\infty \le C_4 \left(1 + \|\eta\|_\infty + \sigma \sup_{t\in[0,T]} |W_t|\right).
\tag{4.10}
$$

2. For any $\alpha \in [0,1], t \in [0,T]$ it holds
$$\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}} \xrightarrow[\epsilon \longrightarrow 0]{C([-T,0])} Y_T^{t,\eta} . \tag{4.11}$$
In fact developing term $Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$, which equals $Y_T^{t,\eta}$, we obtain
$$\left\| \alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}} - Y_T^{t,\eta} \right\|_\infty = \alpha \left\| Y_T^{t+\epsilon,\eta} - Y_T^{t,\eta} \right\|_\infty .$$
The right-hand side converges to zero because of Proposition 4.5.

3. In the sequel we will make an explicit use of the expression below:
$$\left( Y_T^{t+\epsilon,\eta} - Y_T^{t,\eta} \right)(x) = \begin{cases} \eta(x+T-t+\epsilon) - \eta(x+T-t) & x \in [-T, t-T] \\ \eta(x+T-t+\epsilon) - \eta(0) - \sigma W_T(x) + \sigma W_t & x \in [t-T, t-T+\epsilon] \\ \sigma W_t - \sigma W_{t+\epsilon} & x \in [t-T+\epsilon, 0]. \end{cases}$$

We concentrate now to the derivatives of the functional stochastic flow. We fix $\rho \in [-T,0]$. By (4.5) we remind that
$$Y_T^{t,\eta}(\rho) = \begin{cases} \eta(\rho+T-t) & \rho \in [-T, t-T[ \\ X_{T+\rho}^{t,\eta(0)} & \rho \in [t-T, 0]. \end{cases}$$
It is possible to calculate formally the first and second derivatives of $Y_T^{t,\eta}(\rho)$.

**Remark 4.7.** Let $\rho \in [-T,0]$, then it holds $Y_T^{t,\cdot}(\rho) : C([-T,0]) \times \Omega \longrightarrow \mathbb{R}$ and $DY_T^{t,\cdot}(\rho) : C([-T,0]) \times \Omega \longrightarrow (C([-T,0]))^* = \mathcal{M}([-T,0])$. In particular if $f \in C([-T,0])$,
$$_{\mathcal{M}([-T,0])}\langle DY_T^{t,\eta}(\rho) \ , \ f \rangle_{C([-T,0])} = \int_{[-T,0]} f(x) D_{dx} Y_T^{t,\eta}(\rho, \omega).$$

In particular for a fixed $\rho \in [-T,0]$ we have
$$D_{dx} Y_T^{t,\eta}(\rho) = \begin{cases} \delta_{\rho+T-t}(dx) & \rho \in [-T, t-T[ \\ \delta_0(dx) \partial_\xi X_{T+\rho}^{t,\eta(0)} = & \rho \in [t-T, 0] \end{cases} \quad \text{and}$$
$$D^2_{dy\,dx} Y_T^{t,\eta}(\rho) = \begin{cases} 0 & \rho \in [-T, t-T[ \\ \delta_0(dx)\delta_0(dy) \partial^2_{\xi\xi} X_{T+\rho}^{t,\eta(0)} & \rho \in [t-T, 0]. \end{cases} \tag{4.12}$$

**Remark 4.8.** For the functional Brownian stochastic flow, i.e. when $\sigma(t,x) = \sigma$, $\sigma$ a constant, it is trivial to verify using (4.1), that the following holds.
$$\partial_\xi X_t^{s,\xi} = 1, \quad \text{and} \quad \partial^2_{\xi\xi} X_s^{t,\xi} = 0.$$

Avoiding some technicalities it is possible to evaluate the first and second derivatives of the functional flow itself. In the sequel $\eta$ will always be a generic element in $C([-T,0])$. Let $X_t^{s,x}$ be again the standard (real valued) stochastic flow as in (4.1) and the associated functional (Banach valued) stochastic flow $Y_t^{s,\eta}$ as in Definition 4.1.

**Lemma 4.9.** 1. The map $Y_T^{t,\cdot} : C([-T,0]) \times \Omega \longrightarrow C([-T,0])$ acting as $\eta \mapsto Y_T^{t,\eta}$ is of class $C^2\left(C([-T,0])\,; C([-T,0])\right)$ a.s.

2. The derivatives $DY_T^{t,\cdot} : C([-T,0]) \times \Omega \longrightarrow \mathcal{L}\left(C([-T,0]); C([-T,0])\right)$ and $D^2 Y_T^{t,\cdot} : C([-T,0]) \times \Omega \longrightarrow \mathcal{B}\left(C([-T,0]) \times C([-T,0]); C([-T,0])\right)$ are characterized as follows. For $f, g \in C([-T,0])$ we have

$$\rho \mapsto \int_{[-T,0]} D_{dx} Y_T^{t,\eta}(\rho) f(x) = \begin{cases} f(\rho + T - t) & \rho \in [-T, t-T[ \\ f(0)\,\partial_\xi X_{T+\rho}^{t,\eta(0)} = f(0) & \rho \in [t-T, 0] \end{cases} \quad \text{and}$$

$$\rho \mapsto \int_{[-T,0]^2} D^2_{dy\,dx} Y_T^{t,\eta}(\rho) f(x) g(y) = \begin{cases} 0 & \rho \in [-T, t-T[ \\ f(0)g(0)\,\partial^2_{\xi\xi} X_{T+\rho}^{t,\eta(0)} = 0 & \rho \in [t-T, 0]. \end{cases}$$

## 4.3 The existence result for smooth Fréchet terminal condition: existence Theorem

In this section, Theorem 4.10 states the existence result and Fréchet regularity of the solution of the infinite dimensional PDE (1.1) when $\sigma$ is constant and $H$ is $C^3(C([-T,0]))$. In particular will give conditions on the function $H$ such that $u$ defined in (4.3) solves the PDE stated on (1.1). Those conditions are reasonable but they are however not optimal.

**Theorem 4.10.** Let $H \in C^3\left(C([-T,0])\right)$ such that $D^3 H$ has polynomial growth (for instance bounded). Let $u$ be defined by $u(t,\eta) = \mathbb{E}\left[H\big(Y_T^{t,\eta}\big)\right], t \in [0,T], \eta \in C([-T,0])$.

1) Then $u \in C^{0,2}([0,T] \times C([-T,0]))$.

2) Suppose moreover the following for every $\eta \in C([-T,0])$.

   i) The measure $D_{dx} H(\eta)$ is Lebesgue absolutely continuous. We will denote $x \mapsto D_x H(\eta)$ its density and we suppose that $DH(\eta) \in H^1([-T,0])$, i.e. function $x \mapsto D_x H(\eta)$ is in $H^1([-T,0])$.

   ii) $DH$ has polynomial growth in $H^1([-T,0])$, i.e. there is $p \geq 1$ such that

$$\eta \mapsto \|DH(\eta)\|_{H^1} \leq const\left(\|\eta\|_\infty^p + 1\right) . \tag{4.13}$$

   In particular

$$\sup_{t \in [-T,0]} |D_x H(\eta)| \leq const\left(\|\eta\|_\infty^p + 1\right) \leq const\left(\|\eta\|_\infty^p + 1\right) .$$

   iii) The map

$$\eta \mapsto DH(\eta) \qquad \text{considered as} \qquad C([-T,0]) \to H^1([-T,0]) \qquad \text{is continuous.} \tag{4.14}$$

Then $u \in C^{1,2}([0,T] \times C([-T,0]))$ and $u$ is a classical strict solution of (1.1) in $C([-T,0])$, i.e. $u$ solves

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^- \eta(x) + \frac{1}{2}\sigma^2 \langle D^2 u(t,\eta), 1_0 \otimes^2 \rangle = 0 \\ u(T,\eta) = H(\eta). \end{cases}$$

**Remark 4.11.** Contrarily to the (non-degenerate) situation of Section 3, Theorem 4.10 holds even when $\sigma = 0$. In that case one gets a first-order equation; the regularity on $H$ could be relaxed but we are not specifically interested in this refinement.

**Remark 4.12.**
1. Assumption (4.13) implies in particular that $DH$ has polynomial growth in $C([-T,0])$, i.e. there is $p \geq 1$ such that

$$\eta \mapsto \sup_{x \in [-T,0]} |D_x H(\eta)| = \|DH(\eta)\|_\infty \leq const \left( \|\eta\|^p_\infty + 1 \right). \tag{4.15}$$

   Indeed it is well-known that $H^1([-T,0]) \hookrightarrow C([-T,0])$ and for a function $f \in H^1$ it holds $\|f\|_\infty \leq const \|f\|_{H^1}$.
2. By a Taylor's expansion, given for instance by Theorem 5.6.1 in [1], the fact that $D^3 H$ has polynomial growth implies that $H$, $DH$ and $D^2 H$ have also polynomial growth in $C([-T,0])$.
3. $Du(t,\eta)$, $D^2 u(t,\eta)$ and $\partial_t u(t,\eta)$ will be explicitly expressed in term of $H$ at (4.19), (4.21) and (4.49).

*Proof.* By expression (4.3) it is obvious that $u(T,\eta) = H(\eta)$.
*Proof of 1).*

- **Continuity of function $u$ with respect to time $t$.**

We consider a sequence $(t_n)$ in $[0,T]$ such that $t_n \xrightarrow[n\to\infty]{} t_0$. By Assumption, $H \in C^0(C([-T,0]))$. Consequently, by Proposition 4.5

$$H\left(Y^{0,\eta}_{T-t_n}\right) \xrightarrow[n\to\infty]{a.s.} H\left(Y^{0,\eta}_{T-t_0}\right). \tag{4.16}$$

By Remark 4.12.1. $H$ has also polynomial growth, therefore there is $p \geq 1$ such that

$$|H(\zeta)| \leq const \left( 1 + \sup_{x \in [-T,0]} |\zeta(x)|^p \right) \qquad \forall\, \zeta \in C([-T,0]).$$

By (4.9), we observe that

$$|H(Y^{0,\eta}_{T-t})| \leq const \left( 1 + \left\| Y^{0,\eta}_{T-t} \right\|^p_\infty \right) \leq$$
$$\leq const \left( 1 + \sup_{x \in [-T,0]} |\eta(x)|^p + \sigma^p \sup_{t \leq T} |W_t|^p \right).$$

By Lebesgue dominated convergence theorem, the fact that $\sup_{t\le T}|W_t|^p$ is integrable and (4.16), it follows that

$$u(t_n,\eta)=\mathbb{E}\left[H\big(Y^{0,\eta}_{T-t_n}\big)\right]\xrightarrow[n\to\infty]{}\mathbb{E}\left[H\big(Y^{0,\eta}_{T-t_0}\big)\right]=u(t_0,\eta)\ .$$

- **First order Fréchet derivative.**

We express now the derivatives of $u$ with respect to the derivatives of $H$. We start with $Du:[0,T]\times C([-T,0])\longrightarrow\mathcal{M}([-T,0])$. Omitting some details, by integration theory for every $t\in[0,T]$, $u(t,\cdot)$ is of class $C^1\left(C([-T,0])\right)$. By usual composition rules derivation for operator we have

$$D_{dx}H\left(Y^{t,\eta}_T\right)=\int_{[-T,0]}D_{d\rho}H\left(Y^{t,\eta}_T\right)D_{dx}Y^{t,\eta}_T(\rho).$$

$$D_{dx}u\left(t,\eta\right)=\mathbb{E}\left[D_{dx}H\left(Y^{t,\eta}_T\right)\right]=\mathbb{E}\left[\int_{[-T,0]}D_{d\rho}H\left(Y^{t,\eta}_T\right)D_{dx}Y^{t,\eta}_T(\rho)\right].\tag{4.17}$$

We compute explicitly (4.17) using the expression (4.12). Integrating with respect to $\rho$ (for a fixed $x$), we obtain the following.

$$\begin{aligned}D_{dx}u\left(t,\eta\right)&=\mathbb{E}\left[\int_{[-T,t-T[}D_{d\rho}H\left(Y^{t,\eta}_T\right)D_{dx}Y^{t,\eta}_T(\rho)\right]+\mathbb{E}\left[\int_{[t-T,0]}D_{d\rho}H\left(Y^{t,\eta}_T\right)D_{dx}Y^{t,\eta}_T(\rho)\right]\\&=\mathbb{E}\left[\int_{[-T,t-T[}D_{d\rho}H\left(Y^{t,\eta}_T\right)\delta_{\rho+T-t}(dx)\right]+\mathbb{E}\left[\int_{[t-T,0]}D_{d\rho}H\left(Y^{t,\eta}_T\right)\right]\delta_0(dx).\end{aligned}\tag{4.18}$$

Consequently

$$D_{dx}u(t,\eta)=D^{\perp}_{dx}u\left(t,\eta\right)+D^{\delta_0}u\left(t,\eta\right)\delta_0(dx),\tag{4.19}$$

where

$$D^{\perp}_{dx}u(t,\eta)=\mathbb{E}\left[D_{dx-T+t}H\big(Y^{t,\eta}_T\big)\right]\mathbb{1}_{[-t,0[}(x).\tag{4.20}$$

and

$$D^{\delta_0}u\left(t,\eta\right)=\mathbb{E}\left[\int_{[t-T,0]}D_{d\rho}H\big(Y^{t,\eta}_T\big)\right].$$

Indeed the first addend $D^{\perp}_{dx}u\left(t,\eta\right)$ of (4.19), i.e. expression (4.20) comes from (4.18), using the fact that $\delta_{\rho+T-t}(dx)=\delta_{dx-T+t}(d\rho)$ and integrating with respect to $\rho$. The continuity of $(t,\eta)\mapsto D_{dx}u(t,\eta)$ in (4.19) can be justified since the function $[0,T]\times C([-T,0])\to\mathbb{R}$, $(t,\eta)\mapsto D^{\delta_0}u\left(t,\eta\right)$ and function $[0,T]\times C([-T,0])\to\mathcal{M}([-T,0])$ defined by $(t,\eta)\mapsto D^{\perp}u(t,\eta)$ are both continuous. The latter fact follows from the fact that $H\in C^1(C([-T,0]))$, $DH$ with polynomial growth, (4.9), (4.8), the fact that for any given Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments and finally the Lebesgue dominated convergence theorem.

• **Second order Fréchet derivative.**

We discuss the second derivative

$$D^2u\ :[0,T]\times C([-T,0])\longrightarrow (C([-T,0])\hat{\otimes}_\pi C([-T,0]))^*\cong \mathcal{B}(C([-T,0]),C([-T,0])).$$

For every fixed $(t,\eta)$ we get

$$\begin{aligned} D^2_{dx,dy}u(t,\eta) = &\ \mathbb{E}\Big[D^2_{dy-T+t,dx-T+t}H\big(Y^{t,\eta}_T\big)\mathbb{1}_{[-t,0[}(x)\otimes \mathbb{1}_{[-t,0[}(y)\Big]+\\ &+\mathbb{E}\left[D_{dx-T+t}\langle DH\big(Y^{t,\eta}_T\big),\mathbb{1}_{[t-T,0]}\rangle\right]\mathbb{1}_{[-t,0[}(x)\,\delta_0(dy)+\\ &+\mathbb{E}\left[D_{dy-T+t}\langle DH\big(Y^{t,\eta}_T\big),\mathbb{1}_{[t-T,0]}\rangle\right]\mathbb{1}_{[-t,0[}(y)\,\delta_0(dx)+\\ &+\mathbb{E}\left[\langle D^2H\big(Y^{t,\eta}_T\big),\mathbb{1}_{[t-T,0]}\otimes\mathbb{1}_{[t-T,0]}\rangle\right]\delta_0(dx)\,\delta_0(dy)\ . \end{aligned}$$

It is possible to show that all the terms in the first and the second derivative are well defined and continuous using similar techniques used in the first part of the proof. We omit these technicalities for simplicity.

**Remark 4.13.** For illustration, if $D^2H$ is an absolutely continuous measure on $[-T,0]^2$ with density $D^2_{x,y}H=D_xD_yH$, we obtain the following

$$\begin{aligned} D^2_{dx,dy}u(t,\eta) = &\ \mathbb{E}\Big[D_{y-T+t}D_{x-T+t}H\big(Y^{t,\eta}_T\big)\Big]\mathbb{1}_{[-t,0[}(x)\,\mathbb{1}_{[-t,0[}(y)dx\,dy+\\ &+\mathbb{E}\left[\int_{t-T}^0 D_sD_{x-T+t}H\big(Y^{t,\eta}_T\big)ds\right]\mathbb{1}_{[-t,0[}(x)dx\,\delta_0(dy)+\\ &+\mathbb{E}\left[\int_{t-T}^0 D_{y-T+t}D_sH\big(Y^{t,\eta}_T\big)ds\right]\mathbb{1}_{[-t,0[}(y)dy\,\delta_0(dx)+\\ &+\mathbb{E}\left[\int_{[t-T,0]^2} D_{s_1}D_{s_2}H\big(Y^{t,\eta}_T\big)ds_1\,ds_2\right]\delta_0(dx)\,\delta_0(dy)\ . \end{aligned} \tag{4.21}$$

*Proof of 2)*

**Remark 4.14.** Under hypotheses 2) we remark the following.

1. The right-hand side of (4.20) is absolutely continuous in $x$. In other words $D^\perp_{dx}u(t,\eta)=D^{ac}_x u\,(t,\eta)dx$ and

$$D^{ac}_x u(t,\eta)=\mathbb{E}\left[D_{x-T+t}H\big(Y^{t,\eta}_T\big)\right]\mathbb{1}_{[-t,0[}(x)=\begin{cases}0 & x\in[-T,-t[\\ \mathbb{E}\left[D_{x-T+t}H\big(Y^{t,\eta}_T\big)\right] & x\in\ [-t,0[\ .\end{cases} \tag{4.22}$$

2. Since $x\mapsto D_xH(\eta)$ belongs to $H^1$, it has bounded variation, the deterministic forward integral in (1.1) exists because of Proposition 2.3 and it can be expressed through (2.4). We will denote by

$D'H(\eta)$ the derivative in $x$ of function $x \mapsto D_x H(\eta)$, where $D_x H(\eta)$ is the density of the measure $D_{dx}H(\eta)$ for every fixed $\eta$. Since $x \mapsto D_x H(\eta)$ is absolutely continuous then, by (2.2) we have

$$\int_{]-t,0]} D_{dx-T+t} H\left(Y_T^{t,\eta}\right) d^-\eta(x) = \int_{]-t,0]} D_{x-T+t} H\left(Y_T^{t,\eta}\right) d^-\eta(x). \tag{4.23}$$

Previous deterministic integral exists because $x \mapsto D_x H(\eta)$ has bounded variation and by Proposition 2.3 it equals

$$-D_{-T} H\left(Y_T^{t,\eta}\right)\eta(-t) + D_{t-T} H\left(Y_T^{t,\eta}\right)\eta(0) - \int_{-t}^{0} D'_{x-T+t} H\left(Y_T^{t,\eta}\right)\eta(x)dx.$$

- **Derivability with respect to time $t$.**

Let $t \in [0,T], \eta \in C([-T,0])$. We will show that

$$\partial_t u(t,\eta) = -\mathbb{E}\left[\int_{]-t,0]} D_{x-T+t} H\left(Y_T^{t,\eta}\right) d^-\eta(x) + \frac{\sigma^2}{2}\langle D^2 H\left(Y_T^{t,\eta}\right)\ ,\ 1_{]t-T,0]}\otimes^2\rangle\right] .$$

We need to consider $\epsilon$ such that $t+\epsilon \in [0,T]$ and evaluate the limit, if it exists, of

$$\frac{u(t+\epsilon,\eta) - u(t,\eta)}{\epsilon}, \tag{4.24}$$

when $\epsilon \to 0$. Without restriction of generality we will suppose here $\epsilon > 0$; the case $\epsilon < 0$ would bring similar calculations.
The flow property (4.6) gives $Y_T^{t,\eta} = Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$, which allows to write

$$u(t,\eta) = \mathbb{E}\left[H\left(Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\right)\right] . \tag{4.25}$$

We go on with the evaluation of the limit of (4.24). By (4.25) and by differentiability of $H$ in $C([-T,0])$ we have

$$\begin{aligned} H\left(Y_T^{t+\epsilon,\eta}\right) - H\left(Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\right) &= \langle DH\left(Y_T^{t,\eta}\right), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\rangle + \\ &+ \int_0^1 \langle DH\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\right) - DH\left(Y_T^{t,\eta}\right), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\rangle d\alpha = \\ &= \int_{[-T,0]} D_{dx} H\left(Y_T^{t,\eta}\right)\left(Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}(x)\right) + S(\epsilon,t,\eta)\ , \end{aligned} \tag{4.26}$$

where

$$S(\epsilon,t,\eta) = \int_0^1 \langle DH\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\right) - DH\left(Y_T^{t,\eta}\right), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\rangle d\alpha\ .$$

Setting $\gamma = Y_{t+\epsilon}^{t,\eta}$, we need to evaluate

$$Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,\gamma}(x) \qquad x \in [-T,0]\ . \tag{4.27}$$

(4.27) gives

$$Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,\gamma}(x) = \begin{cases} \eta(x+T-t-\epsilon) - \gamma(x+T-t-\epsilon) & x \in [-T, t-T+\epsilon[ \\ \eta(0) - \gamma(0) = -\sigma(W_{t+\epsilon}(0) + W_t) & x \in [t-T+\epsilon, 0]\ , \end{cases} \tag{4.28}$$

because $\gamma(0) = Y_{t+\epsilon}^{t,\eta}(0) = \eta(0) + \sigma(W_{t+\epsilon}(0) - W_t)$. Moreover, by (4.5), we have

$$\gamma(x+T-t-\epsilon) = Y_{t+\epsilon}^{t,\eta}(x+T-t-\epsilon) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T[ \\ \eta(0) + \sigma(W_T(x) - W_t) & x \in [t-T, t-T+\epsilon]\ . \end{cases}$$

Finally we obtain an explicit expression for (4.27); indeed (4.28) gives

$$Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,\gamma}(x) = \begin{cases} \eta(x+T-t-\epsilon) - \eta(x+T-t) & x \in [-T, t-T[ \\ \eta(x+T-t-\epsilon) - \eta(0) - \sigma(W_T(x) + W_t) & x \in [t-T, t-T+\epsilon[ \\ \sigma(W_t - W_{t+\epsilon}) & x \in [t-T+\epsilon, 0]\ . \end{cases} \tag{4.29}$$

Consequently, using (4.25), (4.26) and (4.29), the quotient (4.24) appears to be the sum of four terms.

$$\frac{u(t+\epsilon,\eta) - u(t,\eta)}{\epsilon} = \mathbb{E}\left[\frac{H\big(Y_T^{t+\epsilon,\eta}\big) - H\big(Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\big)}{\epsilon}\right] = I_1(\epsilon,t,\eta) + I_2(\epsilon,t,\eta) + I_3(\epsilon,t,\eta) + \frac{1}{\epsilon}\mathbb{E}\left[S(\epsilon,t,\eta)\right], \tag{4.30}$$

where

$$\begin{aligned} I_1(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-T}^{t-T} D_x H\left(Y_T^{t,\eta}\right) \frac{\eta(x+T-t-\epsilon) - \eta(x+T-t)}{\epsilon} dx\right] = \\ &= -\mathbb{E}\left[\int_{-t}^{0} D_{x-T+t} H\left(Y_T^{t,\eta}\right) \frac{\eta(x) - \eta(x-\epsilon)}{\epsilon} dx\right] \\ I_2(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_x H\left(Y_T^{t,\eta}\right) \frac{\eta(x+T-t-\epsilon) - \eta(0) - \sigma(W_T(x) + W_t)}{\epsilon} dx\right] \\ &- \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_x H\left(Y_T^{t,\eta}\right) \frac{W_t - W_{t+\epsilon}}{\epsilon} dx\right] \\ &= \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_x H\left(Y_T^{t,\eta}\right) \frac{\eta(x+T-t-\epsilon) - \eta(0) - \sigma(W_T(x) + W_{t+\epsilon})}{\epsilon} dx\right] \\ I_3(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{t-T}^{0} D_x H\left(Y_T^{t,\eta}\right) \frac{\sigma(W_t - W_{t+\epsilon})}{\epsilon} dx\right] \end{aligned}$$

and $\frac{1}{\epsilon}\mathbb{E}\left[S(\epsilon,t,\eta)\right]$ is equal to

$$\frac{1}{\epsilon}\int_0^1 \mathbb{E}\left[\int_{-T}^{0} \left(D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\right) - D_x H\left(Y_T^{t,\eta}\right)\right)\left(Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}(x)\right) dx\right] d\alpha\ .$$

$$(4.31)$$

• We will prove that

$$I_1(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} I_1(t,\eta) := I_{11}(t,\eta) + I_{12}(t,\eta) + I_{13}(t,\eta), \tag{4.32}$$

where

$$\begin{aligned} I_{11}(t,\eta) &= \mathbb{E}\left[D_{-T}H\left(Y_T^{t,\eta}\right)\eta(-t)\right] \\ I_{12}(t,\eta) &= \mathbb{E}\left[\int_{-t}^{0} D'_{x-T+t}H\left(Y_T^{t,\eta}\right)\eta(x)dx\right] \\ I_{13}(t,\eta) &= -\mathbb{E}\left[D_{t-T}H\left(Y_T^{t,\eta}\right)\eta(0)\right] \ . \end{aligned}$$

Admitting (4.32), the additivity and Remark 4.14 imply (4.22)

$$I_1(t,\eta) = -\mathbb{E}\left[\int_{]-t,0]} D_{x-T+t}H\left(Y_T^{t,\eta}\right)d^-\eta(x)\right].$$

It remains to show (4.32). In fact $I_1(\epsilon,t,\eta)$ can be rewritten as sum of the three terms

$$\begin{aligned} I_{11}(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-t}^{-t+\epsilon} D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\right] \\ I_{12}(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-t}^{0} \frac{D_{x+\epsilon-T+t}H\left(Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\right] \\ I_{13}(\epsilon,t,\eta) &= -\mathbb{E}\left[\int_{0}^{\epsilon} D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\right] \ . \end{aligned}$$

We can apply dominated convergence theorem. Since $\bar{W}$, $\sup_{x\leq T}|\bar{W}_x|$ has all moments and taking into account (4.15) in Remark 4.12, we get $I_{1i}(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} I_{1i}(t,\eta)$ for $i=1,2,3$ holds.

• $I_2(\epsilon,t,\eta)$ converges to zero when $\epsilon\to 0$. Indeed, Cauchy-Schwarz inequality yields

$$|I_2(\epsilon,t,\eta)|^2 \leq \frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_xH\left(Y_T^{t,\eta}\right)^2dx\right]\frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon}\left(\eta(x+T-t-\epsilon)-\eta(0)-\sigma(W_T(x)+W_{t+\epsilon})\right)^2dx\right] .$$

Again, by usual arguments and again because $\sup_{x\leq T}|\bar{W}_x|$ has all moments and taking into account (4.15) in Remark 4.12, it follows that the first integral converges to $\mathbb{E}\left[D_{t-T}H\left(Y_T^{t,\eta}\right)^2\right]$ and the second integral to zero.

• As third step we prove that

$$I_3(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} -\sigma^2\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right), \mathbb{1}_{]t-T,0]}{\otimes^2}\rangle\right] =: I_3(t,\eta) \ . \tag{4.33}$$

For this, we rewrite $I_3(\epsilon,t,\eta)$ using (A.1), i.e. $W_{t+\epsilon}-W_t=\overline{W}_\epsilon$ and the Skorohod integral to obtain

$$
\begin{aligned}
I_3(\epsilon,t,\eta) &= -\sigma\mathbb{E}\left[\int_{t-T}^0 D_xH\left(Y_T^{t,\eta}\right)\frac{W_{t+\epsilon}-W_t}{\epsilon}dx\right] = -\frac{\sigma}{\epsilon}\mathbb{E}\left[\int_{t-T}^0 D_xH\left(Y_T^{t,\eta}\right)dx\ \cdot\ \overline{W}_\epsilon\right] = \\
&= \frac{\sigma}{\epsilon}\mathbb{E}\left[\int_{t-T}^0 D_xH\left(Y_T^{t,\eta}\right)dx\ \cdot\ \int_0^\epsilon \delta\overline{W}_r\right] \\
&= -\frac{\sigma}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot\int_0^\epsilon \delta\overline{W}_s\right], \qquad (4.34)
\end{aligned}
$$

where $\mathcal{Z}:=\langle DH(Y_T^{t,\eta}),\mathbb{1}_{]t-T,0]}\rangle$.
Denoting by the deterministic function $\mathcal{Y}:=1_{]t-T,0]}(x)$, using Proposition A.4 with $n=1$, it follows that $\mathcal{Z}=\langle DH(Y_T^{t,\eta}),\mathcal{Y}\rangle$ belongs to $\mathbb{D}^{1,2}$ and

$$
D_r^m\mathcal{Z}=\sigma\langle D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{]t-T,0]}(x)\otimes\mathbb{1}_{]r-T+t,0]}(y)\rangle. \qquad (4.35)
$$

By integration by parts on Wiener space, expression (4.35), Fubini's theorem with respect to $r$ and $y$, (4.34) gives

$$
\begin{aligned}
I_3(\epsilon,t,\eta) = -\frac{\sigma}{\epsilon}\mathbb{E}\left[\int_0^\epsilon D_r^m\mathcal{Z}\,dr\right] &= -\frac{\sigma^2}{\epsilon}\mathbb{E}\left[\int_0^\epsilon \langle D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{]r-T+t,0]}(x)\otimes\mathbb{1}_{]t-T,0]}(y)\rangle\,dr\right] \\
&= -\frac{\sigma^2}{\epsilon}\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right),\int_0^\epsilon \mathbb{1}_{]r-T+t,0]}(x)dr\otimes\mathbb{1}_{]t-T,0]}(y)\rangle\right] \\
&= -\frac{\sigma^2}{\epsilon}\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right),\int_t^{t+\epsilon} \mathbb{1}_{]z-T,0]}(x)dz\otimes\mathbb{1}_{]t-T,0]}(y)\rangle\right], \qquad (4.36)
\end{aligned}
$$

where the latter equality comes replacing $z:=r+t$ in the integral.
We observe that

$$
\int_t^{t+\epsilon}\mathbb{1}_{]z-T,0]}(x)dz=\int_t^{t+\epsilon}\mathbb{1}_{[0,x+T[}(z)dz=\begin{cases}\int_t^{t+\epsilon}0dz=0 & x\le t-T \ \Leftrightarrow\ x+T\le t\\ \int_t^{t+\epsilon}\mathbb{1}_{[0,x+T[}(z)dz=x-t & x\in]t-T,t-T+\epsilon]\ \Leftrightarrow\ x+T\in]t,t+\epsilon]\\ \int_t^{t+\epsilon}1dz=\epsilon & x\in]t-T+\epsilon,0]\ \Leftrightarrow\ x+T\in]t+\epsilon,T],\end{cases} \qquad (4.37)
$$

so

$$
\frac{1}{\epsilon}\int_t^{t+\epsilon}\mathbb{1}_{]z-T,0]}(x)dz=\mathbb{1}_{]t-T+\epsilon,0]}(x)+\frac{(x-t)}{\epsilon}\mathbb{1}_{]t-T,t-T+\epsilon]}(x)\ .
$$

Previous expression is bounded by 1. Moreover it converges pointwise to $\mathbb{1}_{]t-T,0]}(x)$. By Remark 4.12 item 1., the fact that $D^2H$ has polynomial growth and that for any given Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments and finally the Lebesgue dominated convergence theorem we conclude that (4.36) converges to $I_3(t,\eta)$, i.e.

$$
I_3(t,\eta)=-\sigma^2\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{]t-T,0]}(x)\otimes\mathbb{1}_{]t-T,0]}(y)\rangle\right].
$$

So the convergence (4.33) is established.

- We study now the term $\frac{1}{\epsilon}\mathbb{E}\left[S(\epsilon,t,\eta)\right]$ in (4.31).

By Lemma 4.3, we get the a.s. equality $Y_T^{t,\eta} = Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$. Using (4.29) and the fact that $H \in C^3\left(C([-T,0])\right)$, (4.31) can be rewritten as the sum of the terms

$$A_1(\epsilon,t,\eta) = \int_0^1 \mathbb{E}\left[\int_{-T}^{t-T} \left(D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_x H\left(Y_T^{t,\eta}\right)\right) \cdot \right.$$
$$\left. \cdot \frac{\eta(x+T-t-\epsilon) - \eta(x+T-t)}{\epsilon} dx\right] d\alpha$$

$$A_2(\epsilon,t,\eta) = \int_0^1 \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} \left(D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_x H\left(Y_T^{t,\eta}\right)\right) \cdot \right.$$
$$\left. \cdot \frac{\eta(x+T-t-\epsilon) - \eta(0) - \sigma W_T(x) + \sigma W_{t+\epsilon}}{\epsilon} dx\right] d\alpha$$

$$A_3(\epsilon,t,\eta) = A_{31}(\epsilon,t,\eta) + A_{32}(\epsilon,t,\eta) + A_{33}(\epsilon,t,\eta) + A_{34}(\epsilon,t,\eta)\ ,$$

where

$$A_{31}(\epsilon,t,\eta) = \frac{\sigma^2}{2}\mathbb{E}\left[\left\langle D^2 H\left(Y_T^{t,\eta}\right), \mathbb{1}_{]t-T+\epsilon,0]} \otimes \mathbb{1}_{]t-T+\epsilon,0]}\right\rangle \cdot \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\right]\ ,$$

$$A_{32}(\epsilon,t,\eta) = \sigma^2 \int_0^1 \mathbb{E}\left[\left\langle \left(D^2 H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D^2 H\left(Y_T^{t,\eta}\right)\right), \mathbb{1}_{]t-T+\epsilon,0]^2}\right\rangle \cdot \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\right] d\alpha\,,$$

$$A_{33}(\epsilon,t,\eta) = \sigma \int_0^1 \mathbb{E}\left[\left\langle \left(D^2 H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D^2 H\left(Y_T^{t,\eta}\right)\right), \right.\right.$$
$$\left.\left. \frac{\eta(y+T-t+\epsilon) - \eta(y+T-t)}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[-T,t-T]}(y)\right\rangle \cdot (W_t - W_{t+\epsilon})\right] d\alpha\,,$$

$$A_{34}(\epsilon,t,\eta) = \sigma \int_0^1 \mathbb{E}\left[\left\langle \left(D^2 H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D^2 H\left(Y_T^{t,\eta}\right)\right), \right.\right.$$
$$\left.\left. \frac{\eta(y+T-t-\epsilon) - \eta(0) - \sigma(W_T(y) + W_{t+\epsilon})}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{]t-T,t-T+\epsilon]}(y)\right\rangle \cdot (W_t - W_{t+\epsilon})\right] d\alpha\,.$$

• Similarly to $I_1(\epsilon,t,\eta)$, the term $A_1(\epsilon,t,\eta)$ can be decomposed into the sum of terms given below.

$$A_{11}(\epsilon,t,\eta)=\mathbb{E}\left[\int_0^1\int_{-t}^{-t+\epsilon} D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\,d\alpha\right],$$

$$A_{12}(\epsilon,t,\eta)=\mathbb{E}\left[\int_0^1\int_{-t}^{0}\frac{D_{x+\epsilon-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\,d\alpha\right]$$
$$-\mathbb{E}\left[\int_0^1\int_{-t}^{0}\frac{D_{x+\epsilon-T+t}H\left(Y_T^{t,\eta}\right)-D_{x-T+t}H\left(Y_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\,d\alpha\right],$$

$$A_{13}(\epsilon,t,\eta)=-\mathbb{E}\left[\int_0^1\int_{-\epsilon}^{0} D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\,d\alpha\right].$$

• We show now that $A_{11}(\epsilon,t,\eta)$ converges to zero.

By Cauchy-Schwarz inequality we have

$$[A_{11}(\epsilon,t,\eta)]^2\le\int_{-t}^{-t+\epsilon}\frac{\eta^2(x-\epsilon)}{\epsilon}dx\,\cdot$$
$$\cdot\,\mathbb{E}\left[\int_0^1\int_{-t}^{-t+\epsilon}\frac{1}{\epsilon}\left[D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_{x-T+t}H\left(Y_T^{t,\eta}\right)\right]^2dx\,d\alpha\right].$$

The integral $1/\epsilon\int_{-t}^{-t+\epsilon}\eta^2(x-\epsilon)dx$ converges to $\eta^2(-t)$ by the finite increments theorem. By hypotheses (4.14) and (4.11) we have

$$\left\|DH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-DH\left(Y_T^{t,\eta}\right)\right\|_{H^1([-T,0])}\xrightarrow[\epsilon\longrightarrow 0]{a.s.}0\,. \tag{4.38}$$

Because of (4.38), it follows that

$$\sup_{x\in[-T,0]}\left|D_xH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_xH\left(Y_T^{t,\eta}\right)\right|\xrightarrow[\epsilon\longrightarrow 0]{a.s.}0\qquad\forall\,x\in[-T,0]\,. \tag{4.39}$$

(4.39) implies that

$$\int_0^1\int_{-t}^{-t+\epsilon}\frac{1}{\epsilon}\left[D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_{x-T+t}H\left(Y_T^{t,\eta}\right)\right]^2dx\,d\alpha\xrightarrow[\epsilon\longrightarrow 0]{a.s.}0\,.$$

Using (4.15), (4.10), (4.8) and the fact that given any Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments and Lebesgue dominated convergence theorem it follows that $A_{11}(\epsilon,t,\eta)$ converges to zero.

• Using the same technique we also obtain that $A_{13}(\epsilon,t,\eta)$ converges to zero.

• We show that $A_{12}(\epsilon,t,\eta)$ converges to zero.

For every fixed continuous function $\zeta$ we can develop

$$D_{x-T+t+\epsilon}H(\zeta)-D_{x-T+t}H(\zeta)=\int_{x-T+t}^{x+\epsilon-T+t}D'_uH(\zeta)\,du\,.$$

It follows that $A_{12}(\epsilon,t,\eta)$ can be rewritten as

$$\mathbb{E}\left[\int_0^1\int_{-t}^0 \frac{1}{\epsilon}\int_{x-T+t}^{x-T+t+\epsilon}\left[D'_uH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_uH\left(Y_T^{t,\eta}\right)\right]\eta(x)\,du\,dx\,d\alpha\right] .$$

Taking the absolute value and considering the fact that $|\eta(x)|\le\|\eta\|_\infty$ we obtain

$$|A_{12}(\epsilon,t,\eta)|\le\mathbb{E}\left[\int_0^1\int_{-t}^0 \frac{1}{\epsilon}\int_{x-T+t}^{x-T+t+\epsilon}\left|D'_uH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_uH\left(Y_T^{t,\eta}\right)\right|\,du\,dx\,d\alpha\right]\|\eta\|_\infty .$$

By Fubini's theorem it follows

$$|A_{12}(\epsilon,t,\eta)|\le\mathbb{E}\left[\int_0^1\int_{-T}^{-T+t}\left|D'_uH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_uH\left(Y_T^{t,\eta}\right)\right|\,du\,d\alpha\right]\|\eta\|_\infty .$$

Now using Cauchy-Schwarz inequality we have

$$\begin{aligned}|A_{12}(\epsilon,t,\eta)|^2&\le T\,\mathbb{E}\left[\int_0^1\int_{-T}^{-T+t}\left(D'_uH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_uH\left(Y_T^{t,\eta}\right)\right)^2\,du\,d\alpha\right]\ \|\eta\|^2_\infty\le\\ &\le T\,\mathbb{E}\left[\int_0^1\left\|D'H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'H\left(Y_T^{t,\eta}\right)\right\|^2_{L^2([-T,0])}\,d\alpha\right]\ \|\eta\|^2_\infty .\end{aligned}$$

Convergence (4.38) implies in particular

$$\left\|D'H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'H\left(Y_T^{t,\eta}\right)\right\|^2_{L^2([-T,0])}\xrightarrow[\epsilon\longrightarrow0]{a.s.}0 .$$

Again using (4.15), (4.10), (4.8) the fact that given any Brownian motion $\bar W$, $\sup_{x\le T}|\bar W_x|$ has all moments and Lebesgue dominated convergence theorem we have that $A_{12}(\epsilon,t,\eta)$ converges to zero.

• This concludes the proof that $A_1(\epsilon,t,\eta)$ converges to zero.

• Term $A_2(\epsilon,t,\eta)$ also converges to zero. In fact Cauchy-Schwarz implies that

$$\begin{aligned}|A_2(\epsilon,t,\eta)|^2\le\int_0^1\frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon}\left(D_xH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_xH\left(Y_T^{t,\eta}\right)\right)^2dx\right]\cdot\\ \cdot\frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon}\left(\eta(x+T-t-\epsilon)-\eta(0)-\sigma W_T(x)+\sigma W_{t+\epsilon}\right)^2dx\right]d\alpha .\end{aligned}$$

The continuity of $DH$ (see (4.14)), the fact that it has polynomial growth in the sense of Remark 4.12.1., (4.10) and Lebesgue dominated convergence theorem imply that the first expectation converges to zero. The second expectation converges to zero by the same arguments together with the fact that $\sup_{x\le T}|\bar W_x|$ has all moments.

• We show now that $A_{31}(\epsilon,t,\eta)$ converges to

$$\frac{\sigma^2}{2}\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{]t-T,0]^2}\rangle\right]=:A_{31}(t,\eta) . \tag{4.40}$$

At this level we need two technical results.

**Lemma 4.15.** The random variable $B(\epsilon) := \frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}$ weakly converges in $L^2(\Omega)$ to 1 when $\epsilon \to 0$.

*Proof.* In fact, $\mathbb{E}\left[B(\epsilon)^2\right] = 3$, so that $(B(\epsilon))$ is bounded in $L^2(\Omega)$ . Therefore there exists a subsequence $(\epsilon_n)$ such that $(B(\epsilon_n))$ converges weakly to some square integrable variable $B_0$. In order to show that $B_0 = 1$ and to conclude the proof of the lemma it is enough to prove that

$$\mathbb{E}\left[B(\epsilon) \cdot Z\right] \longrightarrow \mathbb{E}[Z]$$

for any r.v. $Z$ of a dense subset $\mathcal{D}$ of $L^2(\Omega)$. We choose $\mathcal{D}$ and the r.v. $Z$ such that $Z = \mathbb{E}[Z] + \int_0^T \xi_s dW_s$ where $(\xi_s)_{s\in[0,T]}$ is a bounded previsible process. We have

$$\mathbb{E}\left[B(\epsilon) \cdot Z\right] = \mathbb{E}\left[B(\epsilon)\right] \mathbb{E}\left[Z\right] + \mathbb{E}\left[\frac{(W_{t+\epsilon} - W_t)^2}{\epsilon} \int_0^T \xi_s dW_s\right].$$

Since $\mathbb{E}\left[B(\epsilon)\right] \mathbb{E}\left[Z\right] = \mathbb{E}\left[Z\right]$, we only need to show that

$$\mathbb{E}\left[\frac{(W_{t+\epsilon} - W_t)^2}{\epsilon} \int_0^T \xi_s dW_s\right] \xrightarrow[\epsilon\longrightarrow 0]{} 0\ . \tag{4.41}$$

Since $\int_0^T \xi_s dW_s$ is a Skorohod integral, integration by parts on Wiener space (2.1) implies that the left-hand side of (4.41) equals

$$\mathbb{E}\left[\frac{2}{\epsilon}\int_0^T \xi_s (W_{t+\epsilon} - W_t) 1\!\!1_{[t,t+\epsilon]}(s) ds\right] = \mathbb{E}\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \xi_s ds\ (W_{t+\epsilon} - W_t)\right]\ ;$$

this converges to zero since $\xi$ is bounded. □

**Lemma 4.16.** Let $H$ be an Hilbert space equipped with a product $\langle\cdot,\cdot\rangle$. Let $(Z_n)_n$ and $(Y_n)_n$ be two sequences in $H$ such that $Z_n$ converges strongly to $Z$ and $Y_n$ converges weakly to $Y$. Then $\langle Z_n, Y_n\rangle$ converges to $\langle Z, Y\rangle$.

*Proof.* By Cauchy-Schwarz inequality we obtain

$$|\langle Z_n, Y_n\rangle - \langle Z, Y\rangle| = |\langle Z_n - Z, Y_n\rangle + \langle Z, Y_n - Y\rangle| \le \|Z_n - Z\|_H \|Y_n\|_H + |\langle Z, Y_n - Y\rangle| \xrightarrow[\epsilon\longrightarrow 0]{} 0,$$

since $\|Z_n - Z\|_H$ goes to zero by the strong convergence hypothesis of $(Y_n)$, $\|Y_n\|_H$ is bounded because weakly convergent and $\langle Z, Y_n - Y\rangle$ goes to zero by definition of weak convergence of $(Y_n)_n$ and the fact that $Z \in H$. □

In order to show the convergence of $2A_{31}(\epsilon, t, \eta) = \sigma^2\mathbb{E}\left[\mathcal{Z}(\epsilon)\cdot \frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\right]$ to $2A_{31}(t,\eta)$ we use Lemma 4.16 setting the Hilbert space $H$ equal to $L^2(\Omega)$. We only need to show the strong convergence in $H$ of $\mathcal{Z}(\epsilon)$ to $\mathcal{Z} := \langle D^2 H\left(Y_T^{t,\eta}\right), 1\!\!1_{]t-T,0]} \otimes 1\!\!1_{]t-T,0]}\rangle$. Taking into account $1\!\!1_{]t-T+\epsilon,0]}{\otimes^2} \to 1\!\!1_{]t-T,0]}{\otimes^2}$ pointwise and Lebesgue dominated convergence theorem, it is not difficult to show now that $\mathbb{E}\left[\left(\mathcal{Z}(\epsilon) - \mathcal{Z}\right)^2\right]$ converges to zero, i.e. the strong convergence in $L^2(\Omega)$. Finally by an immediate application of Lemma 4.15, the term $A_{31}(\epsilon,t,\eta)$ expressed in (4.40) converges to $\frac{\sigma^2}{2}\mathbb{E}[\mathcal{Z}]$ which equals $A_{31}(t,\eta)$.

• The term $A_{32}(\epsilon,t,\eta)$ converges to zero. In fact using $\mathbb{1}_{]t-T+\epsilon,0]^2} \le \mathbb{1}_{[t-T,0]^2}$ and then the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\langle D^2H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)-D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{]t-T+\epsilon,0]^2}\rangle\cdot\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\right]\\
&\le \mathbb{E}\left[\langle D^2H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)-D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{[t-T,0]^2}\rangle\cdot\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\right]\\
&\le \sqrt{\mathbb{E}\left[\left|\langle D^2H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)-D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{[t-T,0]^2}\rangle\right|^2\right]}\cdot\sqrt{3}\\
&\le \sqrt{\mathbb{E}\left[\left\|D^2H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)-D^2H\left(Y_T^{t,\eta}\right)\right\|^2_{\left(C([-T,0])\hat{\otimes}_\pi^2\right)^*}\cdot\left\|\mathbb{1}_{[t-T,0]^2}\right\|^2\right]}
\end{aligned}
$$

. The latter term converges to zero because $D^2H\in C^0\left(C([-T,0])\right)$ and $D^2H$ has polynomial growth as we have seen in Remark 4.12 item 1.

• We show that $A_{33}(\epsilon,t,\eta)$ converges to zero. We rewrite $A_{33}(\epsilon,t,\eta)$ as $\sigma(A_{332}(\epsilon,t,\eta)-A_{331}(\epsilon,t,\eta))$, where

$$
A_{331}(\epsilon,t,\eta)=\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right),\frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\rangle(W_{t+\epsilon}-W_t)\right]
$$

$$
\begin{aligned}
A_{332}(\epsilon,t,\eta)=\int_0^1\mathbb{E}\Bigg[&\langle D^2H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right),\\
&\frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\rangle(W_{t+\epsilon}-W_t)\Bigg]d\alpha\,.
\end{aligned}
$$

We will show that both $A_{331}(\epsilon,t,\eta)$ and $A_{332}(\epsilon,t,\eta)$ converge to zero.

Denoting

$$
\mathcal{Z}:=\langle D^2H\left(Y_T^{t,\eta}\right)\ ,\ \mathcal{Y}\rangle\ \text{ where }\ \mathcal{Y}:=\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\ , \tag{4.42}
$$

we rewrite

$$
A_{331}(\epsilon,t,\eta)=\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot(W_{t+\epsilon}-W_t)\right].
$$

Using Proposition A.4, and that $H\in C^3(C([-T,0]))$, with polynomial growth we get that $\mathcal{Z}$ belongs to $\mathbb{D}^{1,2}$ and

$$
\begin{aligned}
D^m_r\mathcal{Z}&=\sigma\langle D^3H(Y_T^{t,\eta})\ ,1_{]r-T+t,0]}\otimes\mathcal{Y}\rangle\quad+\quad\langle D^2H(Y_T^{t,\eta})\ ,\ D^m_r\mathcal{Y}\rangle\ .\\
&=\sigma\langle D^3H\left(Y_T^{t,\eta}\right)\ ,\ 1_{]r-T+t,0]}(z)\otimes\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\mathbb{1}_{[-T,t-T]}(y)\rangle,
\end{aligned}
\tag{4.43}
$$

because $D^m_r \mathcal{Y}$ is zero. Using (4.42), Skorohod integral formulation, integration by parts on Wiener space (2.1), (4.43) and successively Fubini's theorem with respect to the variables $r$ and $z$ and then integrating with respect to $r$, we obtain

$$\begin{aligned} A_{331}(\epsilon,t,\eta) &= \frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot(W_{t+\epsilon}-W_t)\right] = \frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot\overline{W}_\epsilon\right] = \frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot\int_0^\epsilon \delta\overline{W}_u\right] = \\ &= \frac{1}{\epsilon}\mathbb{E}\left[\int_0^\epsilon D^m_r\mathcal{Z}\,dr\right] = \\ &= \frac{\sigma}{\epsilon}\mathbb{E}\Big[\int_0^\epsilon \langle D^3H\left(Y^{t,\eta}_T\right)\,, \\ &\qquad \mathbb{1}_{]r-T+t,0]}(z)\otimes\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\rangle\,dr\Big] \\ &= \frac{\sigma}{\epsilon}\mathbb{E}\Big[\,\langle D^3H\left(Y^{t,\eta}_T\right)\,, \\ &\qquad \int_0^\epsilon \mathbb{1}_{]r-T+t,0]}(z)\,dr\ \otimes\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\rangle\Big]\,. \end{aligned} \tag{4.44}$$

Analyzing the term $\int_0^\epsilon \mathbb{1}_{]r-T+t,0]}(z)\,dr$ analogously to (4.36) and (4.37) we can establish the convergence of $A_{331}(\epsilon,t,\eta)$. In fact the third order Fréchet derivative of $H$, denoted by $D^3H$, is a map from $C([-T,0])$ to the dual of the triple projective tensor product of $C([-T,0])$, i.e. $\left(C([-T,0])\hat{\otimes}^3_\pi\right)^*$. We recall that, given a general Banach space $E$ equipped with its norm $\|\cdot\|_E$ and $x,y,z$ three elements of $E$, then the norm of an elementary element of the tensor product $x\otimes y\otimes z$ which belongs to $E\otimes^3$ is $\|x\|_E\cdot\|y\|_E\cdot\|z\|_E$. We remark that the trilinear form $(\phi,\varphi,\psi)\mapsto\langle D^3H\left(Y^{t,\eta}_T\right),\phi\otimes\varphi\otimes\psi\rangle$ extends from $C([-T,0])\times C([-T,0])\times C([-T,0])$ to $\phi,\varphi,\psi:[-T,0]\longrightarrow\mathbb{R}$ Borel bounded. Indeed the application is a measure in each component. Consequently

$$\begin{aligned} &|\langle D^3H\left(Y^{t,\eta}_T\right)\,,\ \mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-t,0]}(y)\left[\eta(y+\epsilon)-\eta(y)\right]\otimes\mathbb{1}_{]r-T+t,0]}(z)\rangle| \leq \\ &\leq \sup_{\|\phi\|_\infty\leq 1,\|\varphi\|_\infty\leq 1,\|\psi\|_\infty\leq 1} |\langle D^3H\left(Y^{t,\eta}_T\right),\phi\otimes\varphi\otimes\psi\rangle|\cdot\varpi_\eta(\epsilon) = \left\|D^3H\left(Y^{t,\eta}_T\right)\right\|_{\left(C([-T,0])\hat{\otimes}^3_\pi\right)^*}\cdot\varpi_\eta(\epsilon)\xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0, \end{aligned}$$

since $\varpi_\eta(\epsilon)$ is the modulus of continuity of $\eta$. By the polynomial growth of $D^3H$, (4.8), the fact that for any given Brownian motion $\bar{W}$, $\sup_{x\leq T}|\bar{W}_x|$ has all moments and finally the Lebesgue dominated convergence theorem we conclude that (4.44) converges to zero, therefore $A_{331}(\epsilon,t,\eta)$ converges to zero.
At this point we should establish the convergence to zero of $A_{332}(\epsilon,t,\eta)$. This can be done using, again as above, integration by parts on Wiener space (2.1). However there are several technicalities that we omit.

• We show finally that $A_{34}(\epsilon,t,\eta)$ converges to zero.

We rewrite term $A_{34}(\epsilon,t,\eta)$ as

$$\begin{aligned} A_{34}(\epsilon,t,\eta) = \sigma\int_0^1 \mathbb{E}\Big[\langle\left(D^2H\left(\alpha Y^{t+\epsilon,\eta}_T+(1-\alpha)Y^{t,\eta}_T\right)-D^2H\left(Y^{t,\eta}_T\right)\right), \\ \frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\rangle\cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha \end{aligned}$$

as $A_{34}(\epsilon,t,\eta)=\sigma(A_{341}(\epsilon,t,\eta)-A_{342}(\epsilon,t,\eta))$ where

$$A_{341}(\epsilon,t,\eta)=\int_0^1 \mathbb{E}\Big[\langle D^2H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right), \\ \frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\rangle\cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha\ ,$$

$$A_{342}(\epsilon,t,\eta)=\int_0^1 \mathbb{E}\Big[\langle D^2H\left(Y_T^{t,\eta}\right), \\ \frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\rangle\cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha= \\ =\mathbb{E}\Big[\langle D^2H\left(Y_T^{t,\eta}\right), \frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\rangle \\ \cdot(W_t-W_{t+\epsilon})\Big]\,.$$

Firstly we show that $A_{342}$ converges to zero. It holds in fact

$$A_{342}(\epsilon,t,\eta)=\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot(W_t-W_{t+\epsilon})\right]=\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot\overline{W}_\epsilon\right]=\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot\int_0^\epsilon\delta\overline{W}_r\right],$$

where

$$\mathcal{Z}:=\langle D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\left[\eta(y+T-t-\epsilon)-\eta(0)-\sigma W_T(y)+\sigma W_{t+\epsilon}\right]\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\rangle\ .$$

Since $D^2H$ has polynomial growth and it is of class $C^1$, by Proposition A.4, $\mathcal{Z}\in\mathbb{D}^{1,2}$. Then the integration by parts on Wiener space gives

$$A_{342}(\epsilon,t,\eta)=\frac{1}{\epsilon}\,\mathbb{E}\left[\int_0^\epsilon D_r^m\mathcal{Z}\,dr\right]\ . \tag{4.45}$$

According to Proposition A.4, equation (A.4) for $n=2$ and setting

$$\mathcal{Y}:=\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\left[\eta(y+T-t-\epsilon)-\eta(0)-\sigma W_T(y)+\sigma W_{t+\epsilon}\right]\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\ ,$$

we get the following expression for the Malliavin derivative of $\mathcal{Z}$ in the Wiener space associated with $(\bar{W}_r)$, for $r\in[0,T-t]$:

$$D_r^m\mathcal{Z}=\langle D^3H\left(Y_T^{t,\eta}\right),\mathcal{Y}\otimes\mathbb{1}_{]r-T+t,0]}(z)\rangle+\langle D^2H\left(Y_T^{t,\eta}\right),D_r^m\mathcal{Y}\rangle. \tag{4.46}$$

Replacing (4.46) in (4.45) we get that $A_{342}(\epsilon,t,\eta)$ equals a sum of $A_{3421}(\epsilon,t,\eta)$ and $A_{3422}(\epsilon,t,\eta)$ with

$$A_{3421}(\epsilon,t,\eta)=\frac{1}{\epsilon}\,\mathbb{E}\left[\int_0^\epsilon\langle D^3H\left(Y_T^{t,\eta}\right),\mathcal{Y}\otimes\mathbb{1}_{]r-T+t,0]}(z)\rangle dr\right]\ ,$$

$$A_{3422}(\epsilon,t,\eta)=\frac{1}{\epsilon}\,\mathbb{E}\left[\int_0^\epsilon\langle D^2H\left(Y_T^{t,\eta}\right),D_r^m\mathcal{Y}\rangle dr\right]\ . \tag{4.47}$$

The term $A_{3421}(\epsilon,t,\eta)$ converges to zero. In fact, similarly to the method developed in detail in (4.37), we have

$$\begin{aligned} A_{3421}(\epsilon,t,\eta) &= \frac{1}{\epsilon}\,\mathbb{E}\left[\int_0^\epsilon \langle D^3 H\left(Y_T^{t,\eta}\right), \mathcal{Y} \otimes \mathbb{1}_{]z-T+t,0]}(r)\rangle \; dr\right] \\ &= \frac{1}{\epsilon}\,\mathbb{E}\left[\langle D^3 H\left(Y_T^{t,\eta}\right), \mathcal{Y} \otimes \int_0^\epsilon \mathbb{1}_{]z-T+t,0]}(r)\; dr\rangle\right] \end{aligned}$$

and

$$\frac{1}{\epsilon}\int_0^\epsilon \mathbb{1}_{]z-T+t,0]}(r)\rangle \; dr \le \frac{\epsilon \wedge (z+T-t)}{\epsilon} \xrightarrow[\epsilon\longrightarrow 0]{} 1 \; .$$

Then by polynomial growth of $D^3H$, (4.10), the usual property that given any Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments, the convergence of $\mathcal{Y}$ to zero and through the application of Lebesgue dominated convergence theorem we conclude that first term in $A_{3421}(\epsilon,t,\eta)$ converges to zero.
Concerning the term $A_{3422}(\epsilon,t,\eta)$ we firstly need to compute the Malliavin derivative of $\mathcal{Y}$:

$$\begin{aligned} D_r^m\; \mathcal{Y} &= \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes D_r^m \left[\eta(y+T-t-\epsilon) - \eta(0) - \sigma W_T(y) + \sigma W_{t+\epsilon}\right] \mathbb{1}_{]t-T,t-T+\epsilon]}(y) \\ &= \sigma \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes D_r^m \left[W_{t+\epsilon} - W_{T+y}\right] \mathbb{1}_{]t-T,t-T+\epsilon]}(y) \\ &= \sigma \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes D_r^m \left[\overline{W}_{\epsilon} - \overline{W}_{T+y-t}\right] \mathbb{1}_{]t-T,t-T+\epsilon]}(y) \\ &= \sigma \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[T+y-t,\epsilon]}(r)\cdot \mathbb{1}_{]t-T,t-T+\epsilon]}(y), \end{aligned} \tag{4.48}$$

since by usual property of Malliavin derivative $D_r^m\left[\overline{W}_\epsilon - \overline{W}_{T+y-t}\right] = \mathbb{1}_{[T+y-t,\epsilon]}(r)$. Now replacing (4.48) in (4.47) we have, similarly to the method developed in detail in (4.37),

$$A_{3422}(\epsilon,t,\eta) = \frac{1}{\epsilon}\,\mathbb{E}\left[\langle D^2 H\left(Y_T^{t,\eta}\right), \int_0^\epsilon D_r^m \mathcal{Y}\; dr\rangle\right]$$

and

$$\frac{\sigma}{\epsilon}\int_0^\epsilon D_r^m \mathcal{Y}\; dr \le \sigma\frac{\epsilon\wedge(T+y-t)}{\epsilon} \xrightarrow[\epsilon\longrightarrow 0]{} 0\; .$$

We remark that $T+y-t\in[0,\epsilon]$ since $y\in[t-T,t-T+\epsilon]$. Again by polynomial growth of $D^2H$, (4.10), by the usual property that for the Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments and applying Lebesgue dominated convergence theorem we conclude that first term in $A_{3422}(\epsilon,t,\eta)$ converges to zero. Finally $A_{342}(\epsilon,t,\eta)$ converges to zero.

By similar arguments, even though technically a little bit more involved, also $A_{341}(\epsilon,t,\eta)$ converges to zero. This finally proves that $A_{34}(\epsilon,t,\eta) \xrightarrow[\epsilon\longrightarrow 0]{} 0$.

• We are now able to express $\partial_t u : [0,T]\times C([-T,0]) \longrightarrow \mathbb{R}$. For $t\in[0,T]$, $\partial_t u(t,\eta)$ is given by the convergence of term (4.30) to a sum of three terms different form zero:

$$\partial_t u(t,\eta) = I_1(t,\eta) + I_3(t,\eta) + A_{31}(t,\eta)\; ,$$

i.e.

$$\partial_t u(t,\eta) = -\mathbb{E}\left[\int_{]-t,0]} D_{x-T+t} H\left(Y_T^{t,\eta}\right) d^-\eta(x) + \frac{\sigma^2}{2}\langle D^2 H\left(Y_T^{t,\eta}\right), \mathbb{1}_{]t-T,0]}{\otimes}^2\rangle\right]. \tag{4.49}$$

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Finally taking into account (4.23), the second order Fréchet derivative (4.21) and the time derivative (4.49) it finally follows that $u$ solves (1.1). □

# A Appendix: Malliavin and Fréchet derivatives

We need some technical results concerning link between Fréchet and Malliavin derivatives in a separable Banach space that for the moment we set to be equal to $\mathbb{R}$. We need to apply Malliavin calculus related to the Brownian motion. Let $T > 0$ and $t \in [0, T]$ be fixed. We recall that

$$\overline{W}_x := W_{t+x} - W_t, \quad x \in [0, T-t]\ . \tag{A.1}$$

So the Wiener space will be $C([0, T-t])$ with variable parameter in $[0, T-t]$ and based on $\overline{W}$. We consider the window Brownian element $\overline{W}_{T-t}(\cdot)$ with values in $C([-(T-t), 0])$, defined as

$$\overline{W}_{T-t}(x) = W_{t+T-t}(x) - W_t = W_{T+x} - W_t \quad x \in [-(T-t), 0].$$

**Lemma A.1.** Let $G : C([-(T-t), 0]) \longrightarrow \mathbb{R}$ of class $C^1$ such that $DG$ has polynomial growth. Let $\mathcal{Y} \in \mathbb{D}^\infty$. Then

$$G\left(\overline{W}_{T-t}(\cdot)\right)$$

belongs to $\mathbb{D}^{1,2}$ and

$$D_r^m\left(G(\sigma\overline{W}_{T-t}(\cdot)\ \mathcal{Y}\right) = \sigma\int_{]r-(T-t),0]} (D_{dy}G)\left(\sigma\overline{W}_{T-t}(\cdot)\right)\mathcal{Y} \quad + \quad G\left(\sigma\overline{W}_{T-t}(\cdot)\right) D_r^m\mathcal{Y} \qquad r \in [0, T-t]\ a.e. \tag{A.2}$$

*Proof.* The proof of this result needs some boring technicalities involving the approximation of a continuous function and its polygonal approximation. Formula (A.2) is stated in a particular case for instance in [8], Example 1.2.1. □

A consequence of previous lemma is the possibility of the differentiating

$$h = F\left(Y_T^{t,\eta}\right), \quad F : C([-T, 0]) \longrightarrow \mathbb{R}$$

of class $C^1$ Fréchet. We remark that

$$Y_T^{t,\eta} = G_\eta\left(\sigma \overline{W}_{T-t}(\cdot)\right),$$

where

$$G_\eta : C[-(T-t), 0] \longrightarrow C([-T,0]) \text{ given by } G_\eta(\gamma) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T[ \\ \eta(0) + \gamma(T-t+x) & x \in [t-T, 0]. \end{cases}$$

By Lemma A.1, if $\mathcal{Y} \in \mathbb{D}^\infty$,

$$D_x^m(h\mathcal{Y}) = \sigma \int_{]x-T+t,0]} D_{dy}(F \circ G_\eta)\left(\sigma \overline{W}_{T-t}(\cdot)\right)\mathcal{Y} \quad + \quad F \circ G_\eta\left(\sigma \overline{W}_{T-t}(\cdot)\right)\, D_x^m \mathcal{Y} \qquad x \in [0, T-t]. \tag{A.3}$$

**Remark A.2.** We remark that $\forall\, \gamma \in C([-T+t, 0])$

$$D\,(F \circ G_\eta) \in \mathcal{M}([-T+t, 0]).$$

We have, for $\zeta \in C([-T+t,0])$,

$$\int D_{dy}(F \circ G_\eta)(\gamma)\ \zeta(y) = \int_{[t-T,0]} D_{dy}F\left(G_\eta(\gamma)\right)\ \zeta(y).$$

So (A.3) gives, for $x \in [0, T-t]$,

$$\begin{aligned} D_x^m(h\mathcal{Y}) &= \sigma \int_{]x-T+t,0]} (D_{dy}F)\left(G_\eta(\sigma \overline{W}_{T-t})\right)\mathcal{Y} \quad + \quad (F \circ G_\eta)\left(\sigma \overline{W}_{T-t}(\cdot)\right)\ D_x^m \mathcal{Y} \\ &= \sigma \int_{]x-T+t,0]} (D_{dy}F)\left(Y_T^{t,\eta}\right)\ \mathcal{Y} \quad + \quad F\left(Y_T^{t,\eta}\right) D_x^m \mathcal{Y}. \end{aligned}$$

At this point we have proved the following.

**Proposition A.3.** Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ of class $C^1$-Fréchet with polynomial growth. Let $\mathcal{Y} \in \mathbb{D}^\infty$. Then $H(Y_T^{t,\eta})\mathcal{Y}$ belongs to $\mathbb{D}^{1,2}$ and

$$D_r^m\left(H(Y_T^{t,\eta})\ \mathcal{Y}\right) = \sigma \int_{]x-T+t,0]} (D_{dy}H)\,(Y_T^{t,\eta})\ \mathcal{Y} \quad + \quad F\left(Y_T^{t,\eta}\right) D_r^m \mathcal{Y}\ .$$

Previous proposition admits a generalization to the case when $H : C([-T,0]) \longrightarrow \mathbb{R}$ is replaced by a functional

$$C([-T,0]) \longrightarrow \underbrace{\left(C([-T,0])\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C([-T,0])\right)}_{n \text{ times}} \qquad n \geq 1.$$

Typically an example will be $D^n H$. We recall that

$$\Big(\underbrace{C([-T,0])\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C([-T,0])}_{n \text{ times}}\Big)^*$$

can be isomorphically identified with the space of $n$-multilinear continuous functionals on $C([-T,0])$. Proposition A.3 can be generalized as follows.

**Proposition A.4.** Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ of class $C^{n+1}$-Fréchet such that $D^{n+1}F$ has polynomial growth. Let $\mathcal{Y} \in \mathbb{D}^{\infty}\Big(\underbrace{C([0,T-t])\hat{\otimes}_{\pi}\cdots\hat{\otimes}_{\pi}C([0,T-t])}_{n \text{ times}}\Big)$. Then $\langle D^n H(Y_T^{t,\eta}), \mathcal{Y}\rangle$ belongs to $\mathbb{D}^{1,2}$. Moreover, for a.e. $r \in [0,T-t]$ we have

$$D_r^m\left(\langle D^n H(Y_T^{t,\eta})\ ,\ \mathcal{Y}\rangle\right) = \sigma\langle D^{n+1}H(Y_T^{t,\eta})\ , 1_{]r-T+t,0]}\otimes\mathcal{Y}\rangle \quad + \quad \langle D^n H(Y_T^{t,\eta})\ ,\ D_r^m\mathcal{Y}\rangle\ . \tag{A.4}$$

**Remark A.5.** The function $1_{]r-T+t,0]}$ can be considered as a test function $\zeta_0$. Indeed for fixed $\zeta_1,\cdots,\zeta_n \in C([-T,0])$,

$$\zeta_0 \mapsto D^{n+1}H(Y_T^{t,\eta})\,(\zeta_0\otimes\zeta_1\otimes\cdots\zeta_n)$$

is a measure.

*Proof.* Avoiding to state too abstract results, the proof of Proposition A.4 is based on a generalization of Lemma A.1 replacing the value space $\mathbb{R}$ with the separable Banach space $B$, setting $B = C([-T,0])\hat{\otimes}_{\pi}\cdots\hat{\otimes}_{\pi}C([-T,0])$. □

**Lemma A.6.** Let $B$ be a separable Banach space. Let

$$G : C([-T+t,0]) \longrightarrow B^*$$

of class $C^1$ Fréchet with polynomial growth. Let $\mathcal{Y} \in \mathbb{D}^{\infty}(B)$.
Then

$$G\left(\overline{W}_{T-t}(\cdot)\right)\ \mathcal{Y}\ \in \mathbb{D}^{1,2}(B)$$

and

$$\begin{aligned} D_r\left({}_{B^*}\langle G\left(\overline{W}_{T-t}(\cdot)\right)\ ,\ \mathcal{Y}\rangle_B\right) &= {}_{(C([-T,0])\hat{\otimes}_{\pi}B)^*}\ \langle DG\left(\overline{W}_{T-t}(\cdot)\right), 1_{]r-T+t,0]}\otimes\mathcal{Y}\rangle_{C([-T,0])\hat{\otimes}_{\pi}B} \\ &+ \quad \langle G\left(\overline{W}_{T-t}(\cdot)\right)\ ,\ D_r^m\mathcal{Y}\rangle \end{aligned}$$

**Remark A.7.**
1. $DG : C([-T+t,0]) \longrightarrow (C([-T,0])\hat{\otimes}_{\pi}B)^*$.
2. Proposition A.4 will be used for $n = 1,2,3$.
3. $D_r^m\mathcal{Y} \in B$ for almost all $r$.

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