Optimal control problems with horizon tending to infinity and lacking controllability assumption

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M. Q., & J. Renault, On Existence of a limit value in some non expansive optimal control problems,(*submitted*) (2009)
An Optimal Control Problem

\[ V_t(y_0) := \inf_{u \in U} \frac{1}{t} \int_{s=0}^{t} h(y(s, u, y_0), u(s)) \, ds, \]

where \( s \mapsto y(s, u, y_0) \) denotes the solution to

\[ y'(s) = g(y(s), u(s)), \quad y(0) = y_0. \]

\( g : IR^d \times U \to IR^d \) Lipschitz, \( U \) compact, \( g h \) bounded.

**PROBLEM** : Existence of a limit of \( V_t(y_0) \) as \( t \to +\infty \).

No ergodicity condition here (Lions-Papanicolaou-Varadhan, Arisawa-Lions, Bettiol, Alvarez-Bardi Capuzzo-Dolcetta, Artstein-Gaitsgory, Fathi...) The limit may depend on the initial condition.
1. Introduction and examples
2. A controllability approach
3. Existence of limit value in nonexpansive case
4. Generalisations
Introduction

Definition 1  **The problem**  \( \Gamma(y_0) := (\Gamma_t(y_0))_{t > 0} \) **has a limit value** if

\[
V(y_0) := \lim_{t \to \infty} V_t(y_0) = \lim_{t \to \infty} \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^{t} h(y(s, u, y_0), u(s)) ds.
\]

Definition 2  **The problem**  \( \Gamma(y_0) \) **has a uniform value** if it **has a limit value** \( V(y_0) \) and if:

\[
\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^{t} h(y(s, u, y_0), u(s)) ds \leq V(y_0) + \varepsilon.
\]
Examples

- Example 1: here $y \in IR^2$ (seen as the complex plane $i^2 = -1$), there is no control

$$y'(t) = i \, y(t),$$

$$V_t(y_0) \xrightarrow{t \to \infty} \frac{1}{2\pi |y_0|} \int_{|z| = |y_0|} h(z) \, dz,$$

and since there is no control, the value is uniform.

- Example 2: in the complex plane again, but now $g(y, u) = i \, y \, u$, where $u \in U$ a given bounded subset of IR, and $h$ is continuous in $y$. 
Assumptions and Notations

\begin{align*}
\begin{cases}
\text{The function } h : \mathbb{IR}^d \times U \longrightarrow \mathbb{IR} \text{ is measurable and bounded} \\
\exists L \geq 0, \forall (y, y') \in \mathbb{IR}^{2d}, \forall u \in U, \| g(y, u) - g(y', u) \| \leq L \| y - y' \| \\
\exists a > 0, \forall (y, u) \in \mathbb{IR}^d \times U, \| g(y, u) \| \leq a(1 + \| y \|) \\
(\text{HK}) \ \exists \text{a compact invariant set } K \text{ for the control system}
\end{cases}
\end{align*}

\text{Average cost induced by } u \text{ between } 0 \text{ and } t \text{ by:}

\begin{align*}
\gamma_t(y_0, u) := & \frac{1}{t} \int_0^t h(y(s, u, y_0), u(s)) \, ds, \\
V_t(y_0) = & \inf_{u \in U} \gamma_t(y_0, u).
\end{align*}

\text{for } m \geq 0, \gamma_{m,t}(y_0, u) := \frac{1}{t} \int_m^{m+t} h(y(s, u, y_0), u(s)) \, ds,
Suppose that \( \exists T > 0, \forall (y_1, y_2) \in K, \exists t \leq T, \forall u \in U, \exists v \in U, \|y(t, u, y_1) - y(t, v, y_2)\| = 0 \).

Then for any \( t \geq T \) and any \( \Psi \in C(K) \) the maps

\[
y_0 \mapsto V_t^\Psi (y_0) := \inf_{u \in U} \int_{s=0}^{t} h(y(s, u, y_0))ds + \Psi(y(t, u, y_0)),
\]

are equicontinuous with a modulus of continuity which does not depend on \( t \) and \( \Psi \) (but only on the Lipschitz constants of \( h \) and \( f \)).
Thus $V_t^\Psi$ is more regular than $\Psi$. This also could be obtained and generalized using HJB results with coercive concave hamltonians.

- **Example 3:** $g(y, u) = -y + u$, where $u \in U$ a given bounded subset of $IR^d$, and $h$ is continuous in $y$.
- **Example 4:** in $IR^2$. The initial state is $y_0 = (0, 0)$ and $U = [0, 1]$, and the cost is $h(y) = 1 - y_1(1 - y_2)$.

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$ 

One can easily observe that the reachable set $G(y_0) \subset [0, 1]^2$.

If $u = \varepsilon > 0$ constant, $y_1(t) = 1 - \exp(-\varepsilon t)$ and $y_2(t) = \varepsilon y_1(t)$. So we have $V_t(y_0) \xrightarrow{t \to \infty} 0$. Existence of a Uniform Value
No ergodicity:

\[ \{ y \in [0, 1]^2, \lim_{t \to \infty} V_t(y) = \lim_{t \to \infty} V_t(y_0) \} = [0, 1] \times \{0\}, \]

and starting from \( y_0 \) it is possible to reach no point in \((0, 1] \times \{0\}\).
A first result in Nonexpansive case

Denote by $G(y_0) := \{y(t, u, y_0), t \geq 0, u \in \mathcal{U}\}$ the reachable set

**Theorem 3** $h(y, u) = h(y)$ only depends on the state, $G(y_0)$ is bounded (invariant),

\[ \forall (y_1, y_2) \in G(y_0)^2, \quad \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{U}} < y_1 - y_2, g(y_1, u) - g(y_2, v) > \leq 0. \]

Then $\Gamma(y_0)$ has a limit value $V_t(y_0) \xrightarrow{t \to +\infty} V^*(y_0)$. The convergence of $(V_t)_t$ to $V^*$ is uniform over $G(y_0)$, and the value of $\Gamma(y_0)$ is uniform.
A Crucial Technical Lemma

We define $V^{-}(y_0) := \lim \inf_{t \to +\infty} V_t(y_0)$,
$V^{+}(y_0) := \lim \sup_{t \to +\infty} V_t(y_0)$.

Lemma 4 For every $m_0$ in $\mathbb{IR}^+$, we have:

$$\sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \geq V^{+}(y_0) \geq V^{-}(y_0) \geq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

Definition 5

$$V^*(y_0) = \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$
Sketch of the proof of the first result

Lemma 6 \( \forall T > 0, \forall \varepsilon > 0, \forall (y_1, y_2) \in G(y_0)^2, \forall u \in \mathcal{U}, \exists v \in \mathcal{U} \)

\[ \forall t \in [0, T], \|y(t, u, y_1) - y(t, v, y_2)\| \leq \|y_1 - y_2\| + \varepsilon. \]

Proposition 7 \( \forall \varepsilon > 0, \exists m_0, \sup_{t > 0} \inf_{m \leq m_0} V_{m, t}(y_0) \leq \sup_{t > 0} \inf_{m \geq 0} V_{m, t}(y_0) + 2\varepsilon \)
• \((V_T(y_0))_{T>0}\) is equicontinuous (Lemma 6 + continuity of \(h\))
• Define \(G^m(y_0) := \{y(t, u, y_0), t \leq m, u \in \mathcal{U}\}\) the reachable set in time \(m\).

\[
\forall \varepsilon, \exists m_0, \forall z \in G(y_0), \exists z' \in G^{m_0}(y_0) \text{ such that } \|z - z'\| \leq \varepsilon.
\]

• We have \(\inf_{m \geq 0} V_{m, t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}\), and \(\inf_{m \leq m_0} V_{m, t}(y_0) = \inf\{V_t(z), z \in G^{m_0}(y_0)\}\). By steps 1 and 2 \(\inf\{V_t(z), z \in G^{m_0}(y_0)\} \leq \inf\{V_t(z), z \in G(y_0)\} + 2\varepsilon\).
Theorem 8 \( \exists \ C^1 \Delta : IR^d \times IR^d \rightarrow IR_+ \), vanishing on the diagonal \( (\Delta(y, y) = 0) \) and symmetric \( (\Delta(y_1, y_2) = \Delta(y_2, y_1)) \)

\( h(y, u) = h(y) \) only depends on the state,

\( G(y_0) \) is bounded (invariant),

\( \forall (y_1, y_2) \in G(y_0)^2, \forall u \in U, \exists v \in U. \)

\[
< g(y_1, u), \frac{\partial}{\partial y_1} \Delta(y_1, y_2) > + < g(y_2, v), \frac{\partial}{\partial y_2} \Delta(y_1, y_2) > \leq 0
\]

Then \( \Gamma(y_0) \) has a limit value \( V_t(y_0) \xrightarrow{t \rightarrow +\infty} V^*(y_0) \). The convergence of \( (V_t)_t \) to \( V^* \) is uniform over \( G(y_0) \), and the value of \( \Gamma(y_0) \) is uniform.
• This result can be applied to example 4, with $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ ($L^1$-norm). In this example, we have for each $y_1$, $y_2$ and $u$: $\Delta(y_1 + tg(y_1, u), y_2 + tg(y_2, u)) \leq \Delta(y_1, y_2)$ as soon as $t \geq 0$ is small enough.

Example 4: in $IR^2$. The initial state is $y_0 = (0, 0)$ and $U = [0, 1]$, and the cost is $h(y) = 1 - y_1(1 - y_2)$.

$$y'(s) = g(y(s), u(s)) = \left( \begin{array}{c} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{array} \right).$$
Further Generalizations

Theorem 9 (H1) $h$ is uniformly continuous in $y$ on $\bar{Z}$ uniformly in $u$. And for each $y$ in $\bar{Z}$, either $h$ does not depend on $u$ or the set $\{(g(y,u), h(y,u)) \in \mathbb{R}^d \times [0,1], u \in U\}$ is closed.

(H2): $\exists \Delta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^+$, vanishing on the diagonal ($\Delta(y, y) = 0$) and symmetric ($\Delta(y_1, y_2) = \Delta(y_2, y_1)$), and a uniformly continuous function $\hat{\alpha} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ s.t. $\hat{\alpha}(t) \rightarrow 0$ satisfying:

a) \forall sequence $(z_n)_n \subset \bar{Z}, \forall \varepsilon > 0, \exists n, \lim \inf_p \Delta(z_n, z_p) \leq \varepsilon.$

b) $\forall (y_1, y_2) \in \bar{Z}^2, \forall u \in U, \exists v \in U$ such that

$D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0, h(y_2, v) - h(y_1, u) \leq \hat{\alpha}(\Delta(y_1, y_2)).$

Then $\Gamma(y_0)$ has a uniform value $\lim_{t \rightarrow \infty} V_t = V^*$. 
Remarks

• Although \( \Delta \) may not satisfy the triangular inequality nor the separation property, it may be seen as a “distance” adapted to the problem \( \Gamma(y_0) \).

• \( D \uparrow \) is the contingent epi-derivative (which reduces to the upper Dini derivative if \( \Delta \) is Lipschitz) \( D\uparrow\Delta(z)(\alpha) = \lim \inf_{t \to 0^+} \frac{1}{t} (\Delta(z + t\alpha') - \Delta(z)) \). If \( \Delta \) is differentiable, the condition \( D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0 \) just reads: \(< g(y_1, u), \frac{\partial}{\partial y_1} \Delta(y_1, y_2) > + < g(y_2, v), \frac{\partial}{\partial y_2} \Delta(y_1, y_2) > \leq 0 \).
• The assumption: “\[ \{(g(y,u), h(y,u)) \in IR^d \times [0,1], \ u \in U \} \] closed” could be checked for instance if \( U \) is compact and if \( h \) and \( g \) are continuous with respect to \( (y,u) \).

• \( H2a \) is a precompacity condition. It is satisfied as soon as \( G(y_0) \) is bounded. cf Renault 2008

• Notice that \( H2 \) is satisfied with \( \Delta = 0 \) if we are in the trivial case where \( \inf_u h(y,u) \) is constant.
On Uniform Value

**Definition 10** \( \Gamma(y_0) \text{ has a uniform value if} \ \exists V(y_0) \text{ and if:} \)

\[
\forall \varepsilon > 0, \exists u \in U, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^{t} h(y(s, u, y_0), u(s))ds \leq V(y_0) + \varepsilon.
\]

- **Example 5:** in \( IR^2 \), \( y_0 = (0, 0) \), control set \( U = [0, 1] \), \( y'(t) = (y_2(t), u(t)) \), and \( h(y_1, y_2) = 0 \) if \( y_1 \in [1, 2] \), \( = 1 \) otherwise.

  We have \( u(s) = y_2'(s) = y_1''(s) \),

  **Interpretation:** \( u \) ”acceleration”, \( y_2 \) ”speed”, \( y_1 \) the ”position”.

  If \( u = \varepsilon \) constant, then \( y_2(t) = \sqrt{2\varepsilon y_1(t)} \ \forall t \geq 0. \)

**Limit Value:** \( V_T(y_0) \xrightarrow{T \to \infty} 1/2 \)

No Uniform Value.
We define \( \Theta_{\lambda}(y_0) := \inf_{u \in U} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} h(y(s, u, y_0), u(s)) ds, \)

Theorem 11 (Oliu-Barton Vigeral 2010) the following uniform limit in \( K \) exists \( \lim_{\lambda \to 0^+} \Theta_{\lambda}(y_0) \)

iff

the following uniform limit in \( K \) exists \( \lim_{t \to \infty} V_t(y_0) \)

Question Application to different concepts of means
Open Problems

Differential Game at horizon $t$:

$$V_t(y_0) := \inf_{u \in U} \sup_{v \in V} \frac{1}{t} \int_{s=0}^{t} h(y(s, u, v, y_0), u(s), v(s)) ds,$$

where $s \mapsto y(s, u, y_0)$ denotes the solution to

$$y'(s) = g(y(s), u(s), v(s)), \quad y(0) = y_0.$$

**OPEN PROBLEM**: Existence of a limit of $V_t(y_0)$ as $t \to \infty$.

Only Partial results:

- When the Hamiltonian is coercive (hence ergodicity and the limit is $y$ independent) *Alvarez-Bardi ...*
- For nonconvex and non coercive Hamiltonian in $IR^2$ *Cardaliagu...*
Averaging Problem for singularly perturbed system

\begin{align}
\begin{cases}
\text{i)} \quad x'(s) &= f(x(s), y(s), u(s)), \quad x(0) = x, \quad s \in [0, T] \\
\text{ii)} \quad \varepsilon y'(s) &= g(x(s), y(s), u(s)), \quad y(0) = y,
\end{cases} 
\end{align} 

(1)

Change of variable $\tau = \frac{t}{\varepsilon}$, $(X(\tau), Y(\tau), U(\tau)) = (x(\varepsilon \tau), y(\varepsilon \tau), u(\varepsilon \tau))$

\begin{align}
\begin{cases}
X'(\tau) &= \varepsilon f(X(\tau), Y(\tau), U(\tau)), \quad X(0) = x, \quad \tau \in [0, \frac{T}{\varepsilon}] \\
Y'(\tau) &= g(X(\tau), Y(\tau), U(\tau)), \quad Y(0) = y.
\end{cases} 
\end{align} 

(2)

Take $\varepsilon = 0$ in (2). We have the following associated system:

\begin{align}
y'(\tau) &= g(x, y(\tau), u(\tau)), \quad y(0) = y, 
\end{align} 

(3)

$y_x(\cdot, u, y)$ denotes the unique solution of (3).
Averaging method

We suppose that \( f \) and \( g \) are Lipschitz and there is a compact set \( M \times N \) which is invariant by (1) for all \( \varepsilon \).

\[
A(x, y, S, u) = \frac{1}{S} \int_{0}^{S} f(x, y_x(\tau, u, y), u(\tau))d\tau,
\]

\[
F(x, y, S) = \{ A(x, y, S, u); u \in \mathcal{U} \}
\]

Theorem 12 **Gaitsgory, Grammel** If \( \exists \gamma : IR \to IR_+ \) with
\[
\lim_{S \to +\infty} \gamma(S) = 0
\]
and a Lipschitz set-valued map \( \bar{F} : M \to IR^{bb} \) with compact convex nonempty values such that

\[
d(co \text{ cl} F(x, y, S), \bar{F}(x)) \leq \gamma(S), \forall (x, y) \in M \times N, \forall S > 0,
\]

then \( \forall x, y \) the solutions of the differential inclusion

\[
x'(s) \in \bar{F}(x(s)), x(0) = x.
\]

approximate the solutions of the singularly perturbed system (1) in the following sense:
For any $\varepsilon > 0$, and any $T > 0$ there exists $M(T, \varepsilon) > 0$ with $\lim_{\varepsilon \to 0} M(T, \varepsilon) = 0$ such that

a) For any family of solutions $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ to (1) there exists a solution $x(\cdot)$ to (4) such that

$$\sup_{t \in [0, T]} \| x_\varepsilon(t) - x(t) \| \leq M(T, \varepsilon).$$

b) Conversely fix $x(\cdot)$ a solution to (4) then for any $\varepsilon$ small enough there exists a solution $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ to (1) such that

$$\sup_{t \in [0, T]} \| x_\varepsilon(t) - x(t) \| \leq M(T, \varepsilon).$$

cf also Wattbled, M.Q Wattbled ...

QUESTION ases and conditions where $\overline{F}$ may depend on $y$. 


[9] V. Gaitsgory, M. Quincampoix *Linear programming analysis of deterministic infinite horizon optimal con-
trol problems (discounting and time averaging cases), to appear in Siam Journal of Control and Opt.


[13] Quincampoix, M. and J. Renault (submitted) On Existence of a limit value in some non expansive optimal control problems,

Thank You for your Attention