Controllability of the Schroedinger equation via geometric methods

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The physical problem

External field → Quantum System (atom or molecule)
System governed by a PDE

Applications:

- induce chemical reactions (material science)
- Nuclear Magnetic Resonance (after excitation one measure the decay to obtain images)
- quantum computers ......
Two words on Quantum Mechanics

State: \( \psi \in S \subset \mathcal{H} \) (unit sphere on a complex Hilbert space)

Evolution: \( i \frac{d}{dt} \psi(t) = H_0 \psi(t) \) (Schrödinger)

\( H_0 \) is a self-adjoint operator acting on \( \mathcal{H} \) called Hamiltonian.
(If the system is isolated then \( H_0 \) does not depend on the time.)
\( \psi(t) \) is called the wave function
If \( H_0 \) has discrete (nondegenerate) spectrum: \( H_0 \phi_j = E_j \phi_j \)
\( \rightarrow \) eigenvalues \( E_0, E_1, \ldots \)
\( \rightarrow \) normalized eigenfunctions \( \phi_0, \phi_1 \ldots \)
(they form an orthonormal base of \( \mathcal{H} \), since \( H_0 \) is self-adjoint)
then \( \psi(t) = \sum_j c_j(t) \phi_j \), where \( c_j(t) = \langle \phi_j, \psi(t) \rangle \)

Physical interpretation:
\( \rightarrow \) possible results of measures (of energy) are \( E_0, E_1, \ldots \)
\( \rightarrow |c_j(t)|^2 \) is the probability of measuring energy \( E_j \).
Since \( H_0 \) is self-adjoint, \( \sum_j |c_j(t)|^2 = \text{const} = 1 \).
External fields (control): \( u_1(\cdot), \ldots u_m(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R}) \)

Evolution: \( i \frac{d}{dt} \psi(t) = H(t)\psi(t) \),

where \( H(t) = H_0 + \sum_{j=1}^{m} u_j(t)H_j \) and \( H_j \ (j = 1, 2, \ldots, m) \) represent the coupling between the system and the external fields. \( H_0 \) is called the \textbf{free} or \textbf{drift} Hamiltonian.
Two kinds of systems:

- **Atoms and molecules.** In this case $\mathcal{H}$ is infinite dimensional and typically $H_0 = -\Delta + V(x)$

- **Spin Systems.** In this case $\mathcal{H}$ is finite dimensional. For applications also finite dimensional approximation of infinite dimensional systems are important.

![Diagram showing classical potential and finite dimensional reduction]

- Finite dimensional reduction
  - finite dimensional approximation
  - only few levels are coupled with the external field
Plan

- the controllability problem for the Schrödinger equation as a PDE and what is known
- a constructive result (based on geometric techniques applied to the finite dimensional approximations)
- applications
  - harmonic oscillator
  - rotation of a planar molecule and numerical simulations

Generalizations:
- simultaneous controllability
- controllability of the density matrix
The controllability problem
The controllability problem

\[ i\dot{\psi}(t) = (A + u(t)B)\psi(t), \quad \psi(t) \in S \subset H, \quad A, B \text{ selfadjoint} \]

- \( A \) is called the drift. Typically \( A = -\Delta + V(x) \).
- \( B \) is the “control operator”

Fixed \( \psi_0 \) and \( \psi_1 \), prove that there exists a control \( u(.) \in \mathcal{U} \) that steers the system from \( \psi_0 \) to \( \psi_1 \).

- can we do it exactly/approximately?
- can we get explicit expression of controls?
- can time be short? can we evaluate their norm? can we measure their complexity?
- can we follow a given trajectory? (motion planning problem)
Controllability results for the Schrödinger equation

Finite dimension (many results, essentially based on Lie-algebraic methods)

- Agrachev, Albertini, Altafini, Beauchard, Bloch, Bonnard, Chambrion, Charlot, U.B., Brockett, Dahleh, D’Alessandro, Gauthier, Glaser, Guerin, Jauslin, Khaneja, Le Bris, Mirrahimi, Rabitz, Rouchon, Schrimmer, Sugni, Turinici, ... (controllability, optimal control, motion planning, stabilization)

...... proving that the Lie algebra generated by $A$ and $B$ is full rank......

.........then periodicity of $A$ implies controllability.........

For finite dimensional systems there are many other interesting results e.g.

- optimal control
- results for non unitary evolution
- observability
- ....
Controllability results in $\infty$ dimension

- controllability results with interior or boundary control [Zuazua 2003, Zhang-Rosier, Tenebaum-Tucsnak ...]

- Controllability results for the bilinear Schroedinger equation

\[
 i\dot{\psi} = (A + uB)\psi
\]

(negative results, up to 2003)

- non-exact controllability in $S \cap H^2 \cap H^1_0$ 
  (Ball-Marsden-Slemrod, Turinici);
- non controllability of the linearization (Rouchon)
- non controllability of the harmonic oscillator driven by $u(.)x$ (Mirrahimi Rouchon)

→ all finite dimensional approximations of the harmonic oscillator are controllable
positive results (2003–)

- Control of the Eberly-Law System (Brockett-Bloch-Rangan, see also Ervedoza-Puel).
- exact contr. in $H^7(\Omega)$ in $\Omega = (-1/2, 1/2) + \text{Dirichlet B.C.}$, for $i\partial_t \psi(t, x) = (-\partial_x^2 + u(t)x)\psi(t, x)$ (Beauchard-Coron);
- approximate controllability by adiabatic methods using intersections of eigenvalues (Adami, Mason, Sigalotti, Chittaro, U.B.);
- approximate controllability via geometric methods
- approximate controll. by Lyapunov methods
  → Mirrahimi (mixed spectrum)
  → Nersesyan ($H^s$-approximate controllability for discrete spectrum)
Part 2: a controllability result
The abstract result

“generically” all bilinear Schrödinger equations (with discrete spectrum) are controllable


Mathematical framework

We will consider control systems of the form

$$i \frac{d\psi}{dt} = (A + uB)\psi, \quad \psi \in \mathcal{S} \subset H, \quad u \in U, \quad (A, B, U)$$

with the following ingredients

- $H$ complex Hilbert space;
- $U \subset \mathbb{R}$;
- $A, B$ self-adjoint operators on $H$ possibly unbounded;
- $D(A), D(B)$ domains of $A, B$;
- $(\phi_n)_{n \in \mathbb{N}}$ orthonormal basis of $H$ made of eigenvectors of $A$;
- $\phi_n \in D(B)$ for every $n \in \mathbb{N}$.

The hypotheses above guarantee that $\forall u \in U$ the sum $A + uB$ is well defined on span($\phi_n$)$_{n \in \mathbb{N}}$ and can be uniquely extended to a self-adjoint operator. In particular $e^{it(A+uB)} : H \to H$ is a well-defined group of unitary transformations.
Solutions

If $B$ is unbounded, what is a solution to the control system, corresponding to a certain time varying control is not a trivial problem.

Since we would like to work with $B$ unbounded, we use a very primitive class of control functions, i.e. piecewise constant.

We call $\psi : t_k \mapsto e^{it_k(A+u_kB)} \circ \cdots \circ e^{it_1(A+u_1B)}(\psi_0)$ the solution of the control system $(A, B, U)$ starting from $\psi_0$ associated to the piecewise constant control $u_1 \chi_{[0,t_1]} + u_2 \chi_{[t_1,t_1+t_2]} + \cdots$

If $B$ is bounded and $u \in L^1([0,T], U)$, then there exists a unique weak (and mild) solution $\psi \in C([0,T], H)$ which coincides with $\psi(\cdot)$ when $u$ is piecewise constant. Moreover, if $\psi_0 \in D(A)$ and $u \in C^1([0,T], U)$ then $\psi$ is differentiable and it is a strong solution [Ball-Marsden-Slemrod]
Approximate controllability

As recalled Ball-Marsden-Slemrod (adapted by Turinici to the skew-adjoint case) proved that exact controllability does not hold in infinite dimension.

Definition
We say that \((A, B, U)\) is **approximately controllable** if for every \(\psi_0, \psi_1 \in S\) and every \(\varepsilon > 0\) there exist a \(u : [0, T] \to U\) such that

\[
\|\psi_1 - \Upsilon^u_T(\psi_0)\| < \varepsilon
\]

\[
\Upsilon^u_T = e^{it_k(A+u_k B)} \circ \ldots \circ e^{it_1(A+u_1 B)}
\]

Our aim is to find general conditions that guarantee the approximate controllability.
The Geometric Technique

\[ i\dot{\psi}(t) = (A + u(t)B)\psi(t), \quad \psi(t) \in S \subset H, \quad A, B \text{ selfadjoint} \]

Crucial hypotheses:

- The only crucial hypothesis is that \( A \) has discrete spectrum (then we have “generic” conditions)
- \( B \) need not to be bounded
- \( A \) does not need to be \(-\Delta + V(x)\) on a bounded or unbounded domain of \( \mathbb{R}^n \) (ex the Lapace Beltrami on a Riemannian manifold or the Eberly and Law operator)

Features

- Based on geometric techniques applied to the Galerkin approximations
- It is constructive (we can provide explicit expressions of controls)
- It can be generalized to study
  - Simultaneously several systems with the same control
  - Systems for which the initial condition is not known precisely (control of the density matrix)

Proof: finite dimensional technique applied to the finite dimensional approximations
Main result

$(\lambda_n)_{n \in \mathbb{N}}$ eigenvalues of $H_0$ corresponding to $(\phi_n)_{n \in \mathbb{N}}$.

**Theorem (approximate controllability)**

*If there exists a non resonant chain of connectedness then, for every $\delta > 0$, for every $\epsilon > 0$, For every $\psi_0, \psi_1$, there exist a piecewise constant control $u : [0, T] \rightarrow [0, \delta]$ such that*

$$\|\psi_1 - \Upsilon^u_T(\psi_0)\| < \epsilon$$

$$\begin{align*}
\lambda_0 & \xrightarrow{\langle \phi_0, B\phi_3 \rangle \neq 0} \lambda_3 \\
& \quad \xrightarrow{\langle \phi_3, B\phi_2 \rangle \neq 0} \lambda_2 \\
& \quad \xrightarrow{\langle \phi_2, B\phi_4 \rangle \neq 0} \lambda_4 \\
|\lambda_3 - \lambda_0| & \neq |\lambda_2 - \lambda_3| \neq |\lambda_4 - \lambda_2| \neq \ldots \neq "all the other nontrivial couplings" \\
\rightarrow & \text{ approximate controllability with small positive controls}
\end{align*}$$
The existence of a non-resonant chain of connectedness

is “generic” for systems of the form $-\Delta + V(x) + uW(x)$ (Mario Sigalotti and Paolo Mason, 2010)

if it is not satisfyed for $A + uB$ it may be satisfied for $(A + \mu B) + (u - \mu)B$ very useful for academinc examples, e.g.:

→ harmonic oscillator $(2k + 1)$
Time and $L^1$ estimates

- **Time estimates**

$$T_u \geq \frac{1}{\delta} \sup_{k \in \mathbb{N}} \frac{||\langle \phi_k, \psi_0 \rangle - \langle \phi_k, \psi_1 \rangle|| - \epsilon}{\left( \sum_j |\langle \phi_j, B\phi_k \rangle|^2 \right)^{1/2}},$$

→ upper bounds are very difficult to obtain

- **Example of a $L^1$ estimate**

If there the non resonant chain contains $(1, 2)$, then

$$\|u\|_{L^1} = \int_0^T |u(t)| \, dt \leq \frac{5\pi}{4|\langle \phi_1, B\phi_2 \rangle|}$$

→ at the moment we do not have estimates on the ”total variation” of the control (roughly speaking the number of switchings)
Main steps in the proof

Techniques:

- finite dimensional technique applied to the finit dimensional approximations of order $n$. Getting $u^n(.)$.
- trying to use $u^n(.)$ in the infinite dimensional system. It is not working since we cannot control the tails.
- we substitute $u^n(.)$ with a sequence of controls $u^n_k(.)$ such that in the limit the $N - n$ tails are zero ($N > n$).
- passing to the limit for $N$ and $n$ tending suitably to infinity.

The technique is constructive
Part 2: Examples
Example: 1D harmonic oscillator

Consider

\[ i\partial_t \psi(t, x) = (-\partial_x^2 + x^2 + u(t)W(x))\psi(t, x), \]

where \( x \in \mathbb{R} \).

Mirrahimi and Rouchon proved the non-controllability of the system when \( W(x) = x \).

The spectrum of \(-\partial_x^2 + x^2\) is

\[ \{ \lambda_k = 2k + 1 \mid k \geq 0 \}, \]

and therefore there is no a nonresonant chain of connectedness.

Even if the non-resonance hypotheses are not satisfied by the operators \( A \) one can anyway check them for \( A_\mu = -\partial_x^2 + x^2 + \mu W \).

Although the spectrum of \( A_\mu \) is not in general explicitly computable, we can nevertheless deduce its non-resonance by standard perturbation arguments (for suitable \( W \)).
Example: 1D harmonic oscillator

The derivative of $\lambda_k(u)$ with respect to $u$ at $u = 0$ is

$$
\lambda_k'(0) = \int_{\mathbb{R}} \phi_k(x)^* W(x) \phi_k(x) \, dx.
$$

Notice that $\phi_k(x)$ is explicitly known (in terms of Hermite polynomials). If the elements of the sequence $(\lambda_{k+1}'(0) - \lambda_k'(0))_{k \in \mathbb{N}}$ are $\mathbb{Q}$-linearly independent then for almost every $\mu$ the elements of the sequence $(\lambda_{k+1}(\mu) - \lambda_k(\mu))_{k \in \mathbb{N}}$ are $\mathbb{Q}$-linearly independent as well.

We look for $W$ such that $(\lambda_k'(0))_{k \in \mathbb{N}}$ is non-resonant. In the Mirrahimi-Rouchon case, $\lambda_k'(0) = 0$ for every $k$.

**Proposition**

(1) If $W$ is even, then the system is not approximately controllable.

(2) If $W$ has the form $W : x \mapsto e^{ax^2+bx+c}$, with $a, b, c \in \mathbb{R}$ such that $a < 0$ and the two numbers $\sqrt{1 - a}$ and $b$ are algebraically independent, then the system is approximately controllable.
Controllability of the orientation of a quantum planar molecule

\[ i \frac{\partial \psi(\theta, t)}{\partial t} = \left( -\frac{\partial^2}{\partial \theta^2} + u(t) \cos(\theta) \right) \psi(\theta, t), \quad \theta \in S^1 \]
Eigenvalues, Eigenvectors, Coupling

Eigenvalues of \( A = -\partial_\theta^2 \) on \( S^1 \): 0, 1, 1, 4, 4, 9, 9, \ldots

Corresponding eigenvectors: \( \phi_m(\theta) = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } m = 0, \\ \frac{\cos(m\theta/2)}{\sqrt{\pi}} & \text{if } m > 0 \text{ even}, \\ \frac{\sin((m+1)\theta/2)}{\sqrt{\pi}} & \text{if } m \text{ odd}. \end{cases} \)

Coupling: \( b_{jk} = \langle B\phi_j, \phi_k \rangle = \int_{S^1} \cos(\theta)\phi_j(\theta)\phi_k(\theta) d\theta, \)

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0 & 0 & \frac{1}{\sqrt{2}} & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{2} & 0 & \cdots \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots \\
0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},
\]

\( 0 \leftrightarrow 2 \leftrightarrow 4 \leftrightarrow 6 \ldots \)
\( 1 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 7 \ldots \)

- the coupling hypothesis is not satisfied \( \Rightarrow \) the parity of \( \psi \) cannot change \( \Rightarrow \) no global controllability
We just look at the even part

$$A = \begin{pmatrix} 0 & 0 & \ldots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 4 & \cdots \\ \vdots & \ddots & \ddots & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & \ldots \\ 1/\sqrt{2} & 0 & 1/2 & \cdots \\ 0 & 1/2 & 0 & 1/2 \\ \vdots & \ddots & 1/2 & 0 \end{pmatrix}$$

We try to steer the system from the first even eigenstate to the second even eigenstate

→ the system is controllable since $\lambda_{k+1} - \lambda_k = (k + 1)^2 - k^2 = 2k - 1$ and all of them are different.
Numerical simulations to go from the first to the second even eigenstates
Moduli of the first coordinates for $0 \leq t \leq 20$
First coordinates for $0 \leq t \leq 20$
Second coordinate for $0 \leq t \leq 420$
Moduli of coordinates 1, 2, 3, 8, 10 for $0 \leq t \leq 420$
Generalizations
Generalizations 1: Control of systems with the same control

Often in physics you would like to control more than one system with the same control (e.g., two types of molecule)

\[ i\dot{\psi}_1 = (A_1 + u(t)B_1)\psi_1 \]
\[ i\dot{\psi}_2 = (A_2 + u(t)B_2)\psi_2 \]

Theorem (Thomas Chambrion)

*If each of the two systems satisfy the hypotheses of the controllability theorem and the concatenation of the two spectra is $\mathbb{Q}$-linearly independent, then we have simultaneous approximate controllability.*

Recall that \((\lambda_{n+1} - \lambda_n)_{n \in \mathbb{N}}\) are $\mathbb{Q}$-linearly independent (or non-resonant) if for every $N \in \mathbb{N}$ and \((q_1, \ldots, q_N) \in \mathbb{Q}^N \setminus \{0\}\) one has

\[ \sum_{n=1}^{N} q_n (\lambda_{n+1} - \lambda_n) \neq 0. \]
A density matrix is a weighted sum of projectors,

\[ \rho = \sum_{j=1}^{\infty} P_j \Pi_j, \]

where \( P_j \in [0, 1] \), \( \sum_j P_j = 1 \), and \( \Pi_j \) is the orthogonal projector on the space spanned by a unitary vector \( \varphi_j \in H \), i.e. \( \Pi_j = \varphi_j \varphi_j^* \).

Density matrices are used when

- the initial wave function is not known precisely; or
- one deals with an ensemble of identical systems that cannot be prepared precisely in the same state.

Without loss of generality it is possible to require that \( \{ \varphi_j \}_{j \in \mathbb{N}} \) is an orthonormal basis.
The time evolution of the density matrix is

\[ \rho(t) = U(t)\rho(0)U^*(t) \]

where \( U(t) \) is the operator of temporal evolution.

**Definition**

Two density matrices \( \rho_0 \) and \( \rho_1 \) are said to be *unitarily equivalent* if there exists a unitary transformation \( U \) of \( H \) such that \( \rho_1 = U\rho_0U^* \).

**Theorem**

Let \( \rho_0 \) and \( \rho_1 \) be two unitarily equivalent density matrices. Then, under the hypotheses of the approximate controllability theorem, for every \( \varepsilon > 0 \) there exist \( k \in \mathbb{N}, t_1, \ldots, t_k > 0 \) and \( u_1, \ldots, u_k \in (0, \delta) \) such that setting

\[ V = e^{it_k(A+u_kB)} \circ \ldots \circ e^{it_1(A+u_1B)} \],

one has \( \|\rho_1 - V\rho_0V^*\| < \varepsilon \).
Concluding remarks

- A sufficient criterion for approximate controllability of the Schroedinger equation as a PDE.
  - it works for generic systems
  - arbitrary small controls.

- It provides
  - an explicit construction of the control (effective numerical computations);
  - easily computable estimates of the $L^1$ norm of the control.

- it works for the density matrix.

- it works for simultaneous controllability
The end
Scheme of the proof

CRUCIAL IDEAS:
- Diagonalization of $B$
- elimination of the drift

STEPS
- STEP 1: time reparameterization;
- STEP 2: controllability of the Galerkin approximations;
- STEP 3: elimination of the drift and controllability of higher order Galerkin approximations;
- STEP 4: approximate controllability of the original system
STEP 1: TIME REPARAMETRIZATION

\[ i\dot{\psi}(t) = (A + u(t)B)\psi(t) \]

- we need to diagonalize \( B \) (on a countable base)
- we need to eliminate the drift \( A \)

Problem: \( B \) in general has no discrete spectrum
For $u \neq 0$, clearly, $A + uB = u \left( \left( \frac{1}{u} \right) A + B \right)$. Hence

$$i \frac{d\psi}{dt} = (uA + B)\psi \quad u \in (0, \delta) \quad \Leftrightarrow \quad i \frac{d\psi}{d(tu)} = \left( \frac{1}{u} A + B \right)\psi \quad u \in (0, \delta)$$

(1)

With the change of notation $tu \to t$, $1/u \to u$, we get

$$i \frac{d\psi}{dt} = (uA + B)\psi, \quad u \in (1/\delta, \infty)$$

(2)

and the approximate-controllability theorem for (1) is therefore equivalent to the approximate controllability of (2).
STEP2: Controllability of the Galerkyn approximations

For $j, k \in \mathbb{N}$, let $a_{jk}$ and $b_{jk}$ be the components of $A$ and $B$ in the base $(\phi_m)_{m \in \mathbb{N}}$.

Galerkyn approximation of order $n$:

$$\frac{d\psi}{dt} = u A^{(n)} x + B^{(n)} \psi, \quad \psi \in S_n, \quad u > \delta,$$

where $S_n$ denotes the unit sphere of $\mathbb{C}^n$.

$\lambda_2 - \lambda_1, \ldots, \lambda_n - \lambda_{n-1}$ $\mathbb{Q}$-linearly independent $+$ $b_{j,j+1} \neq 0$

$\Downarrow$

the Lie algebra generated by $A^{(n)}$ and $B^{(n)}$ has max. dimension

$\Downarrow$

$(\Sigma_n)$ is controllable.
STEP 3: Controllability of higher order Galerkin approximations

If $u(.)$ is the control steering the Galerkin approximation from the initial to the final state, then one would like to use it in the infinite dim. system. Unfortunately, it does not work because one cannot control the tails.

- Elimination of the drift
- We substitute $u(.)$ with a sequence of controls $u_k(.)$ such that in the limit the $N - n$ tails are zero ($N > n$)
If $t \mapsto \psi(t)$ is a solution of $(\Sigma_N)$ corresponding to a control function $u(.)$ and
\[ v(t) = \int_0^t u(\tau) d\tau, \]
then
\[ Y(t) = e^{-v(t)A^{(N)}} \psi(t) \]
is a solution of
\[ \dot{Y} = M(t)Y(t), \quad \text{where} \quad M(t) = e^{-v(t)A^{(N)}} B^{(N)} e^{v(t)A^{(N)}} \]
control of the $N - n$ tails

**Claim**

There exists a sequence $u_k : [0, T] \to (\delta, \infty)$ piecewise constants such that the sequence

$$t \mapsto M_k(t) = e^{-v_k(t)A^{(N)}} B^{(N)} e^{v_k(t)A^{(N)}} ,$$

where $v_k(t) = \int_0^t u_k(\tau) d\tau$, converges to

$$t \mapsto M(t) = \begin{pmatrix}
    e^{-v(t)A^{(n)}} B^{(n)} e^{v(t)A^{(n)}} & 0_{n \times (N-n)} \\
    0_{(N-n) \times n} & G(t)
\end{pmatrix},$$

where $v(t) = \int_0^t u(\tau) d\tau$, $G(t)$ is continuous and $M_k \to M$ in the following integral sense,

$$\int_0^t M_k(\tau) d\tau \to \int_0^t M(\tau) d\tau$$

as $k \to \infty$ uniformly with respect to $t \in [0, T]$. 
STEP 4: controllability of the original system

- one easily gets approximate controllability for moduli of the components of the wave function (i.e. controllability of the probability)
- phases can be controlled with quick pulses
Genericity is a measure of frequency and robustness.

A complete metric space \( X \) \( \Rightarrow \) a Baire space, i.e., \( \bigcap_{n \in \mathbb{N}} O_n \) is dense if each \( O_n \subset X \) is open and dense.

**Residual set**: intersection of countably many open and dense subsets of a Baire space

A boolean function \( P : X \rightarrow \{0, 1\} \) on a Baire space \( X \) is called a **generic property** if there exists a residual subset \( Y \) of \( X \) such that every \( x \) in \( Y \) satisfies property \( P \), that is, \( P(x) = 1 \).

The sufficient conditions for controllability are in the form of a **countable family of non-vanishing relations**. The idea is then to associate to each of them a set \( O_n \).
The hypotheses of the controllability theorem are verified generically for:

- with respect to $W$ A.C. for fixed $\Omega$ and $V$ A.C.
- with respect to $W$ A.C. for fixed $\Omega$ and $V$, A.C.
- with respect to $\Omega \in C^m$ for fixed $V, W$ A.C.

We endow these spaces with the $C^m$ (for the boundary of $\Omega$) and $L^\infty$ for $V$ and $W$. 