Model Predictive Control of Nonholonomic Vehicle Formations

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1. The Problem
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The Problem

**To control a formation of nonholonomic vehicles**

Objective: devise trajectories, and corresponding controls for all vehicles, maintaining some (pre-specified) relative positions between them.

Nonholonomic systems At a given instant cannot move in certain directions (although they can reach any point in the state space). Example: wheeled vehicles, airplanes, robot manipulators, ...
Which are the main challenges of this problem?

- **Nonholonomy**: Simple controllers (with continuous feedback) cannot, in general, stabilize the system.
- **Constraints**: Input saturation, collision avoidance (between vehicles and with obstacles).
- **Distributed control**: Stability of the formation, communication delays or failure, computational burden in each agent.
Why is MPC a good tool for this problem?

- **MPC can overcome nonholonomy challenges**
  - It involves *planning*, not just reactive control.
  - Can generate required nonlinear, discontinuous feedback.

- **MPC can overcome constraints challenges**
  - It is known to be a (if not the main) technique to deal appropriately with constraints.
  - Simply dealt with within the optimization

- **Appropriate for distributed control**
  - Computational burden can be distributed among agents, while kept at feasible levels
  - Overall stability can be guaranteed
  - In case of communication delays or failures, we have the computed open-loop control as backup plan

Moreover

- It has desirable performance characteristics (optimization-based method)
- It has intrinsic robustness
Nonholonomic Systems

Are completely controllable but instantaneously cannot move in certain directions (Ex: wheeled vehicles, robot manipulators, rolling sphere, ...)

Example: A car-like vehicle

\[
\begin{align*}
\dot{x} &= v \cdot \cos \theta \\
\dot{y} &= v \cdot \sin \theta \\
\dot{\theta} &= v \cdot c
\end{align*}
\]

with control inputs \( v \) and \( c \) satisfying

\[
v \in [0, v_{\text{max}}] \quad \text{and} \quad c \in [-c_{\text{max}}, c_{\text{max}}].
\]

(Minimum turning radius \( R_{\text{min}} = c_{\text{max}}^{-1} \))

- Nonholonomic constraint: velocity vector is orthogonal to the wheel axle
  \((\dot{x}(t), \dot{y}(t))^T (\sin \theta(t), -\cos \theta(t)) = 0\)
- Linearizing around any point \((x_0, y_0, \theta_0, v = 0)\) results in an uncontrollable system.
- Cannot be stabilized by a continuous (time-invariant) feedback.
Cannot be stabilized by a continuous feedback.

Suppose \( x = 0 \) and \( \theta = \pi \)
If \( y \geq 2R_{min} \) ⇒ turn left.
If \( y < 2R_{min} \) ⇒ turn right.
which is a discontinuous feedback.
Cannot be stabilized by a continuous feedback.

Suppose $x = 0$ and $\theta = \pi$
If $y \geq 2R_{\text{min}} \Rightarrow$ turn left.
If $y < 2R_{\text{min}} \Rightarrow$ turn right.
which is a discontinuous feedback.

More precisely ...

**Theorem (Brockett 83)**

*If the system $\dot{x} = f(x, u)$, with $u \in U$, admits a continuous stabilizing feedback, then for any neighbourhood $\mathcal{X}$ of zero, the set $f(\mathcal{X}, U)$ contains a neighbourhood of zero.*

In the car example, if
$\mathcal{X} := \{(x, y, \theta) : |x| < 1, |y| < 1, |\theta| < \pi/6\}$
then
$f(\mathcal{X}, U)$ has no points of the form $(\dot{x}, \dot{y}, \dot{\theta})$ with $\dot{x} < 0$. 
The trajectory $x$, solution to

$$\dot{x}(t) = f(x(t), k(x(t)))) \quad x(0) = x_0$$

is only defined, in classical (Caratheodory) sense, if (among other requirements) if the right-hand side is a locally Lipschitz continuous function of $x$.

Using Filippov solution-concept some controllable systems cannot be stabilized. (Ryan 94, Coron&Rosier 94).

Problem solved if we use “sampling-feedback” solution concept (Clarke,Ledyaev,Sontag,Subbotin 97)
Discontinuous Feedback: “sampling-feedback” solution

(Clarke, Ledyaev, Sontag, Subbotin IEEE TAC97)

- Take a sequence of sampling instants \( \pi := \{t_i\}_{i \geq 0} \) in \([0, +\infty)\) with \( t_0 < t_1 < t_2 < \ldots \)

\[
\dot{x}(t) = f(x(t), k(x(\lfloor t \rfloor_\pi))), \quad x(0) = x_0
\]

where \( \lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\} \)

i.e. The feedback is not a function of the state at every instant of time, rather it is a function of the state at the last sampling instant.

This solution concept combines naturally with our MPC sampled-data framework.
The Problem
- Challenges
- Control of Nonholonomic Systems

The MPC framework

Control of vehicle formations

Path-following

Mission objective
Consider a sequence \( \{t_i\}_{i \geq 0} \) s.t. \( t_{i+1} = t_i + \delta, \ \delta > 0 \):

1. Measure state of the plant \( x_{t_i} \)
2. Get \( \bar{u} : [t_i, t_i + T] \mapsto \mathbb{R}^m \) solution to the OCP:

\[
\begin{align*}
\text{Minimize} & \quad \int_{t_i}^{t_i+T} L(t, x(t), u(t)) \, dt + W(x(t_i + T)) \\
\text{subject to} & \quad \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [t_i, t_i + T] \\
& \quad x(t_i) = x_{t_i} \\
& \quad u(t) \in U(t) \quad \text{a.e. } t \in [t_i, t_i + T] \\
& \quad x(t) \in X \quad \text{a.e. } t \in [t_i, t_i + T] \\
& \quad x(t_i + T) \in S
\end{align*}
\]

3. Apply to the plant the control \( u^*(t) := \bar{u}(t) \) in the interval \([t_i, t_i + \delta]\). (the remaining control \( \bar{u}(t), \ t > t_i + \delta \) is discarded)

4. Repeat for the next sampling time \( t_i = t_i + \delta \)
A Sufficient Condition for Stability

Choose the design parameters: time horizon $T$, objective functions $L$ and $W$, and terminal constraint set $S$, to satisfy:

**SC1–SC4** $S$ is closed, contains the origin is reachable in time $T$... $W$ is locally Lipschitz continuous and positive definite ...

**SC5** $\forall x_t \in S$, $\exists \tilde{u} : [t, t + \delta] \rightarrow U$ [or $\tilde{u}_t \in U$] such that

$$W(x(t + \delta; t, x_t, \tilde{u})) - W(x_t) \leq - \int_t^{t+\delta} L(x(s), \tilde{u}(s)) ds,$$  

**(SC5a)**

$$[\text{or } \langle \xi, f(x_t, \tilde{u}_t) \rangle \leq -L(x_t, \tilde{u}_t), \forall \xi \in \partial P W(x_t)]$$

and

$$x(s; t, x_t, \tilde{u}) \in S, \quad s \in [t, t + \delta).$$  

**(SC5b)**

$$[\text{or } \langle \xi, f(x_t, \tilde{u}_t) \rangle \leq 0, \forall \xi \in \mathcal{N}_S^P(x_t)]$$

Then, the MPC strategy is stabilizing (i.e. for a sufficiently small inter-sample time $\delta$ we have $\|x^*(t)\| \rightarrow 0$ as $t \rightarrow +\infty$).
Consider the system
\[
\begin{align*}
\dot{x}(t) &= (u_1(t) + u_2(t)) \cdot \cos \theta(t) \\
\dot{y}(t) &= (u_1(t) + u_2(t)) \cdot \sin \theta(t) \\
\dot{\theta} &= (u_1(t) - u_2(t)).
\end{align*}
\]
with \( \theta(t) \in [-\pi, \pi] \), and the controls \( u_1, u_2(t) \in [-1, 1] \).

Stability achieved choosing:

- \( S \) to be the set of points s.t. the heading angle \( \theta \)
  is pointing towards the origin of the plane \( (x, y) = (0, 0) \).

- \( T = \pi/2 \).

- \( L(x, y, \theta) := x^2 + y^2 + \theta^2 \),

- \( W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t))dt \)

where \( \bar{t} \) is the time to reach the origin.

SC is satisfied. MPC is stable.
Example: A car-like vehicle

Consider the system

\[
\begin{align*}
\dot{x} &= v \cdot \cos \theta \\
\dot{y} &= v \cdot \sin \theta \\
\dot{\theta} &= v \cdot c
\end{align*}
\]

with control inputs \(v\) and \(c\) satisfying

\[v \in [0, v_{\text{max}}]\] and \(c \in [-c_{\text{max}}, c_{\text{max}}]\).

Stability achieved choosing \(S\) as in picture

\[T = 2\pi R_{\text{min}} / v_{\text{max}}.\]

\[L(x, y, \theta) := x^2 + y^2 + \theta^2,\]

\[W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t)) dt\]

where \(\bar{t}\) is the time to reach the origin.

SC is satisfied. MPC is stable.
Problem Objective

What do we mean by “to control”?

- point-to-point motion (what we’ve seen so far)
- trajectory tracking (much easier)
- path-following (one more degree of freedom)
- accomplish mission objective (why not explore further degrees of freedom?)

**Trajectory tracking:** The aim is to follow, as close as possible, a trajectory in state space (i.e., a geometric path in the cartesian space together with an associated timing law) starting from a given initial configuration.

**Path following:** The robot must reach and follow a geometric path in the cartesian space starting from a given initial configuration, but no associated timing law is specified.
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Control of formations

**Method:** A two-layer model predictive control (MPC) scheme

- The top layer controls the trajectory of the whole group of vehicles. *Based on Nonlinear MPC allowing discontinuous feedbacks.*
- The bottom layer controls the relative position between vehicles. *Based on MPC with a linearized model (solves constrained LQR).*
Why two layers?

Because there are two intrinsically different control problems:

- The nonholonomic vehicles require a nonlinear controller allowing discontinuous feedbacks (point-to-point motion).
- However, while in motion, the relative position between the vehicles in a formation can be changed in all directions (as if they were holonomic). Can be controlled by a linear controller. (trajectory tracking)

In some robotics literature called leader-follower formation control.

**Example:** Consider a car in a parking manoeuvre (nonholonomic), or in an overtaking manoeuvre (can move in all directions relative to the other car).
The two layer MPC scheme

- The top layer, the trajectory controller, applied to the group of vehicles as a whole.
  - Is a nonlinear MPC controller allowing discontinuous feedbacks. The control is computed centrally and a unique control is broadcasted to all vehicles.

- The bottom layer, the formation controller, applied to each vehicle individually.
  - Aims to compensate for small changes around a nominal trajectory maintaining the relative positions between vehicles
  - can be adequately carried out by a linear controller accommodating input and state constraints
  - This has the advantage that the control laws for each vehicle are much simpler to compute (are piecewise affine feedback laws and can even be pre-computed off-line [Bemporad, Morari, Dua, Pistikopoulos 2002]) They are computed and implemented in a distributed way in each vehicle.

In the end the two controls are added and applied to the actuators in each vehicle.
Our vehicle: A differential-drive mobile robot

\[
\begin{align*}
\dot{x}(t) &= (u_1(t) + u_2(t)) \cdot \cos \theta(t) \\
\dot{y}(t) &= (u_1(t) + u_2(t)) \cdot \sin \theta(t) \\
\dot{\theta} &= (u_1(t) - u_2(t)).
\end{align*}
\]

with constrained controls \( u_1, u_2(t) \in [-1, 1] \).
Linearized vehicle model for the Formation Controller

We consider a linearized model for each of the differential drive mobile robots:

- with the $z_1$ axis aligned with the velocity of the *reference vehicle*
- for the nominal velocity $v_n = (u_{1n} + u_{2n})/2$, with $u_{1n} = u_{2n}$ being the nominal velocities of the wheels.

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix} =
\begin{bmatrix}
0 & w_n & 0 \\
-w_n & 0 & v_n \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} +
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v \\
w
\end{bmatrix}
\]
Formation Model

- Consider $M$ vehicles and a reference vehicle, leader (that might not exist) that follows exactly the trajectory predicted by the trajectory controller.
- Consider that the $M + 1$ vehicles are nodes of a directed graph $G = (V, E)$, that is a directed tree rooted at the reference vehicle. That is, all vertices are connected, there are no cycles, and all edges are directed in a path away from the root.

- To each vertex $i$ associate a triple $z^i = (z^i_1, z^i_2, z^i_3)$ with the relative position w.r.t. the reference vehicle.
- To each edge $(i, j) \in E$ associate the pair $\tilde{z}^{ij} = (\tilde{z}^{ij}_1, \tilde{z}^{ij}_2)$ with the desired relative position in the plane of vehicle $i$ with respect to its parent node $j$.

The performance index for each vehicle $i$ with parent $j$ is

$$J^i(z) = \int_{t=0}^{T} \|(\tilde{z}^{ij}_1, \tilde{z}^{ij}_2, 0) - (z^i(t) - z^i(t))\|^2_Q + \|(v(t), w(t))\|^2_R dt$$

$$\| (\tilde{z}^{ij}_1, \tilde{z}^{ij}_2, 0) - (z^i(N) - z^i(N))\|^2_P$$
Stability of the formation

- If we choose the design parameter $L, W, T,$ and $S$ for the trajectory controller as seen previously, and
- if we choose the matrices $Q$ and $R$ as p.d. and $P$ solving the Riccati equation $A'P + PA - PB R^{-1} B'P + Q = 0$, in the objective function of the formation controller. Then,
- The trajectory of the reference vehicle is stable.
- The trajectory of each vehicle is stable with respect to the desired relative position to its parent vehicle.
- Since there is exactly one path from each vehicle to the reference vehicle, stability of any vehicle follows recursively
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Path-following

Given a path parameterized by \( \tau \mapsto (x_d(\tau), y_d(\tau)) \) for \( \tau \in [0, T_d] \):

1. Find an initial point \( Q_0 \) in the path to start the trajectory of the virtual reference vehicle. Given the initial (current) position of our vehicle \((x_0, y_0)\), find the path parameter value that corresponds to a point in the path that is the closest to \((x_0, y_0)\).

\[
\tau_0 = \text{argmin}_{\tau \in [0, T_d]} \| (x_d(\tau), y_d(\tau)) - (x_0, y_0) \|
\]

2. Select a speed profile at which the path is to be followed

\[ \nu : [0, T] \mapsto IR_+ \]

3. Find a feedback control to track the trajectory of the virtual reference vehicle \( t \mapsto (x_r(t), y_r(t)) \)
Path-following with MPC

Minimize over $\nu \in \mathbb{R}_+$ and $u$

$$
\int_0^{T_d/\nu} \left( \|q(t) - q_d(\tau(t))\|^2_Q + \|u(t)\|^2_R \right) dt + \|q(T_d/\nu)\|^2_P
$$
subject to:

\begin{align*}
\dot{q}(t) &= f(t, q(t), u(t)) & \text{a.e.} t \in [0, T_d/\nu] \\
q(0) &= q_{t_i} \\
\tau_0 &= \arg\min_{\tau \in [0, T_d]} \|q_d(\tau) - q_{t_i}\| \\
\tau(t) &= \tau_0 + \nu t \\
u(t) &\in U & \forall t \in [0, T_d/\nu].
\end{align*}

where the path is parameterized by

$$
t \mapsto q_d(\tau), \quad \tau \in [0, T_d]$$
Path-following example

Consider a differential-drive robot to follow an 8-shaped path

\[
\begin{align*}
    x_d(\tau) &= \sin(\tau/10), \quad \tau \in [0, 38\pi] \\
    y_d(\tau) &= \sin(\tau/20), \quad \tau \in [0, 38\pi]
\end{align*}
\]

By selecting \( \nu \) and an initial point in the path we define the trajectory for a virtual reference vehicle

\[
\begin{align*}
    x_r(t) &= \sin(\tau(t)/10) \\
    y_r(\tau) &= \sin(\tau(t)/20) \\
    \tau(t) &= \tau_0 + \int_0^t \nu(s)ds
\end{align*}
\]

and by inverse kinematics we obtain the control of the reference vehicle

\[
\begin{align*}
    \theta_r(t) &= \text{ATAN2} \left( \frac{\dot{y}_r}{\dot{x}_r}, \frac{\dot{x}_r}{\dot{y}_r} \right) \\
    v_r(t) &= \pm \sqrt{\dot{x}_d^2(t) + \dot{y}_d^2(t)} \\
    w_r(t) &= \frac{\ddot{y}_d(t)\dot{x}_d(t) - \ddot{x}_d(t)\dot{y}_d(t)}{\dot{y}_d^2(t) + \dot{x}_d^2(t)}, \quad \text{a.e.}
\end{align*}
\]
Apply the MPC with a linearized model: Minimize
\[
\int_0^{T_d/\nu} \left( \|q(t) - q_d(\tau(t))\|^2_Q + \|u(t)\|^2_R \right) dt + \|q(T_d/\nu)\|^2_P,
\]
subject to:
\[
\dot{e}(t) = Ae(t) + Bu(t) \quad a.e. t \in [0, T_d/\nu]
\]
\[
e(0) = R_\theta(q_d(\tau_0) - q(t_i))
\]
\[
\tau_0 = \arg\min_{\tau \in [0, T_d]} \|q_d(\tau) - q(t_i)\|
\]
\[
\tau(t) = \tau_0 + \nu t \quad \forall t \in [0, T_d/\nu]
\]
\[
|u_1(t)| \leq U_{\text{max}} \quad \forall t \in [0, T_d/\nu]
\]
\[
|u_2(t)| \leq U_{\text{max}} \quad \forall t \in [0, T_d/\nu].
\]

where \( Q \) and \( R \) are chosen to be p.d., and \( P \) solves the Riccati equation
\[
A'P + PA - PBR^{-1}B'P + Q = 0,
\]
guaranteeing stability.
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Mission Objective: Minimize the estimation error of a target (ball) position and velocity.
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Model Predictive Control of Vehicle Formations

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