On the numerical approximation of first order Hamilton Jacobi equations

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Abstract

We review some methods for the numerical approximation of time dependent and steady first order Hamilton Jacobi equations. Most of the discussion focuses on conformal triangular type meshes but we show how to extend this to the most general meshes. We review some first order monotone schemes and also high order ones specially dedicated to steady problems.

1 Introduction

This paper describes some of the schemes that are currently used to compute approximated solutions of first order Hamilton Jacobi equations, namely the steady Dirichlet problem

\[ H(x, u, Du) = 0 \quad x \in \Omega \subset \mathbb{R}^d \]
\[ u = \varphi \quad x \in \partial \Omega \]

or the Cauchy problem

\[ \frac{\partial u}{\partial t} + H(x, u(x), Du) = 0 \quad x \in \mathbb{R}^d, t > 0 \]
\[ u(x, 0) = u_0(x). \]

In many problems of physical interest, one needs to compute the solution of such an equation. One of the simplest examples is the computation of a distance function, more sophisticated examples consist in evaluating the arrival time of a front (wave front, flame front, etc) in a non homogeneous media. Similar problems also exist in control theory, thermodynamics, etc.

We will concentrate on the numerical approximation of these equations on conformal triangular type meshes. This is a more general situation than the standard Cartesian meshes where this problem was considered originally, but still less general than the case of non conformal meshes. However, we sketch how to extend the schemes we describe here to the most general case. Indeed the situation we consider is an intermediate one, it is general enough to be obliged to imagine solutions that are generic enough and not too specific so that the structure of the mesh does not play a too rigid role.

In the first part, we quickly recall the notion of viscosity solution for HJ equations, either for the Cauchy problem or for the steady one with Dirichlet boundary conditions. Then, in two particular cases we recall the exact solution. The next section is devoted to the numerical approximation of the Cauchy problem. In the third section we discuss the approximation of the Dirichlet problem. The fourth section considers a general formulation for high order discretisation. The bridge between Cartesian meshes and non conformal meshes is sketched in the fifth section. The last section is devoted to some numerical applications.

As we have already said, our point of view is quite biased. There are basically two classes of approximation techniques. The first one tries to directly use the notion of viscosity solution, see section 2, this our point
of view. In the second class of methods, one tries to exploit the formal link between some systems of conservation laws and the HJ equations. The link is that if one differentiates the equation

$$\frac{\partial u}{\partial t} + H(Du) = 0$$

with respect to $x$ and $y$, denoting $p_i = \frac{\partial u}{\partial x_i}$ and $p = Du$, we have

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial x}(p) = 0.$$

This is the point of view of the papers that extend finite volume or Discontinuous Galerkin methods, see [1] for an example.

2 Short review on the HJ equations and viscosity solutions

We consider the Cauchy problem: find $u \in C^0(\Omega)$, the space of continuous function on the open subset $\Omega \subset \mathbb{R}^d$, such that

$$H(x, u, Du) = 0 \quad x \in \Omega \subset \mathbb{R}^d$$
$$u = g \quad x \in \partial\Omega$$

in the viscosity sense. In (1), $(x, s, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto H(x, s, p)$ is uniformly continuous.

Before going further, let us briefly review the notion of viscosity solution for (1). For any function $z$, we consider the upper semi–continuous (u.s.c) and lower semi–continuous (l.s.c) envelopes of $z$ with respect to all the variables. They are defined by

$$z^*(x) = \limsup_{x \to y} z(y)$$
$$z_*(x) = \liminf_{x \to y} z(y).$$

Following [2], we introduce the function $G$

$$G(x, s, p) = \begin{cases} H(x, s, p) & x \in \Omega \\ s - g(x) & x \in \partial\Omega. \end{cases}$$

The computation of $G_*$ and $G^*$ is easy and we have:

$$\begin{cases} G_*(x, s, p) = G^*(x, s, p) = H(x, s, p) & \text{if } x \in \Omega \\ G_*(x, s, p) = \min(H(x, s, p), s - g(x)) & \text{if } x \in \partial\Omega \\ G^*(x, s, p) = \max(H(x, s, p), s - g(x)) & \text{if } x \in \partial\Omega \end{cases}$$

(2)

A locally bounded u.s.c function $u$ defined on $\overline{\Omega}$ is a viscosity sub-solution of (1) if and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local maximum of $u - \phi$, then

$$G_*(x_0, u(x_0), D\phi(x_0)) \leq 0.$$  

(3)

Similarly, $u$, a locally bounded, l.s.c. function defined on $\overline{\Omega}$ is a viscosity super-solution of (1) if and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local minimum of $u - \phi$, then

$$G^*(x_0, u(x_0), D\phi(x_0)) \geq 0.$$  

(4)

A viscosity solution is simultaneously a sub– and a super-solution of (1). This can be generalized to other types of boundary conditions such as Neumann, etc.
In the case of the Cauchy problem,
\[
\frac{\partial u}{\partial t} + H(x, u(x), Du) = 0 \quad x \in \mathbb{R}^d, t > 0
\]
where \( u_0 \) belongs to the set of bounded and uniformly continuous functions, \( BUC(\mathbb{R}^2) \). One can adapt easily the arguments raised for the steady problem. Here, \( G \) is simply
\[
G(x, s, p) = pt + H(x, s, px), x \in \mathbb{R}^d, s \in \mathbb{R}^d, p = (pt, px) \in \mathbb{R} \times \mathbb{R}^d
\]
so that \( G_\ast = G^\ast = G \). Subsolutions (resp. supersolutions) are elements of \( BUC(\mathbb{R}^d \times [0, T]) \) where \( T > 0 \), so that inequality (3) (resp. (4)) holds.

All this can be extended to the Cauchy–Dirichlet problem
\[
\frac{\partial u}{\partial t} + H(x, u(x), Du) = 0 \quad x \in \Omega \subset \mathbb{R}^d, t > 0 \]
\[
u(x, 0) = u_0(x) \quad x \in \Omega \]
\[
u(x, t) = g(x, t) \quad x \in \partial \Omega, t > 0
\]
under standard assumptions on the open subset \( \Omega \), \( g \) and \( H \) and \( u_0 \), one can prove existence and uniqueness of the viscosity solutions of (1), (5) and (6), see [2]. In particular, this is true if the Hamiltonian \( H \) is convex in \( p \in \mathbb{R}^d \) and if \( \partial \Omega \) Lipschitz continuous.

In this paper, we assume that (1) has a uniqueness principle, that is any sub–solution \( u \) and any super–solution \( v \) of (1) satisfy
\[
\forall x \in \Omega, \quad u(x) \leq v(x)
\]
and
\[
\forall x \in \mathbb{R}^d, t > 0, \quad u(x, t) \leq v(x, t)
\]
in the case of the Cauchy problem.

### 3 Some exact solutions

Two examples are considered. Either we look for the steady problem (1) with a convex Hamiltonian, or we look for the Cauchy problem (5) with either a convex (or concave Hamiltonian) or a convex (or concave) initial condition.

The main tool is the Legendre transform. If \( f \) is a convex function such that
\[
\lim_{||x|| \to +\infty} \frac{f(x)}{||x||} = +\infty
\]
we define the Legendre transform of \( f \) by
\[
f^\ast(p) = \sup_{y \in \mathbb{R}^d} (p \cdot y - f(y)).
\]
If the supremum is reached at \( y^\ast \), we have the relation
\[
f^\ast(p) + f(y^\ast) = p \cdot y^\ast.
\]
This shows that \( f^\ast(p) \) can be seen as the abscissa of the tangent of the graph of \( f \) at \( y^\ast \). This graphic interpretation helps to see that, if \( f \) is regular enough, the graph of \( f \) is the envelope of its tangent, so that
\[
(f^\ast)^\ast = f.
\]
Of course this relation is generalizable to a convex \( f \) when it satisfies (9).

All this generalizes to a concave functions (since \( -f \) is convex),
\[
f^\ast(p) = -(-f)^\ast(p) = \inf_{y \in \mathbb{R}^d} (-y \cdot p - f(y)).
\]
3.1 For the steady problem

We assume that the Hamiltonian is given by

\[ H(x, u, p) = \sup_{v \in V} \{-b(x, v) \cdot p + \lambda u - f(x, v)\} \]

where the space of controls \( V \) is compact, and we have standard assumptions on \( b, f \) and \( \lambda > 0 \), see [2]. For the Dirichlet condition, the solution of (1) is given by the dynamical programing principle, for any \( T > 0 \),

\[
\begin{align*}
    u(x) &= \inf_{v(\cdot)} \left[ \int_{0}^{\min(T, \tau)} f(y_x(t), v(t)) e^{-\lambda t} dt \right. \\
    &\left. + 1_{\{T<\tau\}} u(y_x(T)) e^{-\lambda T} + 1_{\{T\geq\tau\}} \varphi(y_x(\tau)) e^{-\lambda \tau} \right]
\end{align*}
\]

The trajectory \( y_x(. \) satisfies \( y_x(0) = x \in \Omega \) and

\[
\frac{d}{dt} y_x(t) = b(y_x(t), v(t)) \text{ for } t > 0.
\]

They are defined if \( f \) is regular enough, say Lipschitz continuous. The exit time \( \tau \) is

\[
\tau = \inf\{t \geq 0, y_x(t) \notin \Omega\}.
\]

Details can be found in [3, 2]

3.2 For the Cauchy problem

The analytical expression for the solution of (5), when \( H \) only depends on \( p \in \mathbb{R}^d \), is given in [4],

1. when \( H \) is uniformly Lipschitz continuous and \( u_0 \) convex,

\[
u(x, t) = \sup_{p \in \mathbb{R}^d} \left[ x \cdot p - u_0^*(p) - tH(p) \right],
\]

when \( u_0 \) is concave,

\[
u(x, t) = \inf_{p \in \mathbb{R}^d} \left[ -x \cdot p + u_0^*(p) - tH(p) \right].
\]

2. when \( u_0 \) is uniformly continuous, we have for a convex Hamiltonian

\[
u(x, t) = \inf_{p \in \mathbb{R}^d} \left[ u_0(y) + tH^*\left(\frac{y - x}{t}\right) \right],
\]

and for a concave Hamiltonian

\[
u(x, t) = \sup_{p \in \mathbb{R}^d} \left[ u_0(y) + tH^*\left(\frac{x - y}{t}\right) \right].
\]

The formula (11) and (12) reflect the Huygens’s principle, while (13) and (14) are consequence of the dynamical programing principle (10).

Note that if \( u_0 \) is linear in \( x \), \( u_0(x) = A + p \cdot x \), we have

\[ u(x, t) = u_0(x) - tH(p). \]

These results are only valid for special initial conditions or particular Hamiltonian. We have the more general results
Lemma 1 (Bardi-Osher[5]). If \( u_0 = u_0^{\text{conv}} + u_0^{\text{conc}} \) where \( u_0^{\text{conv}} \) (resp. \( u_0^{\text{conc}} \)) is convex (resp. concave), then the solution \( u \) of (5) satisfies
\[
\forall (x,t) \in \mathbb{R}^d \times [0,T], \quad \psi_2(x,t) \leq u(x,t) \leq \psi_1(x,t)
\]
with
\[
\psi_1(x,t) = \inf_{q \in \mathbb{R}^d} \sup_{p \in \mathbb{R}^d} \left( x \cdot p - \left( u_0^{\text{conv}} \right)^*(p) - \left( u_0^{\text{conc}} \right)^*(q) - tH(p-q) \right)
\]
and
\[
\psi_2(x,t) = \sup_{p \in \mathbb{R}^d} \inf_{q \in \mathbb{R}^d} \left( -x \cdot q - \left( u_0^{\text{conv}} \right)^*(p) - \left( u_0^{\text{conc}} \right)^*(q) - tH(q-p) \right)
\]
Proof. The proof uses the fact that \( u_0^{\text{conc}}(x) = \sup_{p \in \mathbb{R}^d} \left( x \cdot p - \left( u_0^{\text{conv}} \right)^*(p) \right) \) so that for any \( p \),
\[
u_0^{\text{conc}}(x) \geq v_{p,0}(x) := x \cdot p - \left( u_0^{\text{conv}} \right)^*(p).
\]
Then we solve the Cauchy problem for \( v_0 + u_0 \) which is convex, use the comparison principle (8), and take the maximum. This gives the first inequality. The second one is obtained in a similar way. \( \square \)

Lemma 2 ([6]). If \( H = H_{\text{conv}} + H_{\text{conc}} \) where \( H_{\text{conv}} \) (resp. \( H_{\text{conc}} \)) is convex (resp. concave) uniformly continuous, the solution of the Cauchy problem (5) satisfies
\[
\Phi_2(x,t) \leq u(x,t) \leq \Phi_1(x,t)
\]
with
\[
\Phi_1(x,t) = \inf_{q \in \mathbb{R}^d} \sup_{y \in \mathbb{R}^d} \left[ u_0(y) + tH_{\text{conv}}^*(q) + tH_{\text{conc}}^* \left( \frac{y-x}{t} + q \right) \right]
\]
\[
\Phi_2(x,t) = \sup_{p \in \mathbb{R}^d} \inf_{q \in \mathbb{R}^d} \left[ u_0(y) + tH_{\text{conv}}^*(q) + tH_{\text{conc}}^* \left( \frac{x-y}{t} + q \right) \right]
\]
The proof is similar and is given in [6].

4 First order approximation of the Cauchy problem

In order to simplify the text, we assume from now on \( d = 2 \), but all the results can be easily generalized to other dimensions, in particular \( d = 3 \). We consider a triangulation of \( \mathbb{R}^2 \), the vertices are \( \{M_i\}_{i=1,n} \), the triangles are \( \{T_j\}_{j=1,m} \). We denote by \( T \) a generic triangle. The vertices of \( T_k \) are \( M_{i_1} \), \( M_{i_2} \) and \( M_{i_3} \), for simplicity we often denote them by \( i_1 \), \( i_2 \), \( i_3 \) or by \( 1, 2, 3 \) when there is no ambiguity. The family of triangulations we consider is shape regular.

Up to our knowledge, the first paper to discuss in detail the approximation of (5) is [7]. As in this reference, (5) is approximated by
\[
u_i^{n+1} = \nu_i^n - \Delta t \mathcal{H}_i, \quad i = 1, \ldots, n, \quad n \in \mathbb{N}^*
\]
\[
u_i^0 = u_0(M_i)
\]
where \( \Delta t > 0 \) is the time step and \( \nu_i^n \) is an approximation of \( u(M_i, n\Delta t) \), and the numerical Hamiltonian \( \mathcal{H}_i \) depends on \( \nu_i^n \), the values of \( \nu_i^n \) where \( j \in \mathcal{V}_i \) (\( \mathcal{V}_i \) is the set of neighbors of \( M_i \) including \( M_i \) by convention), and if necessary on \( M_i \),
\[
\mathcal{H}_i := \mathcal{H}(M_i, \nu_i^n, \{u_j^n\}_{j \in \mathcal{V}_i}).
\]
\( \mathcal{H}_i \) is introduced the notion of consistency. The numerical Hamiltonian \( \mathcal{H}_i \) is consistent when, if \( \nu_i = A + p \cdot \delta M_i \), then what ever \( M_i \) and \( s \in \mathbb{R} \),
\[
\mathcal{H}(M_i, s, \nu_i) = H(M_i, s, p).
\]
(17)
A less restrictive definition, which is helpful for the proof, is given in [8].
Definition 1. We say that the Hamiltonian $\mathcal{H}$ is weakly consistent if for all $x \in \overline{\Omega}$ and $\phi \in C^\infty_c(\overline{\Omega})$ (the set of $C^\infty$ bounded functions),

$$\limsup_{h \to 0, y \to x, \xi \to 0} \mathcal{H}(y, \phi(y) + \xi, \phi + \xi) \leq G^*(x, \phi(x), D\phi(x)) \quad (18)$$

and

$$\liminf_{\rho \to 0, y \to x, \xi \to 0} \mathcal{H}(y, \phi(y) + \xi, \phi + \xi) \geq G_*(x, \phi(x), D\phi(x)). \quad (19)$$

A scheme that satisfies (17), $\mathcal{H}$ is strongly consistent. A strongly consistent scheme is weakly consistent.

The structure of the solution of (5) was first used in [5, 1]. In particular, they used the results of Lemma 1 to define a Godunov–like scheme. In [6], it is shown that, in general, for non structured meshes, the generalization of the Godunov–like scheme of [5] leads to a non consistant Hamiltonian. However, Lemma 2 provides a solution.

Assume that $\{u^n_j\}_{j=1, \ldots, n_i}$ is known and denote by $u^n_i$ the piecewise linear interpolation of this data. For any mesh point $M_i$, we consider set $\{\Omega_i\}_{i=1, \ldots, \omega_i}$ of angular sectors at $M_i$, see figure 1. Each angular sector $\Omega_i$ corresponds to one of the triangles that share $M_i$ and we denote by $U_i$ the gradient of $u^n_i$ in that triangle.

![Figure 1: Illustration of the angular sectors $\Omega_i$, $\theta_i$ and the vectors $\vec{n}_{i+1/2}$ that are needed in numerical Hamiltonians definitions.](image)

The functions $\Phi_1$ and $\Phi_2$ evaluated at $x = M_i$ are

$$\Phi_1(M_i, \Delta t) = u^n_i - \Delta t \min_{q \in \mathbb{R}^2} \max_{1 \leq k \leq \omega_i} \sup_{z \in \Omega_k} \left( U_i \cdot z - H_1^*(z - q) - H_2^*(q) \right)$$

$$\Phi_2(M_i, \Delta t) = u^n_i - \Delta t \max_{q \in \mathbb{R}^2} \min_{1 \leq k \leq \omega_i} \inf_{z \in \Omega_k} \left( U_i \cdot z - H_1^*(z - q) - H_2^*(q) \right)$$

The key remark is that any of the terms multiplied by $\Delta t$, say $\min \max \sup$ reduces to $H(p)$ when $U_i \equiv p \forall i$. Hence, any of these terms defines a consistant numerical Hamiltonian, for example,

$$\mathcal{H}_i := \max \min \inf_{q \in \mathbb{R}^2} \left( U_i \cdot z - H_1^*(z - q) - H_2^*(q) \right), \quad (20)$$

the dependency in $u^n_j$, $j \in \mathcal{V}_i$ appears in the gradients $U_i$. This formula can easily be extended to the more general case $H = H(x, u, Du)$ and simplifies when $H$ is convex, for example

$$\mathcal{H}_i = \max_{1 \leq k \leq \omega_i} \max_{z \in \Omega_k} \left( U_i \cdot z - H^*(z) \right). \quad (21)$$
A second remark is that, by construction, \( \mathcal{H}_t \) defined by (20) or (21) is monotone, that is

**Definition 2 (Monotone Hamiltonians).** We say that \( \mathcal{H} \) is monotone if, whatever \( M_i \in \Sigma, u_j \leq v_j \), and for any \( s \in \mathbb{R} \),

\[
\mathcal{H}(M_i, s, \{u_j\}_{j \in V_i}) \geq \mathcal{H}(M_i, s, \{v_j\}_{j \in V_i}).
\]

The Hamiltonian (20) is monotone by construction if \( \Delta t/h \max_{\|Du_h\|_\infty \leq L \|D_pH(p)\|_\infty \leq 1/2} \) : this is a consequence of (8) and the inequalities

Another key remark is that the value of \( \mathcal{H}_t \) defined by (20) or (21) does not depend on the structure of the mesh, but on the interpolant \( u^n \). In other words, if one splits an angular sector \( \Omega_k \) in two, without changing the value of \( U_k \), the numerical Hamiltonian is not modified. We say that the scheme is intrinsic and we have the following error estimate,

**Theorem 1 ([6]).** Let \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) be continuous and \( u_0 \in BUC(\mathbb{R}^2) \) and Lipschitz continuous (with constant \( L \)). Let \( T \) be a triangulation where \( h \) is the largest radius of the circles of center \( M_i, i = 1, \cdots, n_s \) and contained in all the triangles having \( M_i \) as a vertex. We assume the mesh shape regular, i.e. the minimum angle \( \alpha \) of the triangles \( T \) is uniformly bounded from below.

Let \( u \) be the viscosity solution of (5) and \( \{u^n_i\}_{j=1,\cdots,n_s} \) be defined by (15). Then there exists a constant \( c \) which depend only on \( \alpha, L, T > 0 \) and \( \mathcal{H} \) such that for any \( M_i \) and \( n \) with \( 0 \leq n \Delta t \leq T \),

\[
\left| u^n_i - u(M_i, n\Delta t) \right| \leq c\sqrt{\Delta t}
\]

The proof is an adaptation of the main result of [7] with some technicalities (in particular for showing that the time step only depend on \( u_0 \) specific to unstructured meshes. We refer to [6].

The practical evaluation of the Legendre transform is not always an easy task, so other numerical Hamiltonians exists. The simplest one is the Lax Friedrichs one, which is inspired by the Lax Friedrichs scheme for conservation laws. It has several versions. The first one is

\[
\mathcal{H}_t^{LF}(Du_{h_i}[\Omega_i], \cdots, Du_{h_i}[\Omega_{k_s}]) = H(\bar{U}) - \frac{\epsilon}{h} \oint_{C_h} [u_h(M) - u_h(M_i)]dl,
\]

where \( C_h \) (resp. \( D_h \)) is a circle (disk) of center \( M_i \) and radius \( h \),

\[
\bar{U} = \int_{D_h} Du_h \frac{dx dy}{\pi h^2},
\]

and \( \epsilon \) is larger than any Lipschitz constant of \( H \) divided by \( 2\pi \). This defines a monotone scheme provided that \( \Delta t/h \leq \frac{\epsilon}{2\pi} \).

A different version of the Lax Friedrichs Hamiltonian, that is monotone under the same constraint, is the following :

\[
\mathcal{H}_h^{LF}(Du_{h_i}[\Omega_i], \cdots, Du_{h_i}[\Omega_{k_s}]) = \int_{D_h} \frac{H(Du_h)}{\pi h^2} - \frac{\epsilon}{h} \oint_{C_h} [u_h(M) - u_h(M_i)]dl.
\]

This version can be rewritten as

\[
\mathcal{H}_h^{LF}(Du_{h_i}[\Omega_i], \cdots, Du_{h_i}[\Omega_{k_s}]) = \sum_{0 \leq l \leq k_i} \frac{\theta_i}{2\pi} H(Du_{h_i}[\Omega_i]) + \epsilon \sum_{0 \leq l \leq k_i} \tan \frac{\bar{n}_i}{2} \frac{n_i^{l-1/2} + n_i^{l+1/2}}{2} \cdot Du_{h_i}[\Omega_i].
\]

The vector \( \bar{n}_i^{l+1/2} \) is the unit vector of the edge that separates the angular sectors \( \Omega_l \) and \( \Omega_{l+1} \), the angle \( \theta_i \) is the angle of the angular sector at \( M_i \), see Figure 1. The parameter \( \epsilon \) is the same as in the previous version.
A third version is
\[ \mathcal{H}^{LF}(Du_{h|T_i}, \ldots, Du_{h|T_k}) = \sum_{T \supset M_i} |T| H(Du_{h|T}) + \alpha \sum_{M_j \in T} (u_i - u_j) \sum_{T \ni M_i} \frac{|T|}{\sum_{T \ni M_i} |T|} \]
and \( \alpha \geq h_T \max_p ||D_pH|| \) where \( h_T \) is the largest edge of \( T \).

The main difference between these different formulas is that (22a) and (22b) are intrinsic in the sense given in [6] while (22c) is not. Hence, following the same reference, (22a) and (22b) are convergent and the error estimate is \( O(h^{1/2}) \). For (22c), such an error estimate is not available (at least when following the technique of [6], but it is convergent: this is a simple application of [8].

The advantage of (22c) over the other two versions is its simplicity in coding compared to (22a) and (22b).

5 The Dirichlet problem

The approximation of the Dirichlet problem is not as simple as it looks. An illustration is to find \( u : [0, 1] \to \mathbb{R} \) such that
\[ |u'|-1=0 \text{ in } x \in [0, 1], \quad u(0)=1 \text{ and } u(1)=2 \]
which has no classical solution, but which viscosity solution, defined only in \([0, 1]\) is \( u(x)=x \). We have \( \lim_{x \to 1^-} u(x) = 1 \neq 2 \). In other cases, say \( u(0)=u(1)=0 \), we have \( u(x)=|x-1/2| \) which matches strongly the boundary conditions.

In order to define a scheme, we start from (10), and consider a triangulation of \( \Omega \). First we assume that \( M_i \in \partial \Omega \). In (10), the set of controls can be splitted into two parts: the set \( V_1 \) for which \( T < \tau \), and \( V_2 \) for which \( T \geq \tau \). Hence,
\[ u(x) = \min \{ \inf_{v \in V_1} \cdots, \inf_{v \in V_2} \cdots \}. \]

Let \( \vec{n} \) be the interior normal to \( \Omega \) at \( x \in \Omega \). Since \( T \) is arbitrary, it can be chosen as small as possible. In the limit \( T \to 0 \), the set \( V_1 \) would be the set of controls for which \( b(x,v). \vec{n} > 0 \), i.e. the control for which the trajectory goes into \( \Omega \). The dynamical programing principle \( \inf_{v \in V_1} \cdots - u(x) = 0 \) corresponds to the Hamiltonian
\[ H_b(x,t,p) = \sup_{v \in V_1} \{ b(x,v).p + \lambda t - f(x,v) \}. \]

We also have the relation \( H_b \leq H \).

The "inf" on \( V_2 \) can be approximated, if \( T \) is small, by \( \varphi(y_x(\tau)) \). Since \( T \leq \tau \), and if we can choose controls for which \( T \approx \tau \), we get
\[ \varphi(y_x(\tau)) \simeq \varphi(x) \]
because \( \varphi \) is continuous. We see that (10) can be approximated, at a boundary point, by
\[ 0 = \max(\mathcal{H}^b_i, u(x)-\varphi(x)) \]
where \( \mathcal{H}^b \) is a consistent approximation of \( H_b \).

When \( M_i \notin \partial \Omega \), taking \( T \) small enough, we can see formally that the boundary plays no role so that we can take any consistent Hamiltonian, for example those defined in the previous section.

The scheme is then
\[ S(M_i, u_i, \{u_j\}_{j \in V_i}) = 0 \quad \forall i \] (23)
with
\[ S(x,s, \{u_j\}_{j \in V_i}) = \begin{cases} \mathcal{H}(x,s, \{u_j\}_{j \in V_i}) & \text{if } x \notin \partial \Omega \\ \max(\mathcal{H}_b(x,s, \{u_j\}_{j \in V_i}), s-\varphi(x)) & \text{else.} \end{cases} \] (24)
The scheme (23)–(24) can be extended to other types of boundary conditions. There is an implicit dependency of $S$ with respect to $h$. We extend the definition of $S$ to any $y \in \Omega$ by saying that $S(x, s, \{u_j\}_{j \in V_j}) = S(M_j, s, \{u_j\}_{j \in V_j})$ if $x$ belongs to the dual control volume associated to $M_j$.

We have the following result

**Theorem 2 ([9]).** Assume that

1. $H_b \leq H$,
2. $H, H_b$ are monotone and the solutions of (23) are uniformly bounded,
3. for all $\phi \in C_0^\infty(\overline{\Omega})$, we have

$$
\lim_{h \to 0, y \to x, \xi \to 0} H(y, \varphi(y) + \xi, \varphi + \xi) = H(x, \varphi(x), D\varphi(x))
$$

For any $x$ in a neighborhood of $\partial \Omega$,

$$
\lim_{h \to 0, y \to x, \xi \to 0} H^b(y, \varphi(y) + \xi, \varphi + \xi) = H_b(x, \varphi(x), D\varphi(x))
$$

4. The equation (1) has a uniqueness principle,

Then the family $u_h$ defined by (24) converges locally uniformly to the solution of (1) in $\Omega$.

**Proof.** The key argument of the proof is a convergence result by Barles and Souganidis [8].

Unfortunately, this results is not enough to guaranty a “good” convergence. Take the example at the beginning of the section, a regular mesh $(1/\Delta x = N + 1)$, the Godunov scheme that reduces here to

$$
\mathcal{H}_i = \mathcal{H}(u_{i-1}, u_i, u_{i+1}) = \max\left(\frac{|u_{i+1} - u_i|}{\Delta x}, \frac{|u_{i-1} - u_i|}{\Delta x}\right) - 1
$$

and $\mathcal{H}_b = -\infty$. This amounts to setting $u_0 = 0, u_N = 2$. Theorem 2 applies but numerical experiments indicate that the gradient of the solution is not bounded so that there is no hope to have a convergence like $\Delta x^\alpha$ with $\alpha > 0$ reasonable.

In [9], this problem is studied and it is shown that if $H$ is convex, if the Godunov scheme constructed on the boundary Hamiltonian $H_b$ is constructed, and if a coercivity assumption holds for $H, H_b$ and the associated numerical Hamiltonian, then one can control $Du$, and it is shown in [10] that the error behaves like $h^{-1/2}$. Similar error estimates (for Cartesian meshes) were obtained in [11].

## 6 High order extension

Up to now, all the examples we have given are only first order accurate schemes. There are several ways constructing high order schemes.

One possible construction is a consequence of the following fact. The “$Du$” dependency in the Hamiltonian comes form the term “$\{u_j, j \in V_j\}$” in (16). More precisely, in all the known examples, this dependency occurs through differences, $u_j - u_i$ for $j \in V_j$. These terms can be rewritten in terms of the gradients of $u$ in the triangles surrounding $M_i$ (this remark has already been used in (22a), (22b) and (22c)). One can exploit this remark, as in [1] for example, by modifying the evaluation of the gradients in the triangles. Instead of linear interpolant, one can use higher degree polynomials thanks to the ENO/WENO methodology, [6, 12, 13].

A other solution is the Discontinuous Galerkin strategy [14, 15, 16]. We do not detail this technique here.

A last method is a blending strategy, [17]. The idea is to blend a low order, monotone Hamiltonian ($\mathcal{H}^M$) with a high order Hamiltonian consistent Hamiltonian ($\mathcal{H}^H$). By high order we mean that if $u$ is a smooth solution of (1), then

$$
\mathcal{H}^H(M_i, u_i, \{u_j\}_{j \in V_i}) = O(h^k)
$$

(26)
for $k > 1$. The scheme writes
\[ H(M_i, u_i, \{u_j\}_{j \in V_i}) = \ell_i H^M(M_i, u_i, \{u_j\}_{j \in V_i}) + (1 - \ell_i) H^H(M_i, u_i, \{u_j\}_{j \in V_i}) + \varepsilon(h) \]  
(27)
where $\varepsilon(h) = Ch^k$ for some positive constant $C$ and $\ell_i$ is chosen such that, if $r_i := \frac{H^H}{H^M}$, we have
\[ \ell_i + (1 - \ell_i)r_i \geq \varepsilon'(h). \]  
(28)
where $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$. We have the simple lemma which proof is immediate,

**Lemma 3.** If $H^M$ and $H^H$ are strongly consistent, $H$ defined by (27) is weakly consistent.

The justification of (28) comes from the simple relation
\[ H(M_i, u_i, \{u_j\}_{j \in V_i}) = \left( \ell_i + (1 - \ell_i)r_i \right) H^M(M_i, u_i, \{u_j\}_{j \in V_i}) + \varepsilon(h) \]  
(29)
from which, using once more the technique of the convergence result of [8], one can show for the scheme (23)-(24) where $H$ is given by (27) the following result

**Theorem 3.** ([17]) We consider the scheme (23) where, in (24) $H$ is defined by (27). We assume that
1. $H^M$, $H^H$ and $H_b$ are strongly consistent
2. $H^M$ and $H_b$ are monotone Hamiltonians,
3. $H_b \leq H$,
4. The blending parameter $\ell$ belongs to $[0,1]$ and satisfies
\[ r = \frac{H^H(x, u_h(x), u_h)}{H^M(x, u_h(x), u_h)}, \quad \ell(x) + (1 - \ell(x))r \geq \varepsilon'(h) \]
where the parameters $\varepsilon(h)$ and $\varepsilon'(h)$ satisfies $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$,
5. There exists a unique solution $u_h$ of (23) that satisfies $L^\infty$ bound that is uniform in $h$,
6. The equation (1) has a uniqueness principle.

Then the family $u_h$ defined by the scheme converges locally uniformly to the solution of (1) in $\Omega$.

Examples of the blending parameter are, given constants $\alpha_- \geq 1$, $\alpha_+ > 0$ and $\beta > 0$, $\ell^*$
\[ \ell = \begin{cases} 
\min(1, \alpha_- |r|) & \text{if } r \leq 0 \\
0 & \text{if } 0 \leq r \leq \beta \\
\min(1, \alpha_+(r - \beta)) & \text{else.} 
\end{cases} \]  
(30)
This the one we have chosen in practical applications with $\beta = 0$ and $\alpha_+ = 1$. Implementation details can be found in [17].

## 7 Links between Cartesian and non conformal meshes

It is not difficult to construct numerical Hamiltonians that work on general non conformal meshes. The only key point is to construct *monotone* Hamiltonians. The convergence results of [8] and Theorem 2 can easily be adapted : a close inspection of the proof shows that the structure of the mesh plays no role. What matters is to define, for any mesh point $M_i$, a local interpolation, $\pi$ that operates on $U_i := \{u_j\}_{j \in V_i}$ onto the space of piecewise linear functions, and such that if $u_j \leq v_j, j \in V_j$, then $\pi(U_i) \leq \pi(V_i)$.

Consider Figure 2. The neighbors of $M_i$ are $\{P_i\}_{i=1,...,8}$, from which we construct a local triangulation (dotted lines) that is used to define a piece–wise linear interpolant. It does not need to be continuous. Then we can use our Hamiltonians to define schemes that are clearly consistent, monotone. The tricky part is the choice of the neighbors. Figure 2 shows an extreme case. A probably better choice would have been to choose only $\{P_2, P_3, P_4, P_5\}$ because the aspect ratio of the triangles is larger.

Note that the Hamiltonians of [1], thanks to this set of remarks, are particular cases of our formula.
8 Numerical results

In general, it is difficult to compute analytically the solution of a first order Hamilton–Jacobi equation, and the situation is even worse when the Hamiltonian is not convex (nor concave) because the analogy with hyperbolic systems becomes looser in general. Hence, it becomes more difficult to judge the quality of numerical results. To overcome this difficulty in a special case, we consider $H(p) = (||p|| - 1)^3$ and the problem

\[ H(Du) = 0 \quad \text{on } \Omega, \]
\[ u = 0 \quad \text{on } \Gamma_1, \]
\[ u = 10 \quad \text{on } \Gamma_2, \] (31)

where $\Omega$ is depicted in figure 3. Since $t \mapsto t^3$ is monotone increasing, $u$ is solution of (31) if and only if it is a solution of

\[ ||Dv|| - 1 = 0 \quad \text{on } \Omega, \]
\[ v = 0 \quad \text{on } \Gamma_1, \]
\[ v = 10 \quad \text{on } \Gamma_2. \] (32)

The solution of (31) and (32) is the distance to $\Gamma_1$.

In order to discretize (31), we write $H = H_1 + H_2$ with $H_1(p) = \max(||p|| - 1, 0)^3$ and $H_2(p) = \min(||p|| - 1, 0)^3$. These functions are respectively convex and concave. The numerical Hamiltonian is the Lax Friedrich’s and the boundary Hamiltonian is Godunov’s. The numerical solution is displayed on Figure 4-(a). The solution of (32) with the Godunov Hamiltonian is provided on Figure 4-(b). A close comparison show that they are (almost) identical.

An other application of the boundary conditions is given by the approximation of the following problem,
Figure 3: Computational domain for problem (31). \( \Gamma_1 \) is the inner circle of center \((0, 0)\) and radius \(r = 1\), \( \Gamma_2 \) is the outer circle (center \((0, 0.5)\), radius \(r = 3\)).

 Figure 4: (a) : Solution of problem (31), min=0, max=1.48. (b) : Solution of problem (32), min=0, max=1.504
on the same geometry,

\begin{align}
H(Du) &= 0 \quad \text{on } \Omega, \\
u(x, y) &= 0 \quad (x, y) \in \Gamma_1, \\
u(x, y) &= 3 \cos(2\pi x) \quad (x, y) \in \Gamma_2. 
\end{align}

Since \( H \) is non convex, it is difficult to know \textit{a priori} what would be the value of the solution on the boundary. The computed solution is given on Figure 5–(a). It can be seen that the solution satisfies strongly the boundary condition on \( \Gamma_2 \) and only weakly on \( \Gamma_1 \) (contrarily to the previous example). Note however that they have been numerically \textit{weakly} imposed on \( \Gamma_1 \) and \( \Gamma_2 \). The solution is also in very good agreement with the one obtained from the discretization of

\begin{align}
||Dv|| - 1 &= 0 \quad \text{on } \Omega, \\
v(x, y) &= 0 \quad (x, y) \in \Gamma_1, \\
v(x, y) &= 3 \cos(2\pi x) \quad (x, y) \in \Gamma_2, 
\end{align}

which is displayed on Figure 5–(b).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{(a) : Solution of problem (33), min=-3, max=-1.53. (b) : Solution of problem (34), min=-3, max=-1.47}
\end{figure}

We also show how the high order extension of section 6 works when the Godunov solver with \( P^2 \) interpolation for the high order scheme. The zoom is displayed on Figure 6. Clearly, a very large overshoot exists where \( u \) is not \( C^1 \), there the solution of the blended scheme is monotone and is very similar to the first order one. In the smooth part of the solution, the second order and the blended scheme are very similar (the results by the blended scheme is slightly more dissipative than those of the second order unlimited scheme).

The last Figure show that our implementation of the boundary conditions is effective. If we impose strongly the boundary conditions, as in Figure 7–(a) This has to be compared with Figure 5. The Figure 7 show that there is a strong boundary layer on parts of the outer boundary (where the isolines agglomere). This is not true for the Figure 7–(a). In fact, on some parts of the outer boundary, the compatibility condition of [3] is true, so that one can impose the boundary conditions strongly, and on other parts this is not true and we have to apply them weakly. This partition of the boundary is not known a priori : our implementation take this into account automatically.

9 Conclusions

We have describe several technique for the solution of first order Hamilton Jacobi equations. We have tried to explain the hidden details and the origins of the schemes. Several theoretical results are provided, the
Figure 6: Comparison of the solution of problem (34) with several schemes. In blue: the first order scheme (min = −3, max = −1.993), in green: second order unlimited scheme (min = −3, max = 23.25), in red: second order blended scheme (min = −3, max = −1.996).

Figure 7: (a) Solution of problem (34) when the boundary conditions are imposed strongly, (b) mesh
proofs are given in the references. Once more, there are many other methods for solving these problems, some are very general, some are specially tuned for a specific example such as computing a distance function which is one of the key ingredient of the level set method.

References


CONSTRUCTION OF SIMPLE, STABLE AND CONVERGENT HIGH ORDER SCHEMES FOR STEADY FIRST ORDER HAMILTON JACOBI EQUATIONS

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Abstract. We develop a very simple algorithm that permits to construct compact, high order schemes for steady first order Hamilton Jacobi equations. The algorithm relies on the blending of a first order scheme and a compact high order scheme. The blending is conducted in such a way that the scheme is formally high order accurate. A convergence proof is given. We provide several numerical illustrations that demonstrate the effective accuracy of the scheme. The numerical examples use triangular unstructured meshes, but our method may be applied to other kind of meshes. Several implementation remarks are also given.

1. Introduction. We consider the Cauchy problem: find $u \in C^0(\Omega)$, the space of continuous function on the open subset $\Omega \subset \mathbb{R}^d$, such that

$$H(x,u,Du) = 0 \quad x \in \Omega \subset \mathbb{R}^d \quad u = g \quad x \in \partial \Omega$$

(1.1)

in the viscosity sense. In (1.1), $(x,s,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \mapsto H(x,s,p)$ is uniformly continuous.

Before going further, let us briefly review the notion of viscosity solution for (1.1). For any function $z$, we consider the upper semi-continuous (u.s.c) and lower semi-continuous (l.s.c) envelopes of $z$ with respect to all the variables. They are defined by

$$z^*(x) = \limsup_{x \to y} z(y) \quad \text{and} \quad z_*(x) = \liminf_{x \to y} z(y).$$

Following [1], we introduce the function $G$

$$G(x,s,p) = \begin{cases} H(x,s,p) & x \in \Omega \\ s - g(x) & x \in \partial \Omega. \end{cases}$$
The computation of $G_*$ and $G^*$ is easy and we have:

\[
\begin{aligned}
G_*(x, s, p) &= G^*(x, s, p) = H(x, s, p) & \text{if } x \in \Omega \\
G_*(x, s, p) &= \min(H(x, s, p), s - g(x)) & \text{if } x \in \partial \Omega \\
G^*(x, s, p) &= \max(H(x, s, p), s - g(x)) & \text{if } x \in \partial \Omega
\end{aligned}
\]  

(1.2)

A locally bounded upper semi continuous function $u$ defined on $\overline{\Omega}$ is a viscosity sub-solution of (1.1) if and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local maximum of $u - \phi$, then

\[
G_*(x_0, u(x_0), D\phi(x_0)) \leq 0.
\]  

(1.3)

Similarly, $u$, a locally bounded, l.s.c. function defined on $\overline{\Omega}$ is a viscosity super-solution of (1.1) if and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local minimum of $u - \phi$, then

\[
G^*(x_0, u(x_0), D\phi(x_0)) \geq 0.
\]  

(1.4)

A viscosity solution is simultaneously a sub- and a super-solution of (1.1).

This can be generalized to other types of boundary conditions such as Neumann, etc.

Under standard assumptions on the open subset $\Omega$, $g$ and $H$, one can prove existence and uniqueness of the viscosity solutions of (1.1), see [1]. In particular, this is true if the Hamiltonian $H$ is convex in $p \in \mathbb{R}^d$ and if $\partial \Omega$ lipschitz continuous.

In this paper, we assume that (1.1) has a uniqueness principle, that is any sub-solution $u$ and any super-solution $v$ of (1.1) satisfy

\[
\forall x \in \Omega, \quad u(x) \leq v(x).
\]

Throughout the paper, we consider a family of regular triangulations. The triangulations are denoted by $T^h$, $h$ is the maximum diameter of the elements $K_j$, $j = 1, \ldots, n_e$ and $\{x_{ij}\}_{j=1,\ldots,n_e}$ is the set of vertices of $T^h$. The family of triangulations is also assumed to be shape regular, that is there exist constants $C_1$ and $C_2$ such that the diameter of $K_j$ satisfy

\[
C_1 h \leq \text{diam}(K_j) \leq C_2 h \quad \forall j = 1, \ldots, n_e.
\]  

(1.5)

For a given triangulation, the solution of (1.1) is approximated on a family of degree of freedom denoted by $\Sigma = \{\sigma_l\}_{l=1,\ldots,n_\Sigma}$. Typically, in each element $K$, we consider
a family of points that are unisolvent for some interpolation spaces that are defined later in the text, typically subsets of $\mathbb{P}^k(K)$, the set of polynomials of degree $k \in \mathbb{N}$. The set $\Sigma$ is the collection of these degrees of freedom. This enables to define, from $\{u_i\}_{\sigma_i \in \Sigma}$, an interpolant $u^h$ that we assume to be continuous. This is only possible under constraints on the degrees of freedom, examples are given in section 4.2. It is not necessary yet to go in more details. Last, for any $\sigma_i$, $\mathcal{V}_i$ denote the set of its neighbors.

We are interested in constructing high order convergent schemes for (1.1), that is a functional $\mathcal{H}$ which is defined for any $\sigma \in \Sigma$, such that the approximation $u_j \simeq u(\sigma_j)$ of $u$ at $\sigma_j$ satisfies, for $i = 1, \cdots, n_\Sigma$,

$$\mathcal{H}(\sigma_i, u_i, \{u_j, j \in \mathcal{V}_i\}) = 0. \quad (1.6)$$

In (1.6), $\mathcal{V}_i$ is the set of neighbors of $\sigma_i$. This Hamiltonian has to be consistent in the meaning of Definition 1.1, this brings an implicit dependency with respect to the mesh, and in particular in $h$.

For technical reasons only, we need to extend the definition of the scheme to any point of $\Omega$. This can be done as follows. We consider a subdivision of $\Omega$ of “control volumes” $\mathcal{C}_j$, $\Omega = \bigcup_{j=1}^{n_\Sigma} \mathcal{C}_j$ such that any control volume contains one and only one degree of freedom, and conversely, any degree of freedom is contained in one and only one control volume. After a convenient numbering, we can assume $\sigma_i \in \mathcal{C}_i$. We can also assume that a property similar to (1.5) also hold for this family of control volume. A possible construction of such control volume is given by a Voronoi diagram, see [2] for example. The functional $\mathcal{H}_h$ is extended on $\overline{\Omega}$ by the following: if $x \in \overline{\Omega}$, consider $\sigma \in \Sigma$ such that $x \in \mathcal{C}_\sigma$ and we set

$$\mathcal{H}(x, u^h(x), u^h) = \mathcal{H}(\sigma_i, u_i, \{u_j, j \in \mathcal{V}_i\}).$$

The approximation scheme (1.6) needs to be consistent with (1.1). We follow Barles and Souganidis [3] definition:

**Definition 1.1 (Consistant Hamiltonians).** We say that the Hamiltonian $\mathcal{H}$ is weakly consistent if for all $x \in \overline{\Omega}$ and $\phi \in C^\infty_b(\overline{\Omega})$ (the set of $C^\infty$ bounded functions),

$$\limsup_{h \to 0, y \to x, \xi \to 0} \mathcal{H}(y, \phi(y) + \xi, \phi + \xi) \leq G^*(x, \phi(x), D\phi(x)) \quad (1.7)$$
\[
\lim_{\rho \to 0, y \to x, \xi \to 0} \mathcal{H}(y, \phi(y) + \xi, \phi + \xi) \geq G_\ast(x, \phi(x), D\phi(x)).
\]

(1.8)

We say that \( \mathcal{H} \) is strongly consistent if, whenever \( \phi \) is linear (constant gradient), whatever \( \sigma \in \Sigma, \phi \in \Omega, \)

\[
\mathcal{H}(\sigma, \phi(\sigma), \phi) = H(\sigma, \phi, D\phi(\sigma)).
\]

Note that if \( \mathcal{H} \) is weakly consistent, \( \mathcal{H} \) is strongly consistent.

**Definition 1.2 (Monotone Hamiltonians).** We say that \( \mathcal{H} \) is monotone if, whatever \( \sigma_i \in \Sigma, u_i \leq v_j, \) and for any \( s \in \mathbb{R}, \)

\[
\mathcal{H}(\sigma_i, s, \{u_j\}_{j \in \mathcal{V}_i}) \geq \mathcal{H}(\sigma_i, s, \{v_j\}_{j \in \mathcal{V}_i}).
\]

There exists many numerical schemes devoted to the resolution of (1.1), examples are given by references [4, 5, 6, 7, 8], where the problem (1.1) is directly tackled.

One quite standard way of constructing scheme for (1.1) is to consider the unsteady problem

\[
\begin{align*}
\frac{\partial v}{\partial t} + H(x, v(x), Du) &= 0 & x \in \Omega \subset \mathbb{R}^d, t > 0 \\
v(x, t) &= g(x, t) & x \in \partial \Omega, t > 0 \\
v(x, 0) &= v_0(x) & x \in \Omega, t = 0.
\end{align*}
\]

(1.9)

for some suitable initial condition \( u_0 \) such that the solution of (1.1) is obtained as the limit, when \( t \to +\infty \) of the solution of (1.9). There is a whole industry of numerical schemes for (1.9). Omitting the boundary conditions and in their simplest form, they are of the type

\[
u^{n+1}_i = u^n_i - \Delta t \mathcal{H}(\sigma_i, u^n_i, \{u^j_i, j \in \mathcal{V}_i\})
\]

with \( u^0_i = u_0(\sigma_i) \). Along these lines, one may quote the work of [9, 6, 10] and among many others [11, 12] for Cartesian meshes and [13, 14, 15, 16] for unstructured meshes.

To the best of our knowledge, the only convergence results, with error estimates, are for first order schemes, see [9, 17, 18] for structured meshes and [13] for unstructured meshes. A general method for proving convergence, without error estimates, is given in [3]. All these constructions are strongly related to the different techniques that
have been devised for constructing high order accurate, Godunov type, schemes for conservation laws.

Here, starting from a different construction, we explain how it is possible to construct simple, convergent, high order accurate schemes for the problem (1.1). We also show examples for which the computational stencil is the most possible compact. Our construction relies on the blending of a low order accurate scheme and a high order stable scheme. The structure of the blending parameter is analyzed so that high order accuracy is obtained as well as a convergence proof. These schemes are of course not monotone, but monotonicity preserving. Up to our knowledge, it is the first time where both properties can simultaneously be achieved. We also study the practical implementation of the scheme, and demonstrate its effectiveness on one dimensional and two-dimensional examples. In this paper, we focus on unstructured triangular type meshes. It is clear however that the main result of the paper (i.e. the form of the scheme and the convergence proof) can be used in a more general context.

The structure of this paper is the following: we first start by a general derivation of the scheme. We discuss in detail the structure of the blending parameter. We then provide a convergence proof. The next section is devoted to showing some examples of schemes, and we also discuss the practical implementation of the scheme. The last section is devoted to numerical examples.

2. Derivation of the schemes. We first discuss the scheme for the mesh points in the open set \( \Omega \). The boundary conditions are discussed at the end of this section.

We consider \( \mathcal{H}^M_i := \mathcal{H}^M(\sigma_i, u_i, \{u_j\}_{j \in \mathcal{V}_i}) \) a monotone consistent Hamiltonian and \( \mathcal{H}^H_i := \mathcal{H}^H(\sigma_i, u_i, \{u_j\}_{j \in \mathcal{V}_i}) \) a high order consistent Hamiltonian. By high order we mean that if \( u \) is a smooth solution of (1.1), then

\[
\mathcal{H}^H(\sigma_i, u_i, \{u_j\}_{j \in \mathcal{V}_i}) = O(h^k)
\]

for \( k > 1 \)

Next we consider for some \( \ell \in \mathbb{R} \) the following Hamiltonian

\[
\mathcal{H}(\sigma_i, u_i, \{u_j\}_{j \in \mathcal{V}_i}) = \ell \mathcal{H}^M(\sigma_i, u_i, \{u_j\}_{j \in \mathcal{V}_i}) + (1 - \ell) \mathcal{H}^H(\sigma_i, u_i, \{u_j\}_{j \in \mathcal{V}_i}) + \varepsilon(h)
\]

We have the simple lemma which proof is immediate,

**Lemma 2.1.** If \( \mathcal{H}^M \) and \( \mathcal{H}^H \) are strongly consistent, \( \mathcal{H} \) defined by (2.2) is weakly consistent.
We assume that \( \varepsilon(h) = O(h^k) \). In order to define \( \ell \), we introduce the ratio \( r_i := \frac{h_i}{m_i} \) and rewrites (2.2) as

\[
\mathcal{H}(\sigma_i, u_i, \{u_j\}_{j \in V_i}) = \left( \ell_i + (1 - \ell_i) r_i \right) \mathcal{H}^M(\sigma_i, u_i, \{u_j\}_{j \in V_i}) + \varepsilon(h) \tag{2.3}
\]

and choose \( \ell \) such that

\[
\ell_i + (1 - \ell_i) r_i \geq \varepsilon'(h). \tag{2.4}
\]

where \( \varepsilon'(h)^{-1} \varepsilon(h) = o(1) \). The locus of the points \((r, \ell)\) that satisfy condition (2.4)

\[
\begin{array}{c}
\ell \\
\hline
1 \\
\hline
r
\end{array}
\]

Fig. 2.1. Graph of \( \ell + (1 - \ell) r = 0 \).

lies between the two branches of the hyperbola \( \ell + (1 - \ell) r = 0 \) displayed in Figure 2.1. Then, we can rewrite

\[
\left| \mathcal{H}(x, t, \{u_j\}_{j \in V_i}) \right| \leq \left| \frac{\ell}{r} + 1 - \ell \right| \left| \mathcal{H}^M(x, t, \{u_j\}_{j \in V_i}) \right| + \varepsilon(h). \tag{2.5}
\]

Since \( \varepsilon(h) = O(h^k) \) and if \( \ell \in [0, 1] \), if there exists \( C > 0 \) such that

\[
\left| \frac{\ell}{r} \right| \leq C \tag{2.6}
\]

then the scheme defined by the Hamiltonian (2.2) satisfies

\[
\mathcal{H}(\sigma_i, u_i, \{u_j\}_{j \in V_i}) = O(h^k)
\]

for any smooth solution of (1.1).

Since the tangent at origin of the hyperbola \( \{(r, \ell) \in \mathbb{R}^2, \ell + (1 - \ell) r = 0\} \) is \(-1\), a solution of the problem (2.6) is \( \ell = \max(\ell^*, \varepsilon'(h)) \) where

\[
\ell^* = \begin{cases} 
0 & \text{if } r \geq 0 \\
\min (1, \alpha |r|) & \text{else.}
\end{cases} \tag{2.7}
\]
for any $\alpha \geq 1$.

We have an additional constraint on $\ell$. It comes from the iterative scheme that is needed to compute the solution of

$$
\mathcal{H}(\sigma_i, u_i, \{u_j\}_{j \in V_i}) = 0
$$

where $\mathcal{H}$ is defined by (2.2). In this paper, we employ the following explicit scheme

$$
u^{n+1}_i = u^n_i - \Delta t \mathcal{H}(x_i, t, u^n_i, \{u^n_j\}_{j \in V_i}) \quad \text{for } n \geq 1
\quad u^0_i = u_0(M_i)
$$

(2.8)

Up to our knowledge, all first order monotone Hamiltonians satisfy an $L^\infty$ stability condition under a constraint on the time step of the type

$$\Delta t \leq C h
$$

(2.9)

where $C$ is a constant that depends only on $u_0$ and $H$, not on $h$ the maximum diameter of the mesh elements. From (2.8), (2.3) and (2.4) we also have

$$
u^{n+1}_i = u^n_i - \Delta t \left[ (\ell_i + (1 - \ell_i) r_i) \mathcal{H}^M(\sigma_i, u_i, \{u_j\}_{j \in V_i}) + \varepsilon(h) \right]
$$

so that the scheme is $L^\infty$ stable if

$$\Delta t (\ell_i + (1 - \ell_i) r_i) \leq C h.
$$

In order to have $\Delta t$ not too small, we consider $C' > 0$ and $(r_i, \ell_i)$ and $0 \leq \ell_i \leq 1$ such that

$$0 \leq \ell_i + (1 - \ell_i) r_i \leq C'.
$$

This imposes an additional condition only for $r_i > 0$. There are many ways of imposing this constraint in conjunction with (2.4). For simplicity reasons, given constants $\alpha_\leq \geq 1$, $\alpha_+ > 0$ and $\beta > 0$, we choose the following form for $\ell^*$

$$
\ell^* = \begin{cases}
\min(1, \alpha_- |r|) & \text{if } r \leq 0 \\
0 & \text{if } 0 \leq r \leq \beta \\
\min(1, \alpha_+(r - \beta)) & \text{else}
\end{cases}
$$

(2.10)

The graph of such a function, for $C = 1$ is displayed on Figure 2.2. With $\ell$ defined as in (2.10), there exists $C'$ such that for

$$0 < \Delta t C' h
$$

(2.11)
the scheme defined by (2.8), (2.3) and (2.4) is $L^\infty$ bounded.

Other choices are possible such as

$$l^* = \varphi(r) := \frac{|r|}{1+|r|}$$

or more generally

$$l^* = \varphi(\psi(r))$$

with $\psi(r) \geq r$ and $\psi'(0) = 1$. An example is $\psi(r) = r + r^2$. These possibilities have not been explored.

Since equation (1.1) does not depend on time, boundary conditions must be specified, otherwise the problem is meaningless. Here we follow the technique described in [19]. In order to simplify the text, we consider only first order accurate discretisation of the boundary conditions.

We consider a boundary numerical Hamiltonian $H_b$ that is consistent with a boundary Hamiltonian $H_b$. It is defined for $x \in \partial \Omega$, $s \in \mathbb{R}$ and $p \in \mathbb{R}^d$ and also satisfies

$$\forall x \in \partial \Omega, s \in \mathbb{R}, p \in \mathbb{R}^d, \quad H_b(x, s, p) \leq H(x, s, p).$$

The resulting scheme is: find $\{u_j\}_{j=1,\ldots,n_\Sigma}$ such that

$$\ell_i \mathcal{H}^M(\sigma_i, u_i, \{u_j\}_{j \in V_i}) + (1 - \ell_i) \mathcal{H}^H(\sigma_i, u_i, \{u_j\}_{j \in V_i}) + \varepsilon(h) = 0$$

(2.12a)
\[ \text{if } \sigma_i \in \partial \Omega, \quad \max \left( \mathcal{H}^b(\sigma_i, u_i, \{u_j\}_{j \in V_i}), u_i - g(\sigma_i) \right) = 0. \] (2.12b)

The solution of (2.12) is not an easy task. Following standard techniques, we compute it as the limit when \( n \to +\infty \), if this limit exists, of \( \{u^n_j\}_{j=1,\ldots,n} \), \( n \in \mathbb{N} \) defined by \( u^0_i = u_0(x_i) \) and

\[ \text{if } \sigma_i \in \Omega, \quad u^{n+1}_i = u^n_i - \Delta t \left( \ell_i \mathcal{H}^M (\sigma_i, u_i, \{u_j\}_{j \in V_i}) + (1 - \ell_i) \mathcal{H}^H (\sigma_i, u_i, \{u_j\}_{j \in V_i}) + \varepsilon(h) \right) \] (2.13a)

\[ \text{if } \sigma_i \in \partial \Omega, \quad \max \left( \frac{u^{n+1}_i - u^n_i}{\Delta t} + \mathcal{H}^b(\sigma_i, u_i, \{u_j\}_{j \in V_i}), u^{n+1}_i - g(x_i) \right) = 0. \] (2.13b)

In (2.12) and (2.13), \( \ell \) is defined by (2.10).

In the following, we extend the definition of \( \ell \) to any \( x \in \Omega \) as we have done for the Hamiltonian via the explicit dependence of the ratio \( r \) in \( x \).

3. Convergence proof. We denote by \( S \) the operator

\[ S(h, x, u_h) = \begin{cases} 
\ell(x) \mathcal{H}^M (x, u_h(x), u_h) + (1 - \ell(x)) \mathcal{H}^H (x, u_h(x), u_h) + \varepsilon(h) & \text{if } x \in \Omega \\
\max \left( \mathcal{H}^b(x, u_h(x), u_h), u_h(x) - g(x) \right) = 0. & \text{if } x \in \partial \Omega.
\end{cases} \]

We have

**Theorem 3.1.** We consider the scheme (2.12) and assume that

1. \( \mathcal{H}^M, \mathcal{H}^H \) and \( \mathcal{H}_b \) are strongly consistent
2. \( \mathcal{H}^M \) and \( \mathcal{H}_b \) are monotone Hamiltonians,
3. \( \mathcal{H}_b \leq \mathcal{H} \),
4. The blending parameter \( \ell \) belongs to \([0, 1]\) and satisfies

\[ r = \frac{\mathcal{H}^H (x, u_h(x), u_h)}{\mathcal{H}^M (x, u_h(x), u_h)}, \quad \ell(x) + (1 - \ell(x)) r \geq \varepsilon'(h) \]

where the parameters \( \varepsilon(h) \) and \( \varepsilon'(h) \) satisfy \( \varepsilon'(h)^{-1} \varepsilon(h) = o(1) \),
5. There exists a unique solution of (2.12) \( u_h \) that satisfies \( L^\infty \) bound that is uniform in \( h \),
6. The equation (1.1) has a uniqueness principle.
Then the family $u_h$ defined by (2.12) converges locally uniformly to the solution of (1.1) in $\Omega$.

**Remark 1.** An essential ingredient in theorem 3.1 is that the scheme (2.12) has a unique solution which is bounded in $L^\infty$ uniformly in $h$. In practical calculations, the solution is obtained by an iterative scheme: we look for the limit, when $n \to +\infty$, of the iterative scheme (2.13) (for example (2.13)), if such limit exists. If such a limit exists, the $L^\infty$ bound comes from the CFL condition (2.11), but the existence of the limit is not a trivial statement.

**Proof.** [Proof of Theorem 3.1]

We proceed as in [1]. The sequence $u_h$ is bounded so we can define

$\bar{u}(x) = \limsup_{y \to x, h \to 0} u_h(y)$ and $\underline{u}(x) = \liminf_{y \to x, h \to 0} u_h(y)$

They are defined on $\bar{\Omega}$ because $u_h$ has bounds independent of $h$. We show that the functions $\bar{u}$ and $\underline{u}$ are respectively sub- and super-solutions of (1.1). We proceed in two parts: first we consider the case of an interior point, then the case of a boundary point.

Case of an interior point. In fact, we show first that if $x_0 \in \Omega$ is a local maximum of $\bar{u} - \phi$ for some $\phi \in C^b(\Omega)$, then

$$H(x_0, \varphi(x_0)D\varphi(x_0)) \leq 0,$$

while if $x_0 \in \Omega$ is a local minimum of $\underline{u} - \phi$,

$$H(x_0, \varphi(x_0)D\varphi(x_0)) \geq 0$$

To show the inequality (3.2), we repeat Barles and Souganidis’ arguments, the inequality (3.2) is obtained in the same way. We may assume that $x_0$ is a strict minimum, $\underline{u}(x_0) = \phi(x_0)$, $\phi \leq 2\inf_h \|u_h\|_\infty$ outside of $B(x_0, r)$, where $r$ is such that

$$\underline{u}(x) - \phi(x) \geq \underline{u}(x_0) - \phi(x_0) = 0 \text{ in } B(x_0, r).$$

There exists sequences $h_n$ and $y_n \in \bar{\Omega}$ such that $n \to +\infty$, $h_n \to 0$, $y_n \to x_0$, $u_{h_n}(y_n) \to \underline{u}(x_0)$, and $y_n$ is a global minimum of $u_{h_n} - \phi$. We denote by $\xi_n$ the quantity $u_{h_n}(y_n) - \phi(y_n)$. We have $\xi_n \to 0$, $u_{h_n}(y) \geq \phi(y) + \xi_n$ in $B(x_0, r)$.

Defining $r_n = \frac{\mathcal{H}_h^M(y_n, u_h(y_n), u_h)}{\mathcal{H}_h^M(y_n, u_h(y_n), u_h)}$, we get

$$0 = \ell(y_n)\mathcal{H}_h^M(y_n, u_h(y_n), u_h) + (1 - \ell(y_n))\mathcal{H}_h^H(y_n, u_h(y_n), u_h) + \varepsilon(h_n)$$

$$= (\ell(y_n) + (1 - \ell(y_n))r_n)\mathcal{H}_h^M(y_n, u_h(y_n), u_h) + \varepsilon(h_n)$$

(3.3)
Since
\[ 0 \leq \frac{\varepsilon(h_n)}{\ell(y_n) + (1 - \ell(y_n))r_n} \leq \frac{\varepsilon(h_n)}{\varepsilon'(h_n)} = o(1) \]
we have since \( \ell(y_n) + (1 - \ell(y_n))r \geq \varepsilon'(h) > 0 \), we get all in all
\[ \frac{\varepsilon(h_n)}{\ell(y_n) + (1 - \ell(y_n))r_n} = o(1). \]

because \( \mathcal{H}^M \) is monotone and \( \varepsilon'(h) > 0 \). Hence, if we divide the last equality of (3.3) by \( \ell(y_n) + (1 - \ell(y_n))r_n > \varepsilon'(h_n) > 0 \), we get
\[ 0 \leq \mathcal{H}^M_{h_n}(y_n, u_n(y_n), u_b) + o(1). \quad (3.4) \]

Last, using the monotonicity of \( \mathcal{H}^M_{h_n} \), we end up to
\[ 0 \leq \mathcal{H}^M_{h_n}(y_n, \phi(y_n) + \xi_n, \phi + \xi_n) + o(1). \quad (3.5) \]

Note that in passing from (3.4) to (3.5), we have used the uniform continuity of \( H \).

Thus
\[ 0 \leq \limsup_n \mathcal{H}^M(y_n, \phi(y_n) + \xi_n, \phi + \xi_n) \]
\[ \leq H(x_0, \varphi(x_0), D\varphi(x_0)) \]

This shows that \( \underline{u} \) is a super-solution of (1.1). The same arguments applied to \( \bar{u} \) show that it is a sub-solution of (1.1) in \( \Omega \).

Case of a boundary point. Now we consider the case of \( x_0 \in \partial \Omega \). The first remark is that for \( x \in \partial \Omega \), and any \( \varphi \in C^\infty_b(\Omega) \),
\[
\limsup_{h \to 0, y \to x, \xi \to 0} S(h, x, \varphi + \xi) = \max(H(x, \varphi(x), D\varphi), \\
\max(H_b(x, \varphi(x), D\varphi(x)), \varphi(x) - g(x)))
\]
\[
\liminf_{h \to 0, y \to x, \xi \to 0} S(h, x, \varphi + \xi) = \min(H(x, \varphi(x), D\varphi), \\
\max(H_b(x, \varphi(x), D\varphi(x)), \varphi(x) - g(x)))
\]

The proof for showing that if \( x_0 \in \partial \Omega \) is a local maximum of \( \underline{u} - \phi \) for some \( \phi \in C^\infty_b(\Omega) \), then
\[ \min(H(x_0, \varphi(x_0)D\varphi(x_0)), \max(H_b(x_0, \varphi(x_0), D\varphi(x_0)), \varphi(x_0) - g(x_0))) \leq 0, \quad (3.6) \]
while if \( x_0 \in \bar{\Omega} \) is a local minimum of \( \underline{u} - \phi \),
\[ \max(H(x_0, \varphi(x_0)D\varphi(x_0)), \max(H_b(x_0, \varphi(x_0), D\varphi(x_0)), \varphi(x_0) - g(x_0))) \geq 0 \quad (3.7) \]
can easily be obtained by combining the same arguments and those of [19], Theorem 2.2.

Since $S$ is monotone, we get

$$0 \leq \limsup_{n} S(h_n, y_n, \phi(y_n) + \xi_n) \leq \limsup_{h \to 0, y \to x, \xi \to 0} S(h, y, \phi + \xi) \leq \max(H(x_0, \phi(x_0), D\phi(x_0)), \max(H_b(x_0, \phi(x_0), D\phi(x_0)), F(x, \phi(x_0), D\phi(x_0))).$$

Now we have to check that the condition (3.6) (resp. (3.7)) implies the super-solution (resp. sub-solution) condition.

- **Inequality (3.6).** If $F(x_0, \underline{u}(x_0), D\phi(x_0) \leq 0$, there is nothing to prove. We assume $F(x_0, \underline{u}(x_0), D\phi(x_0) > 0$. We have either

$$H(x_0, \phi(x_0), D\phi(x_0)) \leq 0 \quad \text{(3.8)}$$

or

$$\max(H_b(x_0, \underline{u}(x_0), D\phi(x_0), F(x_0, \underline{u}(x_0), D\phi(x_0)) \leq 0.$$

In the second case, we have necessarily (3.8), and in both cases the inequality holds.

- **Inequality (3.7).** If $F(x_0, \underline{u}(x_0), D\phi(x_0) \geq 0$, there is nothing to prove. Assume $F(x_0, \underline{u}(x_0), D\phi(x_0) < 0$, then we must have either $H(x_0, \underline{u}(x_0), D\phi(x_0)) \geq 0$ or

$$\max(H_b(x_0, \underline{u}(x_0), D\phi(x_0), F(x_0, \underline{u}(x_0), D\phi(x_0)))) \geq 0.$$

Since $F < 0$, this inequality implies $H_b \geq 0$, so that

$$H(x_0, \underline{u}(x_0), D\phi(x_0)) \geq H_b(x_0, \underline{u}(x_0), D\phi(x_0)) \geq 0.$$

Thus, in both cases, we get $H(x_0, \underline{u}(x_0), D\phi(x_0)) \geq 0$, which is what we wanted.

**Conclusion.** All this shows that $\underline{u}$ is a super-solution and $\overline{u}$ is a sub-solution of (1.1). The strong uniqueness principle enables to conclude.

- Remark 2.
1. **Choice of $\varepsilon(h)$ and $\varepsilon'(h)$** Since we must have $\varepsilon(h) = O(h^k)$, $\ell = O(h^k)$ and $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$, a good choice for $\varepsilon(h)$ and $\varepsilon'(h)$ is $\varepsilon(h) = Ch^{k+1}$ and $\varepsilon(h) = C'h^k$ for any constants $C$, $C'$. 

In the numerical applications, however, we have chosen $\varepsilon'(h) = 0$ that seems to work fine.

2. **Consistency of $\mathcal{H}^H$.** The proof does not make any use of the consistency of $\mathcal{H}^H$. Only $\mathcal{H}^H$ matters. However, since $\mathcal{H}^H$ satisfies (2.1), it must be consistent.

4. **Examples of schemes and practical implementation.**

4.1. **First order numerical schemes.** We consider two kinds of first order schemes, the Godunov scheme and the Lax Friedrichs scheme. We recall briefly their construction here. We describe their construction for elements, and the degrees of freedom are the vertices of these elements. To make the text simpler, we implicitly assume that the elements are triangles in 2D, but this is absolutely not essential.

**Godunov Hamiltonian.** If $H = H_1 + H_2$ where $H_1$ (resp. $H_2$) is convex (resp. concave), then we set

$$H^G_h(p_1, \ldots, p_{k_i}) = \inf_{q \in \mathbb{R}^2} \max_{0 \leq l \leq k_i} \sup_{y \in -\Omega_l + q} [p_i \cdot (y - q) - H_1^*(y) - H_2^*(q)]$$

(4.1)

where $\Omega_l$, $l = 1, \ldots, k_1$ are the angular sectors defined by the triangles $T_1, \ldots, T_{k_1}$ at node $M_i$, $H_1^*$ and $H_2^*$ are the Legendre transforms of $H_1$ and $H_2$. We have denoted by $x \cdot y$ the dot product of $x$ and $y$.

If $h$ is the smallest radius of the circles of center $M_i$ contained in $\bigcup_{l=1}^{k_i} T_j$, if $L_1$ and $L_2$ are Lipschitz constants for $H_1$ and $H_2$, then the scheme is monotone provided that the time step satisfies

$$\frac{\Delta t}{h} (L_1 + L_2) \leq \frac{1}{2},$$

one can consult [13] for more details.

In most of the numerical examples below, the Hamiltonian is convex, so (4.1) becomes simpler

$$H^G_h(p_1, \ldots, p_{k_i}) = \max_{0 \leq l \leq k_i} \sup_{y \in -\Omega_l} [p_i \cdot y - H_1^*(y)].$$

(4.2)

We denote by $H_T$ the term $\sup_{y \in -\Omega_l} [p_i \cdot y - H_1^*(y)]$ where the angular sector $\Omega_l$ is the sector of $T$ seen from the vertex $M$.

**Remark 3.** *The implementation of this Hamiltonian is very easy if the evaluation...*
of
\[
\sup_{y \in \Omega} \left[ p_i \cdot y - H^*_1(y) \right]
\]
is simple. We first need to initialize \( \mathcal{H}_i^G := \mathcal{H}_i^G(\sigma_i, u_h^i, u_h) \) to a large negative value. We then make a loop over the elements of the mesh, evaluate for each element the gradient of the piecewise linear interpolant \( u_h \), evaluate for each vertex of the element the quantity
\[
\mathcal{H}_i^T := \sup_{y \in \Omega_i} \left[ p_i \cdot y - H^*_1(y) \right]
\]
where, for the vertex \( \sigma_i \), \( \Omega_i \) is the sector of the element as in Figure 4.1.

Fig. 4.1. The angular sector in the element \( K \) at \( \sigma_i \).

Lax–Friedrichs Hamiltonian. Here we set
\[
\mathcal{H}_h^{LF}(Du_{\Omega_1}, \cdots, Du_{\Omega_k}) = H(\tilde{U}) - \frac{\epsilon}{h} \oint_{C_h} [u(M) - u(M_i)]dl,
\]
where \( C_h \) (resp. \( D_h \)) is a circle (disk) of center \( M_i \) and radius \( h \),
\[
\tilde{U} = \frac{\int_{D_h} Du \ dx \ dy}{\pi h^2},
\]
and \( \epsilon \) is larger than any Lipschitz constant of \( H \) divided by \( 2\pi \).

A different version of the Lax Friedrichs Hamiltonian, that is monotone under the same constraint, is the following:
\[
\mathcal{H}_h^{LF}(Du_{\Omega_1}, \cdots, Du_{\Omega_k}) = \frac{\int_{D_h} H(Du)}{\pi h^2} - \frac{\epsilon}{h} \oint_{C_h} [u(M) - u(M_i)]dl.
\]
This version can be rewritten as

\[ \mathcal{H}_h^{LF}(Du_{\Omega_1}, \ldots, Du_{\Omega_{k_1}}) = \sum_{0 \leq l \leq k_i} \frac{\theta_l^i}{2\pi} H(Du_{\Omega_l}) + \varepsilon \sum_{0 \leq l \leq k_i} \tan \theta_l^i \frac{\vec{n}_{l-1/2}^i + \vec{n}_{l+1/2}^i}{2} \cdot Du_{\Omega_l}. \]

The vector \( \vec{n}_{l+1/2} \) is the unit vector of the edge that separates the angular sectors \( \Omega_l \) and \( \Omega_{l+1} \), the angle \( \theta_l^i \) is the angle of the angular sector at \( \sigma_i \), see Figure 4.2. The parameter \( \varepsilon \) is the same as in the previous version.

![Diagram](image)

**Fig. 4.2.** Illustration of the angular sectors \( \Omega_i \) and the vectors \( \vec{n}_{l+1/2} \) that are needed in (4.3a).

A third version, which is the one we have used in the simulations, is

\[ \mathcal{H}_h^{LF}(Du_{\Omega_1}, \ldots, Du_{\Omega_{k_1}}) = \frac{\sum_{T \supset M_i} |T| H(Du_T) + \alpha \sum_{M_j \in T} (u_i - u_j)}{\sum_{T \supset M_i} |T|} \]  

(4.3c)

and \( \alpha \geq h_T \max_p ||D_p H|| \) where \( h_T \) is the largest edge of \( T \).

The main difference between these different formulas is that (4.3a) and (4.3b) are intrinsic in the sense given in [13] while (4.3c) is not. By the way, the same is true for (4.1). Hence, following the same reference, (4.3a) and (4.3b) are convergent and the error estimate is \( O(h^{1/2}) \). For (4.3c), such an error estimate is not available (at least when following the technique of [13]), but it is convergent : this is a simple application of [3].

The advantage of (4.3c) over the other two versions is its simplicity in coding. As (4.3b), we need to make a loop over the element. For each element, we compute
Du_{ijT}, \alpha and evaluate for each degree of freedom in the element,

\[ |T| H(Du_{ijT}) + \alpha (u_i - u_j). \]

The numerical Hamiltonian is the arithmetic average of these quantities.

The dissipation mechanisms is much simpler than for (4.3b). Considering (4.3a), the loop has to be carried out over each degree of freedom. Then for each of them, we need to make a loop over its neighbors. In the case of (4.3c), the coding is much simpler, this is why we prefer (4.3c).

4.2. High order Hamiltonians. There are many formally high order numerical Hamiltonians, but it is more difficult to construct stable and high order Hamiltonians. For example, if a discrete \( L^2 \)-like stability property is sought, one has to realize that there is no natural counterpart on the continuous side because of the nature of the viscosity solutions: the test functions are not integrated by part as for standard hyperbolic problems. The natural \( L^2 \)-like stability property has to be written on \( Du \), this causes some difficulties that may be overcome, see for example [14] and more recently [20]. In the following, we first consider a formally high order scheme that appears to be stable as long as no singularity of \( Du \) appear. It is in the spirit of the ENO-like scheme of [6, 21] and [13]. A second example is also considered following [15]. The scheme is more compact, but only adapted to homogeneous Hamiltonians. However, we propose an ad-hoc extension to the inhomogeneous case in this paper.

Notations. In this paragraph, we consider elements that are implicitly thought as triangular elements, but this restriction is not essential. We first precise the type of interpolant (reconstruction) of \( \{u_i\}_{i=1,\ldots,n_N} \) we consider in the examples.

Two kinds of Lagrange type interpolants are considered. Since there is no ambiguity when we consider a given interpolation, \( u^h \) represents the interpolation of the continuous function \( u \).

In the first case, in each triangle \( T \), the solution is approximated by a polynomial of degree \( r \), their set is \( \mathbb{P}^r(T) \). Hence, the solution is described by \( \frac{(r+1)(r+2)}{2} \) degrees of freedom. It is known that the points of \( T \) which barycentric coordinates are \( (\frac{1}{r}, \frac{i}{r}, \frac{j}{r}, \frac{k}{r}) \) with \( i, j, k \) positive integers and \( i + j + k = r \) are unsolvant. Examples for \( r = 1, 2 \) are displayed on Figure 4.3. The degrees of freedom \( \sigma \) in the mesh is the collection of these new points.

Other choices are possible such as the interpolants defined in [22]. The simplest
example is described, in any triangle, by the degrees of freedom consisting of those
of the $\mathbb{P}^2$ interpolation plus the centroid of the triangle, see Figure 4.4. Denoting
by $\text{vec}(b)$ the set of functions of the type $\lambda b$ where $\lambda \in \mathbb{R}$, the interpolation space is
$\tilde{\mathbb{P}}^2(T) = \mathbb{P}^2(T) \oplus \text{vec}(b)$. The bubble function is, if $\{\Lambda_j\}_{j=1,3}$ denotes the barycentric
coordinates in $T$, $b = \Lambda_1\Lambda_2\Lambda_3$. This interpolation space is used in the two dimensional
elements. It yields a third order accurate interpolation, and enjoys the following
quadrature relation
\[ \int_T f(x) dx = |T| \sum_j \omega_j f(x_j) \]  \hspace{1cm} (4.4)

with $\omega_j = \frac{1}{20}$ for $j = 1, \ldots, 3$, $\omega_j = \frac{2}{15}$ for $j = 4, \ldots, 6$ and $\omega_7 = \frac{2}{20}$. See [22] for more
details and other examples.

We describe the basis functions for $\mathbb{P}^1(T)$, $\mathbb{P}^2(T)$ and $\tilde{\mathbb{P}}^2(T)$. We follow the
notations of Figure 4.3 and 4.4.
\begin{itemize}
  \item **Case** $\mathbb{P}^1(T)$. We have $N_i = \Lambda_i$.
  \item **case** $\mathbb{P}^2(T)$. We have $N_1 = \Lambda_1(2\Lambda_1 - 1)$ and similar formula for $i = 2, 3$; $N_4 = 4\Lambda_1\Lambda_4$ and similar formula for $i = 5, 6$.
  \item **Case of** $\tilde{\mathbb{P}}^2(T)$. The basis functions are $N_i = \Lambda_1(2\Lambda_1) + 3b$ and similar formula for $i = 2, 3$; $N_4 = 4\Lambda_1\Lambda_3 - 12b$ and similar formula for $i = 5, 6$; $N_7 = 27b$.
\end{itemize}

The main difference between the basis functions of $\mathbb{P}^2(T)$ and those of $\tilde{\mathbb{P}}^2(T)$ is that \[
\int_T N_\sigma dx = 0 \text{ for } \sigma = 1, 2, 3 \text{ in } \mathbb{P}^2(T) \text{ while } \int_T N_\sigma dx > 0 \text{ for any } \sigma \text{ in } \tilde{\mathbb{P}}^2(T). \] This behavior plays an important role in the next paragraph.

Other types of Lagrange interpolants could have been considered.

**4.2.1. ENO/WENO like schemes.** Consider any degree of freedom $\sigma$. It belongs to several triangles $T$, possibly only one if $r \geq 3$. Denote by $\mathcal{V}_\sigma$ the list of these triangles. We define the high order Hamiltonian as

$$
\mathcal{H}_\sigma^H := \mathcal{H}^M(Du^h_{T_k}, T_k \in \mathcal{V}_\sigma)
$$

where $\mathcal{H}^M$ is any of the low order Hamiltonians defined above by (4.1), (4.2), (4.3a) or (4.3b). Other high order Hamiltonians, such as the high order central one of [23, 16] could have been considered.

In the very early draft of this work, (4.5) has been implemented and tested. We do not present the results here, mainly because we want to stress on simplicity aspect of our derivation. In the case of ENO/WENO type scheme, the reconstruction step is very complex, especially for unstructured meshes. In the next section, we present a much simpler method for constructing high order scheme which has led us to abandon the ENO one.

**4.2.2. Compact schemes numerical Hamiltonians.** In this section, for any triangle, the interpolant $u^h$ belongs to $\mathbb{P}^1(T)$ or $\tilde{\mathbb{P}}^2(T)$.

**Derivation.** We follow [15]: in the case of an homogeneous Hamiltonian of degree $k$ in $p$, for example

$$
H(x, u, p) \equiv H(p) - f(x)
$$

with

$$
H(p) = \frac{1}{k} D_p H(p) \cdot p
$$
we can look at

\[ H(x, Du) = 0 \]

as a convection problem with source term,

\[ \vec{\lambda} \cdot Du = f(x) \]  \hspace{1cm} (4.6)

where \( \vec{\lambda} = \frac{1}{
\rho} D_p H(Du). \)

Then we consider the unsteady problem associated to this equation (without the boundary conditions),

\[ \frac{\partial u}{\partial t} + \frac{1}{
\rho} H_p(Du) \cdot Du = f(x) \]

i.e. a convection–like problem which is approximated, via the SUPG method [24], as

\[ \int_{\Omega \times [t_n, t_{n+1}]} \omega \frac{\partial u}{\partial t} + H(x, Du) dx + \int_{\Omega \times [t_n, t_{n+1}]} \left( \frac{\partial \omega}{\partial t} + DH \cdot D\omega \right) \tau \left( \frac{\partial u}{\partial t} + H(x, Du) \right) = 0. \]  \hspace{1cm} (4.7)

where \( \omega \) is any linear combination of the basis functions \( N_{\sigma}(x) \varphi(t) \) with \( \varphi \) linear in time. The positive real number \( \tau \) is typically chosen as

\[ \tau = \left( \left( \frac{2}{\Delta t} \right)^2 + \left( \frac{2 |D_p H|}{h} \right)^2 \right)^{-1/2}. \]

Noticing that

\[ \frac{\partial \omega}{\partial t} = \frac{\omega(x, t_{n+1}) - \omega(x, t_n)}{\Delta t}, \]

using (4.4) we apply mass lumping and get

\[ \frac{u_{n+1}^\sigma - u_n^\sigma}{\Delta t} + \mathcal{H}_\sigma(\sigma, u_\sigma, u_{\xi \in V_\sigma}) = 0 \]

with

\[ \mathcal{H}_\sigma(\sigma, u_\sigma, u_{\xi \in V_\sigma}) := \left( \int_\Omega H(\sigma, Du^h) N_\sigma dx \right. \]

\[ + h \int_\Omega \left[ \frac{D_p H(\sigma, Du^h)}{|D_p H(\sigma, Du^h)|} \cdot D N_\sigma \right] H(\sigma, Du^h) dx \left( \int_\Omega N_\sigma dx \right)^{-1} \]  \hspace{1cm} (4.8)
In (4.8), $h$ is the maximum diameter of the triangle of the mesh. The Hamiltonian (4.8) will be used for (1.1), when $H$ is homogeneous in $p = Du$. Modifications for the inhomogeneous case is considered later.

We have the simple Lemma which can easily be generalized to other types of interpolants.

**Lemma 4.1.** Assume $H$ is $C^1$ and consider the interpolant in $\mathbb{P}^1(T)$ or $\mathbb{P}^2(T)$ where linear functions are preserved. Away from the boundaries, relation (4.8) defines a consistent Hamiltonian in the meaning given by definition 1.1.

**Proof.** If $u$ is linear, $u^h = u$. For any degree of freedom $\sigma$, we have

$$
\left( \int_{\Omega} H(\sigma, Du^h) N_\sigma dx \right) \left( \int_{\Omega} N_\sigma dx \right)^{-1} = H(\sigma, Du(\sigma)).
$$

(4.9a)

because $Du$ is constant.

Since $\sigma$ is not on the boundary of $\Omega$, the support of $N_\sigma$ lies inside of $\Omega$ and thus

$$
\int_{\Omega} DN_\sigma dx = 0.
$$

The second term in (4.8) gives

$$
\int_{\Omega} \left[ \frac{D_p H(\sigma, Du^h)}{||D_p H(\sigma, Du^h)||} \cdot DN_\sigma \right] H(\sigma, Du^h) dx
$$

$$
= H(\sigma, Du) \left[ \frac{D_p H(\sigma, Du^h)}{||D_p H(\sigma, Du^h)||} \right] \cdot \int_{\Omega} DN_\sigma dx \quad (4.9b)
$$

$$
= 0.
$$

Using (4.9a) and (4.9b) and Definition 1.1, the Hamiltonian (4.8) is consistent.

**Remark 4.**

1. The choice of the basis functions is fundamental because $\int_{\Omega} N_\sigma dx$ needs to be non zero.

2. The proof does not depends on the fact that the Hamiltonian is homogeneous. It can be easily adapted to the case where the “dissipation” parameter $\frac{D_p H(x, Du^h)}{||D_p H(x, Du^h)||}$ is replaced by $DH$, a continuous vector-valued function.

We first review the stability properties of (4.8) for homogeneous Hamiltonians, from [15].

**Stability property.** As mentioned several times earlier in the text, the Hamiltonian (4.8) must satisfy some stability property. Here, we recall some results of [15] where
they show that this is indeed the case, at least when a particular integration scheme of (4.7) is chosen.

In [15], the Hamiltonian (4.8), introduced for Hamiltonians that are homogeneous of degree $k$ in the $Du$ variable, have the property that (4.8) and (4.7) satisfy a minimization principle for $f = 0$. Their remark is trivially extended to $f \neq 0$. This ensures a stability condition. For example, the following inequality is shown in [15]:

Setting $t_n = n\Delta t$, we have

$$
\frac{1}{2}||u(t_n^N)||^2_{X^n} + \frac{1}{2} \sum_{n=0}^{N-1} ||u(t_n^+)-u(t_n^-)||^2_{X^n} + \sum_{n=0}^{N-1} ||\sqrt{\gamma}(\partial_u H(x,u,Du))||^2_{\Omega \times [t_n,t_{n+1}]}
+ \frac{1}{2} \sum_{n=0}^{N-1} \sum_{T \in T} \int_{t_n}^{t_{n+1}} u^2 (D \cdot DH) dxdt
= \frac{1}{2}||u(t_0^-)||^2_{X^0}.
$$

(4.10)

This relation shows that the (implicit in time) Petrov Galerkin scheme (4.7) is energy stable.

**Accuracy arguments.** We show now that the scheme (4.8) is high order accurate for the steady problem (1.1) in which we omit the boundary conditions. We assume that the solution $u$ is smooth enough so that this formal calculation is indeed valid. We have, strongly, that for any $x$, $H(x,u(x),Du(x)) = 0$. So, if $I^h u$ denotes the interpolant of the exact solution, we see that the truncation error in term of the gradient of the error $e^h = u^h - I^h u$ satisfies

$$
\int_{\Omega} [H(x,u^h, Du^h) - H(x, I^h u, DI^h u)] N_\sigma +
\int_{\Omega} \left[ \frac{H_p(Du^h)}{||H_p(Du^h)||} \cdot DN_\sigma \right] [H(x,u^h, Du^h) - H(x, I^h u, DI^h u)] dx
= O(h^k) \times \left[ \int_{\Omega} N_\sigma dx \right].$

if $u - I^h u = O(h^k)$ in a suitable norm, $k = 2$ for $P^1(T)$ and $k = 3$ for $P^2(T)$. This relation indicates that $H(x,u^h, Du^h) - H(x, I^h u, DI^h u) = O(h^k)$ in the same norm, so that $De^h = O(h^k)$. This is clearly not a proof, only an indication of the formal accuracy of the scheme. We see that we get an extra order thanks to the fact that the exact solution $u$ satisfies, for any $\sigma$, the residual property

$$
\int_{\Omega} H(x,u,Du) N_\sigma dx + h \int_{\Omega} \left[ \frac{D_p H(x,u^h, Du^h)}{||D_p H(x,u^h, Du^h)||} \cdot DN_\sigma \right] H(x,u,Du) dx = 0.
$$
Case of inhomogeneous Hamiltonians. The difficult problem is to control the term
\[
\left( h \int_{\Omega} \left[ \frac{D_p H(x, Du^h)}{||D_p H(x, Du^h)||} \cdot DN_{\sigma} \right] H(x, Du^h) dx \right) \left( \int_{\Omega} N_{\sigma} dx \right)^{-1}
\] (4.11)
of (4.8). For an homogeneous Hamiltonian of the type \( H(x, u, p) \equiv H(p) - f(x) \) with
\( H \) homogeneous of degree \( > 0 \), this term clearly brings dissipation because
\[
D_p H(x, Du^h) \cdot DN_{\sigma} \; H(x, Du^h) = \frac{1}{p} \left( D_p H(x, Du^h) \cdot DN_{\sigma} \right) \left( D_p H(x, Du^h) \cdot Du^h \right)
- D_p H(x, Du^h) \cdot DN_{\sigma} \; f(x).
\]
The second term plays the role of a source term and does not affect the stability.

In the inhomogeneous case, this argument does not work any longer, but if we write
\[
H(x, u, Du^h) = \mathcal{D}H(x, u^h) \cdot (Du^h - p^{ref}) + H(x, u, p^{ref})
\]
where \( p^{ref} \) is any reference vector, if the mapping \( Du^h \mapsto \mathcal{D}H(x, u^h) \) is continuous
(so that the numerical Hamiltonian remains consistant), we can repeat formally the
stability argument by replacing (4.11) in (4.8) by
\[
\left( h \int_{\Omega} \left[ \frac{\mathcal{D}H(x, u^h)}{||\mathcal{D}H(x, u^h)||} \cdot DN_{\sigma} \right] H(x, u^h, Du^h) dx \right) \left( \int_{\Omega} N_{\sigma} dx \right)^{-1}
\]. (4.12)
A natural choice for \( \mathcal{D}H(x, u^h) \) is
\[
\mathcal{D}H(x, u^h) = \int_0^1 D_p H(x, u^h, sDu^h + (1 - s)p^{ref}) ds,
\] (4.13)
and then, instead of (4.8), we consider the numerical Hamiltonian, and the Hamiltonian
\[
\mathcal{H}_{\sigma}(\sigma, u_\sigma, u_{\xi \in \mathcal{V}_\sigma}) :=
\left( \int_{\Omega} H(\sigma, Du^h) N_{\sigma} dx \right.
\left. + h \int_{\Omega} \left[ \frac{\mathcal{D}H(\sigma, Du^h)}{||\mathcal{D}H(\sigma, Du^h)||} \cdot DN_{\sigma} \right] H(\sigma, Du^h) dx \right) \left( \int_{\Omega} N_{\sigma} dx \right)^{-1},
\] (4.14)
where \( p^{ref} \) is an arbitrary vector. Note that the accuracy arguments of the previous
paragraph still holds, as well as Lemma 4.1.
Implementation details. We end this section by giving some details about the implementation of (4.8). We follow [15]. From (4.8), we see that the numerator and the denominator can be computed following an element based approach. In pseudo code, this gives

1. initialize \( \mathcal{H}_\sigma = 0 \) and \( \psi_\sigma = 0 \) for any degree of freedom,
2. Loop over elements \( T, j_t = 1, \ldots, n_e \)
   (a) Compute \( N_\sigma \) for each degree of freedom on \( T \)
   (b) Compute \( u \) and \( Du \)
   (c) Evaluate
      \[
      \mathcal{H}_\sigma = \mathcal{H}_\sigma + \int_T H(x, u^h, Du^h) N_\sigma dx \\
      + h \int_T \frac{DH(x, u^h)}{|DH(x, u^h)|} \cdot DN_\sigma \cdot DpH(x, u^h, Du^h) \cdot Du^h dx
      \]
      via high order quadrature formula and thanks to (4.13)
   (d) Evaluate
      \[
      \psi_\sigma = \omega_\sigma + \int_T N_\sigma dx
      \]
      via high order quadrature formula,
3. Compute
   \[
   \mathcal{H}_\sigma = \frac{\mathcal{H}_\sigma}{\psi_\sigma}
   \]

Remark 5. When implementing the blended scheme (2.12a), the low order Hamiltonians are implemented by using a sub-triangulation. In the case of \( \mathbb{W}^2(T) \), we use the sub-triangulation of Figure 4.5.

4.3. Practical implementation of the scheme (2.12)-(2.13)-(2.10). The most straightforward implementation is to use the pseudo-time inconsistent scheme like
\[
\psi_\sigma + 1 = \psi_\sigma - \Delta t \mathcal{H}_\sigma
\]
where the time step is limited by a CFL type condition. Since we are computing steady solution, the time step and the time \( t_n \) are only relaxation and iterative parameters. Hence, a better strategy is to use a local time step.

Remember we are using the limiter (2.10), it is useful to see exactly how the scheme is written. Following the notations of section (2), we see that if \( \alpha_- = 1 \)
• if $r > \beta + \frac{1}{\alpha_+}, \mathcal{H}_\sigma = \gamma \mathcal{H}_\sigma^H$ with $\gamma = \frac{1}{\beta} \leq 1$ if $\beta \geq 0$,
• if $r \in [\beta, \beta + \frac{1}{\alpha_+}], \mathcal{H}_\sigma = \gamma \mathcal{H}_\sigma^H$ with $\gamma = 1 + \alpha_+ \frac{(1-r)(r-\beta)}{r} \in [0,1]$ if $\beta \geq 1$,
• if $r \in [0, \beta], \mathcal{H}_\sigma = \mathcal{H}_\sigma^H$,
• if $r \in [-1, 0], \mathcal{H}_\sigma = \gamma \mathcal{H}_\sigma^H$ with $\gamma = -\frac{\mathcal{H}_\sigma^H}{\mathcal{H}_\sigma^M} \in [0,1]$,
• if $r < -\frac{1}{\alpha_-}, \mathcal{H}_\sigma = \mathcal{H}_\sigma^M$.

Thus, we see that if $\beta \geq 1$ and $\alpha_- = 1$, we can always write $\mathcal{H}_\sigma = \mathcal{H}_\sigma^H$ or $\mathcal{H}_\sigma = \gamma \mathcal{H}_\sigma^H$ with $\sigma \in [0,1]$. The scheme, omitting the boundary conditions, is written as

$$u_{\sigma}^{n+1} = u_{\sigma}^n - \Delta t \gamma(u^n)_\sigma \begin{cases} 
\mathcal{H}_\sigma^M & \text{if } r \leq -1 \\
\mathcal{H}_\sigma^H & \text{if } r \geq -1
\end{cases} - \Delta t \times \varepsilon(h)$$

with $0 \leq \gamma(u^n) \leq 1$. This lead to the following choice of the local time step : $\Delta t_\sigma := \Delta t \gamma(u^n)_\sigma$. The optimal stability condition is

$$\Delta t_\sigma = CFL \times \begin{cases} 
\Delta t_{\sigma_{\max}}^{L} & \text{if } r \leq -1 \\
\Delta t_{\sigma_{\max}}^{H} & \text{if } r \geq -1
\end{cases} \quad (4.15)$$

where $\Delta t_{\sigma_{\max}}^{L}$ (resp. $\Delta t_{\sigma_{\max}}^{H}$) is the maximum time step allowed by the first order scheme (resp. the high order scheme).

Hence, in practice, we implement the following scheme (using the fact that $\varepsilon(h) = Ch^k$ with $C > 0$ arbitrary, so we can redefine $\varepsilon$) :

1. For an internal degree of freedom,

$$u_{\sigma}^{n+1} = u_{\sigma}^n - \Delta t_\sigma \mathcal{H}_\sigma \quad (4.16)$$
where
\[
\mathcal{H}_\sigma = \begin{cases} 
\mathcal{H}_\sigma^M + \varepsilon(h) & \text{if } r \leq -1 \\
\mathcal{H}_\sigma^H + \varepsilon(h) & \text{else} 
\end{cases}
\] (4.17)

2. For a degree of freedom on the boundary, \( u_{\sigma}^{n+1} \) is defined as the solution of
\[
\max \left( \frac{u_{\sigma}^{n+1} - u_{\sigma}^{n}}{\Delta t_{\sigma}} + \mathcal{H}_\sigma^L, u_{\sigma}^{n+1} - g(\sigma) \right) = 0
\] (4.18)

with \( \Delta t_{\sigma} = CFL \times \Delta t_{\text{max}} \).

We see that, thanks to the local time stepping strategy, we can use almost everywhere the high order unlimited scheme, without creating spurious oscillations in the solution. In order to illustrate this remark, and the way we have defined the various schemes throughout the paper, we provide now some numerical illustrations.


5.1. Computational strategy. If \( \varepsilon(h) = 0 \), a close examination of (2.2) and (2.8) reveals that if one initializes the calculation with a converged first order solution (resp. a converged second order solution), we get \( \ell = 1 \) (resp. \( \ell = 0 \)) in general. In other words, the high order scheme will provide either a first order solution or a second order solution with possibly oscillations \ldots! The role of the parameter \( \varepsilon(h) \neq 0 \) is precisely to avoid this situation.

In practice, either we initialize by a constant function and run the high order scheme from scratch. This strategy works well for the one dimensional examples below because there is no particular difficulties in the evaluation of \( H \). In the two dimensional cases, because a \( D_pH \) (resp. \( D\mathcal{H} \)) appears in the denominator of (4.8) (resp. (4.14)), we have first run the first order scheme up to a point where there is no difficulty in the evaluation of \( \mathcal{H}^H \), and then we run the high order schemes.

Except in subsection 5.3.2 where \( P^2(T) \) is used, the interpolation is always in \( P^1(T) \).

5.2. One dimensional examples. In this example, we consider the Lax–Friedrichs scheme and the one dimensional version of the high order schemes (4.8) and (4.14).

The test problem is
\[
\begin{align*}
|u'| - n(x) &= 0 \quad x \in [0,1] \\
u(0) &= u(1) = 0
\end{align*}
\] (5.1)
where \( n(x) = 3x^2 + a \) with \( a = \frac{1 - 2x_0^3}{2x_0 - 1} \) and \( x_0 = \frac{\sqrt{2} + 2}{4\sqrt{2}} \). The solution is

\[
  u(x) = \begin{cases} 
    x^3 + ax & \text{if } x \in [0, x_0] \\
    1 + a - ax - x^3 & \text{if } x \in [x_0, 1]
  \end{cases}
\]

In this example, the solution is computed by a blending between the first order Lax Friedrichs scheme and the compact scheme of section 4.2.2 adapted to the one dimensional case. The scheme should be formally second order accurate. Since the Hamiltonian \( x \mapsto |x| \) is homogeneous of degree 1, we use the original formula of [15].

We have represented the solution on Figure 5.1 for only 50 mesh points as well as zooms in \([0.4, 0.6]\) where the solution is smooth (see Figure 5.2) and around the maximum (see Figure 5.3).

We have also computed the \( L^1 \) error in \([0, 1]\) (Figure 5.4) and \( L^\infty \) error in \([\frac{1}{\sqrt{2}}, 1]\) where the solution is still smooth (Figure 5.4). We see that the expected order of accuracy is obtained. The convergence behavior of the blended scheme seems better than those of the unlimited second order scheme.

![Graph](image)

**Fig. 5.1.** Problem (5.1): exact solution and solution obtained by the blended scheme, 50 mesh points.

Then we consider a inhomogeneous and non convex, namely

\[
  H(x, p) = \cos(p)^2 + |p|.
\]
Fig. 5.2. Zoom in \([0.4, 0.6]\) for (5.1)

Fig. 5.3. Zoom around the maximum for (5.1)
The problem reads

\[ H(x, u') = 0 \text{ in } x \in [0, 1], \text{ with } u(0) = u(1) = 0. \]

On Figure 5.5, we display \( H \) in the range \( x \in [-1, 1] \). The numerical solutions are displayed in Figure 5.6 in the range \( x \in [0.5, 0.8] \). The mesh has 50 points. We
have numerically checked that the gradient of the solution may be larger that 1 in absolute value, so that non convex effects do occur. We have chose $p_{ref} = 0$ in the one dimensional version of (4.14):

![Graph showing plots of first, second order, and blended solutions.](image)

**Fig. 5.6. Plot of the first, second order and blended solutions in $x \in [0.5, 0.8]$.**

The iterative convergence has been run to machine accuracy. Once again, the blended solution lies between the first order and second order unlimited solution. In the smooth regions, the two second order solutions are indistinguishable, while at the extrema, the quality of the first order and blended solutions are comparable. We note an overshoot for the second order unlimited solution.

**5.3. Two dimensional examples.** First, we consider an example of an homogeneous Hamiltonian (the eiconal Hamiltonian) and second, a inhomogeneous, but still convex, one. We have not considered any non convex case since it seems difficult to construct example for which existence and uniqueness, with Dirichlet boundary conditions, can be proved, at least from the informations provided in [1].

**5.3.1. Eikonal equation.** We consider the eiconal equation with a Dirichlet boundary condition on the inner boundary of the geometry considered in Figure 5.7. The solution is nothing more than the distance function to this boundary. On Figure 5.8, we superimpose the first order solution, the second order unlimited solution and the limited one. This figure shows that the two second order solutions superimpose in the area where the solution is smooth, that the unlimited exhibits slight oscillations
5.3.2. Inhomogeneous case. The problem is to find $u$ in $\Omega = [0, 1] \times [0, 1] - C$ where $C = \{(x, y) \in [0, 1/2] \times [0, 1/2] \text{ and } (x - 1/2)^2 + (y - 1/2)^2 \geq 1\}$, solution of

$$
\psi(Du) - 1 = 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega.
$$

(5.2)

In (5.2), $\psi$ is defined as

$$
\psi(p) = \begin{cases} 
\frac{||p||^2 + 1}{2} & \text{if } ||p|| \leq 1 \\
||p|| & \text{else.}
\end{cases}
$$
In that case, the third order accurate interpolant of [22] is used, see section 4.2, yielding a formally third order accurate solution. Our purpose is not to check a third order accurate error behavior, but to show that other interpolation techniques are possible and the resulting scheme is still simple to implement. The vector $\mathcal{D}H$ is chosen to be $\frac{\mathcal{D}u}{||\mathcal{D}u||}$ as in the homogeneous case because $\psi' \geq 0$. The mesh is very crude, see Figure 5.9.

![Mesh for the inhomogeneous test case. Only the vertices of the mesh are displayed, the other degrees of freedom (mid-point of edges and centroids) are omitted.](image)

In Figure 5.10, we have plotted the first order (Lax Friedrich), the second order unlimited and the blended solution. On Figure 5.10 we have plotted the solution of all the degrees of freedom.

First we see a very clear improvement of the solution between the first order and higher order solution. The resolution of discontinuities is very sharp.

The second order solution is already very good. We have small wiggles, or what looks like small wiggles.

6. Conclusion, further work. In this paper, we have presented a general and simple method for increasing the order of accuracy of schemes for first order Hamilton Jacobi equations. It relies on a simple blending of a low order and a high order scheme. The structure of the blending has been studied so that the original stability properties of the original first order scheme are kept. The main difficulty is to construct high order schemes that are stable in a meaningful to precise. We have presented some examples, but our examples are clearly not optimal.

This work can be extended in two directions. The first one is about the construction of better high order schemes. There is lots of flexibility in the construction:
the only constraint is the consistency of the high order scheme, there is no conservation-related constraints as for conservation equations since these constraints are meaningless. Due to the generality of our method, we can adapt it of blend low order scheme and Discontinuous-Galerkin like schemes such as those of [14, 20]. This will be done in a future work.

The second direction is about unsteady problems. Consider

$$\frac{\partial u}{\partial t} + H(x, u, Du) = 0,$$

we can pre-discretize it in time first, for example by doing

$$\frac{3}{2}(u^{n+1} - u^n) - \frac{1}{2}(u^n - u^{n-1}) + \Delta t H(x, u^{n+1}, Du^{n+1}) = 0$$

which is a second order in time approximation of the true problem. This equation can be seen as a steady Hamilton-Jacobi problem in the variable $v := u^{n+1}$. Other,
and more accurate pre-discretisation, leading to problems like

\[
o \alpha u^{n+1} + \beta \Delta t H(x, u^{n+1}, Du^{n+1}) = F(\{u^k, k = n, n-1, \ldots, n-k\}) \equiv F(x)
\]

with \(\alpha, \beta > 0\), exist. Because our technique only need the consistency of the Hamiltonian, it can be applied in this context and will lead to high order accurate in space and time schemes. This will also be investigated in a future research.

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