The vanishing viscosity limit for Hamilton-Jacobi equations on networks

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joint with Claudio Marchi (Padova) and Dirk Schieborn (Tübingen).
Three different approaches to HJ equation on networks

- **Achdou, Camilli, Cutrì, Tchou**: Hamilton-Jacobi equations constrained on networks, hal-00503910 (NoDea)


- **Camilli, Schieborn**: Viscosity solutions of eikonal equations on topological networks, arXiv:1103.4041v1 (CVPDE)

All the previous papers aim to extend the concept of viscosity solution to networks, but they differ for the assumptions made on the Hamiltonian at the vertices.
Vanishing Viscosity Approximation

In many problems, the vanishing viscosity method arises as a natural selection principle of the physical admissible solution in the class of weak solutions.

**Goal**

Show that viscosity solution of the Hamilton-Jacobi equation

\[ H(x, u, \partial u) = 0, \quad x \in \Gamma, \]

given in Camilli-Schieborn can be obtained as limit of solutions to the viscous approximation

\[-\varepsilon \partial^2 u_\varepsilon + H(x, u_\varepsilon, \partial u_\varepsilon) = 0, \quad x \in \Gamma,\]

letting the viscosity \( \varepsilon \to 0^+ \)
The definition of Network

A network is a connected set $\Gamma$ consisting of vertices $V := \{v_i\}_{i \in I}$ and edges $E := \{e_j\}_{j \in J}$ connecting the vertices. We assume that the network is imbedded in the Euclidian space so that any two edges can only have intersection at a vertex. A coordinate $\pi_j : [0, l_j] \to \mathbb{R}^N$, $j \in J$ is chosen on the edge $e_j$. 
Some Notations

- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of arcs incident the vertex $v_i$.

- A vertex $v_i$ is a boundary vertex if it has only one incident edge. We denote by $\partial \Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.

- A vertex $v_i$ is a transition vertex if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices.

- The graph is not oriented. The parametrization of the arcs $e_j$ induces an orientation on the edges, expressed by the signed incidence matrix $A = \{a_{ij}\}_{i \in I, j \in J}$

$$a_{ij} := \begin{cases} 
1 & \text{if } v_i \in \tilde{e}_j \text{ and } \pi_j(0) = v_i, \\
-1 & \text{if } v_i \in \tilde{e}_j \text{ and } \pi_j(l_j) = v_i, \\
0 & \text{otherwise.}
\end{cases}$$
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0 & \text{otherwise.}
\end{cases}
\]
The network is an imbedded subset of $\mathbb{R}^n$. It follows that the problem is intrinsically 1-dimensional and we can always work with the variable giving the parametrization of the edges.

Given $u : \Gamma \to \mathbb{R}$, $u^j : [0, l_j] \to \mathbb{R}$ denotes the restriction of $u$ to $\bar{e}_j$, i.e. $u^j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$

- $u$ is **continuous** ($u \in C^0(\Gamma)$) if $u^j \in C([0, l_j])$ for any $j \in J$ and

\[
    u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{for any } i \in I, j, k \in Inc_i
\]

- For $\alpha \in \mathbb{N}$, differentiation is defined with respect to the parameter variable

\[
    \partial_j^\alpha u(x) := \frac{d^\alpha}{dy^\alpha} u^j(y), \quad \text{for } x \in e_j, y = \pi_j^{-1}(x)
\]

and at a vertex $v_i$ by

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\[ H(x, u, \partial u) = 0 \quad x \in \Gamma \]

where Hamiltonian \( H : \Gamma \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is a family \( \{H^j\}_{j \in J} \) with \( H^j : [0, l_j] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) satisfying

(H1) \( H^j \in C^0([0, l_j] \times \mathbb{R} \times \mathbb{R}) \) for \( j \in J \)

(H2) \( H^j(x, r, p) \) is non decreasing in \( r \) for \( p \in \mathbb{R}, x \in [0, l_j], j \in J \)

(H3) \( H^j(x, r, \cdot) \) is convex, coercive in \( p \) for any \( x \in [0, l_j], j \in J \)

(H4) \( H^j(\pi_j^{-1}(v_i), r, p) = H^k(\pi_k^{-1}(v_i), r, p) \) for any \( p \in \mathbb{R}, i \in I, j, k \in \text{Inc}_i \)

(H5) \( H^j(\pi_j^{-1}(v_i), r, p) = H^j(\pi_j^{-1}(v_i), r, -p) \) for any \( p \in \mathbb{R} i \in I, j \in \text{Inc}_i \)

(H1)- (H3) are standard conditions for HJ equation. Assumptions (H4)-(H5) are continuity at the vertices and independence of the orientation of the incident edge.
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(H1)- (H3) are standard conditions for HJ equation. Assumptions (H4)-(H5) are continuity at the vertices and independence of the orientation of the incident edge.
Example

Consider the eikonal equation

$$\lambda u + |\partial u|^2 - f(x) = 0, \quad x \in \Gamma.$$  

where $\lambda \geq 0$, $f \geq 0$ and $f \in C^0(\bar{\Gamma})$, i.e. $f(x) = f^j(\pi_j^{-1}(x))$ for $x \in \bar{e}_j$, $f^j \in C^0([0, l_j])$, and $f^j(\pi_j^{-1}(v_i)) = f^k(\pi_k^{-1}(v_i))$ for any $i \in l, j, k \in Inc_i$. 
In the theory of viscosity solution, the class of test functions is a crucial point.

**Test functions**

- **ϕ** is a test function at \( x \in e_j \), if \( \varphi^j := \varphi \circ \pi_j : [0, l_j] \to \mathbb{R} \) is differentiable at \( y = \pi_j^{-1}(x) \).

- **ϕ** is \((j,k)\)-test function at \( x = v_i \) if \( \partial_j \varphi_j(\pi_j^{-1}(x)) \) and \( \partial_k \varphi_k(\pi_k^{-1}(x)) \) exist. Moreover

\[
 a_{ij} \partial_j \varphi_j(\pi_j^{-1}(x)) + a_{ik} \partial_k \varphi_k(\pi_k^{-1}(x)) = 0, \tag{1}
\]

where \((a_{ij})\) is the incidence matrix.

**Remark**

Condition (1) says that, taking into account the orientation, the function \( \varphi \) is differentiable at \( v_i \) along the direction given by the couple of edges \( e_j \) and \( e_k \) (no condition, except continuity, along the other incident edges).
An u.s.c. function $u$ is called a **viscosity subsolution** if

- for any $x \in e_j$ and any test function $\varphi$ at $x$ for which $u - \varphi$ attains a local maximum at $x$, then

$$H^j(y, u^j(y), \partial_j \varphi^j(y)) \leq 0, \quad y := \pi_j^{-1}(x) \quad (2)$$

- for any $x = v_i, i \in I_T$, and for any $j, k \in Inc_i$, any $(j, k)$-test function $\varphi$ at $x$ for which $u - \varphi$ attains a local maximum at $x$, then

$$H^j(y, u^j(y), \partial_j \varphi^j(y)) \leq 0, \quad y := \pi_j^{-1}(v_i) \quad (3)$$

A l.s.c. function $u$ is called a **viscosity supersolution** if

- for any $x \in e_j$, any test function $\varphi$ at $x$ for which $u - \varphi$ attains a local minimum at $x$, then

$$H^j(y, u^j(y), \partial_j \varphi^j(y)) \geq 0, \quad y := \pi_j^{-1}(x) \quad (4)$$

- for any $x = v_i, i \in I_T$, and for any $j \in Inc_i$, there exists $k \in Inc_i, k \neq j$ such that for any $(j, k)$-test function $\varphi$ at $x$ for which $u - \varphi$ attains a local minimum at $x$, then

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  \] (5)
If supersolutions would be defined similarly to subsolutions, the distance function from the boundary could not be a supersolution (note that the edge $e_1$ giving the shortest path to the boundary is admissible for the other two edges).

**Figure:** The distance function

$$|Du(x)|^2 = 1$$
Three basic results for HJ equations

**Perron**

Let $w \in USC(\Gamma)$ be a subsolution and $W \in LSC(\Gamma)$ a supersolution of (HJ) such that $w \leq W$ and $w^*(x) = W^*(x) = g(x)$ for $x \in \partial \Gamma$. Set $u(x) := \sup\{v(x) : v \text{ is a subsol. of (HJ) with } w \leq v \leq W \text{ on } \Gamma\}$

Then, $u^*$ and $u_*$ are respectively a subsolution and a supersolution to (HJ) in $\Gamma$ with $u = g$ on $\partial \Gamma$.

**Comparison**

Let $u_1$ and $u_2$ be a subsolution and a supersolution of (HJ) such that $u_1(v_i) \leq u_2(v_i)$ for all $v_i \in \partial \Gamma$. Then $u \leq v$ in $\Gamma$.

**Stability**

Assume $H_n(x, r, p) \to H(x, r, p)$ locally uniformly for $n \to \infty$. Let $u_n$ be a solution of $H_n(x, u, \partial u) = 0$ for $x \in \Gamma$. Assume $u_n \to u$ uniformly in $\Gamma$ for $n \to \infty$. Then $u$ is a solution of (HJ).
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Remark

While we do not need to assume convexity of $H$ for the previous results, instead coercivity of $H$ seems to be necessary for this theory. Coercivity gives Lipschitz continuity of the viscosity subsolutions, a regularity that allows to control the behavior of the viscosity solution near the transition vertices when there are several incident edges.
An example

\[
\begin{cases}
|\partial u|^2 = f(x) & x \in \Gamma \\
u = 0 & x \in \partial \Gamma
\end{cases}
\]

(a) Network

(b) Numerical solution

The solution is the distance from the boundary and it is computed via the numerical scheme in:

The viscous HJ equation

Definition

We say that a function \( u \in C^2_K(\Gamma) \), if

- \( u \in C^0(\Gamma) \)
- \( u^j \in C^2([0, l_j]) \) for any \( j \in J \)
- for any \( i \in I_T \), \( u \) satisfies the Kirchhoff condition:

\[
S(u) := \sum_{j \in \text{Inc}_i} a_{ij} \partial_j u(v_i) = 0
\]

We look for a solution \( u \in C^2_K(\Gamma) \) of the viscous HJ equation

\[
\begin{align*}
- \varepsilon \partial^2_j u + H(x, u, \partial u) &= 0 \quad x \in e_j, j \in J \\
u(v_i) &= g_i \quad i \in I_B
\end{align*}
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\[
\begin{cases}
- \varepsilon \partial_j^2 u + H(x, u, \partial u) = 0 & x \in e_j, j \in J \\
u(v_i) = g_i & i \in I_B
\end{cases}
\]
For second order problems, we consider the equation inside the edges and the Kirchhoff condition at the nodes. It is known from the theory of linear elliptic PDE on networks that the Kirchhoff condition (or other transition conditions) is necessary for the maximum principle.
The role of the Kirchhoff condition for the maximum principle

Assume that the function \( w \) satisfies

\[
\partial^2 w(x) \geq 0, \ x \in e_j, \ j \in J, \quad \text{and} \quad S(w) := \sum_{j \in \text{Inc}_i} a_{ij} \partial_j w(v_i) \geq 0, \ i \in I_T
\]

Then \( w \) cannot attain a maximum in \( \Gamma \setminus \partial \Gamma \).

**Proof.** Assume first \( \partial^2 w > 0 \) and \( S(w) > 0 \) and there exists \( x_0 \in \Gamma \setminus \partial \Gamma \) such that \( w \) attains a maximum at \( x_0 \). If \( x_0 \in e_j \) for some \( j \in J \), then it follows that \( \partial_j w(x_0) = 0 \) and \( \partial^2_j w(x_0) \leq 0 \), a contradiction to \( \partial^2 w > 0 \).

If \( x_0 = v_i \) for some \( i \in I_T \), then \( a_{ij} \partial_j w(v_i) \leq 0 \) for all \( j \in \text{Inc}_i \), hence \( S w(v_i) \leq 0 \), a contradiction to \( S(w) > 0 \).

For the general case, we prove that there exists \( \varphi \in C^2(\Gamma) \) such that

\[
\partial^2 \varphi(x) > 0 \quad \text{for} \ x \in e_j, \ j \in J \quad \text{and} \quad S\varphi(v_i) > 0 \quad \text{for} \ i \in I_T \quad \text{and we consider} \quad w_\delta = w + \delta \varphi, \ \delta > 0.
\]

Then \( w_\delta \) cannot attain a maximum in \( \Gamma \setminus \partial \Gamma \) and we conclude sending \( \delta \to 0^+ \).
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Then $w$ cannot attain a maximum in $\Gamma \setminus \partial \Gamma$.

**Proof.** Assume first $\partial^2 w > 0$ and $S(w) > 0$ and there exists $x_0 \in \Gamma \setminus \partial \Gamma$ such that $w$ attains a maximum at $x_0$. If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial^2_j w(x_0) \leq 0$, a contradiction to $\partial^2 w > 0$.

If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in \text{Inc}_i$, hence $Sw(v_i) \leq 0$, a contradiction to $S(w) > 0$.

For the general case, we prove that there exists $\varphi \in C^2(\Gamma)$ such that $\partial^2 \varphi(x) > 0$ for $x \in e_j, j \in J$ and $S\varphi(v_i) > 0$ for $i \in I_T$ and we consider $w_\delta = w + \delta \varphi, \ \delta > 0$. Then $w_\delta$ cannot attain a maximum in $\Gamma \setminus \partial \Gamma$ and we conclude sending $\delta \to 0^+$.
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For the general case, we prove that there exists $\varphi \in C^2(\Gamma)$ such that $\partial^2 \varphi(x) > 0$ for $x \in e_j, j \in J$ and $S\varphi(v_i) > 0$ for $i \in I_T$ and we consider $w_\delta = w + \delta \varphi, \ \delta > 0$. Then $w_\delta$ cannot attain a maximum in $\Gamma \setminus \partial \Gamma$ and we conclude sending $\delta \to 0^+$.
The role of the Kirchhoff condition for the maximum principle

Assume that the function $w$ satisfies

$$\partial^2 w(x) \geq 0, \; x \in e_j, \; j \in J, \quad \text{and} \quad S(w) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i) \geq 0, \; i \in I_T$$

Then $w$ cannot attain a maximum in $\Gamma \setminus \partial \Gamma$.

**Proof.** Assume first $\partial^2 w > 0$ and $S(w) > 0$ and there exists $x_0 \in \Gamma \setminus \partial \Gamma$ such that $w$ attains a maximum at $x_0$. If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial^2_j w(x_0) \leq 0$, a contradiction to $\partial^2 w > 0$.

If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $S(w(v_i)) \leq 0$, a contradiction to $S(w) > 0$.

For the general case, we prove that there exists $\varphi \in C^2(\Gamma)$ such that $\partial^2 \varphi(x) > 0$ for $x \in e_j, j \in J$ and $S\varphi(v_i) > 0$ for $i \in I_T$ and we consider $w_{\delta} = w + \delta \varphi, \; \delta > 0$. Then $w_{\delta}$ cannot attain a maximum in $\Gamma \setminus \partial \Gamma$ and we conclude sending $\delta \to 0^+$. 
The Kirchhoff condition is a first order condition for a second order problem. What about Kirchhoff conditions for the HJ equation?

Is it sufficient to assume only the **continuity of the solution at the vertices** to characterize the viscosity solution? No, there are infinite many a.e. solutions satisfying this condition.

Can we use the Kirchhoff condition to **characterize the viscosity solution** of the HJ equation? No, it is not satisfied by the distance function from the boundary.
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The viscous eikonal equation

Form now on, we consider

\[
\begin{cases}
-\varepsilon \partial_j^2 u + |\partial u|^2 - f(x) = 0 & x \in e_j, j \in J \\
u(v_i) = g_i & i \in l_B
\end{cases}
\]

Remark

Only for the \textbf{existence} of a solution to the second order problem we are forced to consider the particular case of the eikonal equation. The problem is that a complete theory for nonlinear elliptic PDE on network is still not available. Hence to prove \textbf{existence} of a solution to the second order problem we need to resume, via the Hopf-Cole transformation, to the \textbf{linear case}.

For \textbf{uniqueness}, \textbf{a priori estimates} and \textbf{vanishing viscosity limit} we can consider the general case

\[-\varepsilon \partial^2 u_\varepsilon + H(x, u_\varepsilon, \partial u_\varepsilon) = 0.\]
The Hopf-Cole transform

Let \( u_\varepsilon \in C^2_K(\Gamma) \) be a solution of

\[
\begin{aligned}
&-\varepsilon \partial_j^2 u + |\partial u|^2 - f(x) = 0 \quad x \in e_j, \ j \in J \\
u(v_i) = g_i \\
&i \in I_B
\end{aligned}
\]

and set \( w_\varepsilon = e^{-u_\varepsilon/\varepsilon} - 1 \). Then \( w_\varepsilon \in C^2_K(\Gamma) \) (the Hopf-Cole transform preserves the Kirchhoff’s cond.) is a solution of the linear problem

\[
\begin{aligned}
&\varepsilon \partial_j^2 u + f(x)w_\varepsilon - f(x) = 0 \quad x \in e_j, \ j \in J \\
w_\varepsilon(v_i) = e^{-g_i/\varepsilon} - 1 \\
i \in I_B
\end{aligned}
\]
The linear problem

We consider a linear operator $L$ defined on $\Gamma$ by

$$L^j w(x) := a^j(x) \partial^2_j w(x) + b^j(x) \partial_j w(x) - c^j(x) w(x)$$

with $a^j, b^j, c^j, g^j \in C((0, l_j)), j \in J$, and

$$a^j(x) \geq \lambda > 0 \text{ and } c^j(x) \geq 0 \forall x \in (0, l_j), j \in J$$

Theorem

There exists a unique solution $w \in C^2_K(\Gamma)$ of

$$\begin{cases}
    L^j w(x) + g^j(x) = 0 & x \in e_j, j \in J \\
    w(v_i) = \gamma_i & i \in I_B.
\end{cases}$$
Idea of the proof

**Uniqueness**: Consequence of the maximum principle


\[
u(x) = \mathbb{E}_x \left\{ \int_0^\tau e^{-c(Y(s))} g(Y(s)) ds + e^{-c(Y(\tau))} \gamma_{i(\tau)} \right\}
\]

where \( Y(s) \) is a Markov process defined on the graph which on each edge \( e_j \) solves the stochastic differential equation

\[
dY(s) = b^j(Y(s)) ds + a^j(Y(s)) dW(s)
\]

\( \tau = \inf\{t > 0 : Y(t) \in \partial \Gamma\} \), \( i(\tau) \in I_B \) is such that \( Y(\tau) = \nu_{i(\tau)} \in \partial \Gamma \). In the probabilistic interpretation the Kirchhoff condition implies that the process almost surely spends zero time at each vertex \( \nu_i \), \( i \in I_T \).
Idea of the proof

**Uniqueness**: Consequence of the maximum principle


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In the probabilistic interpretation the Kirchhoff condition implies that the process almost surely spends zero time at each vertex $v_i$, $i \in I_T$. 
By the previous result we get via Hopf-Cole transform

**Corollary**

For any $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in C^2_K(\Gamma)$ of

\[
\begin{cases}
-\varepsilon \partial_j^2 u + |\partial_j u|^2 - f(x) = 0 & x \in e_j, j \in J \\
u(v_i) = g_i & i \in I_B
\end{cases}
\]
A comparison result for the semilinear problem

The following comparison result is a key ingredient to get a-priori estimates for the solution of the second order problem

**Theorem**

Assume that $H(x, r, p)$ is strictly increasing in $r$. Let $w_1, w_2 \in C^2(\Gamma)$ be such that

\[
\begin{cases}
  -\varepsilon \partial_j^2 w_1 + H(x, w_1, \partial_j w_1) \geq -\varepsilon \partial_j^2 w_2 + H(x, w_2, \partial_j w_2) & x \in e_j, \ j \in J, \\
  S^i_\beta w_1 \leq S^i_\beta w_2 & i \in I_T \\
  w_1(v_i) \geq w_2(v_i) & i \in I_B
\end{cases}
\]

Then $w_1 \geq w_2$ on $\Gamma$. 

A priori estimates

**Theorem**

Assume that for each \( \varepsilon \), there is a solution \( u_\varepsilon \in C_k^2(\Gamma) \) of

\[
\begin{cases}
  -\varepsilon \partial_j^2 w_\varepsilon + H(x, w_\varepsilon, \partial_j w_\varepsilon) = 0 & x \in e_j, j \in J, \\
  w_\varepsilon(v_i) = g_i & i \in I_B
\end{cases}
\]

Then there is \( \bar{\varepsilon} \) sufficiently small such that for any \( 0 < \varepsilon < \bar{\varepsilon} \), the functions \( u_\varepsilon \) are uniformly bounded and equi-Lipschitz continuous on \( \Gamma \).

**Remark:** The estimates are first obtained at the vertices using the boundary condition and the Kirchhoff condition and, then, inside the edges using the estimate at the vertices.
The vanishing viscosity limit

**Theorem**

Let $u_n := u_{\varepsilon n} \in C^2_K(\Gamma)$ be a sequence of solutions of

$$
\begin{align*}
-\varepsilon \partial^2_j u_{\varepsilon} + H(x, u_{\varepsilon}, \partial_j u_{\varepsilon}) &= 0 & x \in e_j, j \in J, \\
u_{\varepsilon}(v_i) &= g_i & i \in I_B
\end{align*}
$$

such that $u_n$ and $\partial u_n$ are uniformly bounded on $\Gamma$. If $u_n$ converges uniformly to a function $u \in C(\Gamma)$, then $u$ is a solution of

$$
\begin{align*}
H(x, u, \partial u) &= 0 & x \in e_j, j \in J, \\
u(v_i) &= g_i & i \in I_B
\end{align*}
$$
Idea of the proof

To prove the supersolution condition we use the following idea:

1. Consider \( i \in I_T \). The Kirchhoff condition \( \sum_{j \in \text{Inc}_i} a_{ij} \partial_j u_n(v_i) = 0 \) says that we can find a subsequence \( u_{n_k} \) and an index \( j \in \text{Inc}_i \) s.t.
   (assume \( a_{ij} = 1 \))
   \[
   \partial_j u_{n_k}(v_i) \leq 0 \quad \forall n
   \]

2. Formally, at the limit, \( \partial_j u(v_i) \leq 0 \), hence the function \( u \) is decreasing along the edge \( e_j \). This means that \( e_j \) is a minimizing edge for the "distance" from the boundary, hence it is admissible for all the other edges \( e_k, k \in \text{Inc}_i \).

Note that the Kirchhoff condition is used only in the proof of the supersolution property of the limit.
Idea of the proof

To prove the supersolution condition we use the following idea:

- consider $i \in I_T$. The Kirchhoff condition $\sum_{j \in Inc_i} a_{ij} \partial_j u_n(v_i) = 0$ says that we can find a subsequence $u_{n_k}$ and an index $j \in Inc_i$ s.t. (assume $a_{ij} = 1$)

$$\partial_j u_{n_k}(v_i) \leq 0 \quad \forall n$$

- Formally, at the limit, $\partial_j u(v_i) \leq 0$, hence the function $u$ is decreasing along the edge $e_j$. This means that $e_j$ is a minimizing edge for the “distance” from the boundary, hence it is admissible for all the other edges $e_k$, $k \in Inc_i$

Note that the Kirchhoff condition is used only in the proof of the supersolution property of the limit.
Corollary

Let \( u_n := u_{\varepsilon n} \in C_K^2(\Gamma) \) be the sequence of solutions of

\[
\begin{cases}
-\varepsilon \partial_j^2 u_{\varepsilon} + |\partial u|^2 - f(x) = 0 & x \in e_j, j \in J, \\
u_{\varepsilon}(v_i) = g_i & i \in I_B
\end{cases}
\]

Then \( u_n \) converges uniformly to the solution \( u \in C(\Gamma) \) of

\[
\begin{cases}
|\partial u|^2 - f(x) = 0 & x \in e_j, j \in J, \\
u(v_i) = g_i & i \in I_B
\end{cases}
\]
Remark:
The method of the **semi-relaxed limits**, which avoids the use of a-priori estimates, does not work with this definition of viscosity solution. Suppose that we want to test $u^*(x) = \limsup_{x_\varepsilon \to x, \varepsilon \to 0} u_\varepsilon(x_\varepsilon)$ at a vertex $x$ by means of $(j, k)$-admissible test function $\varphi$. We cannot exclude that the possibility that the approximating sequence of points $x_\varepsilon$ which defines $u^*(x)$ belongs to a third different edge ($e_m$ in the figure) and we cannot test $u_\varepsilon$ at $x_\varepsilon$ by means of $\varphi$. 

![Diagram with vertices and edges]
Conclusions

- Various **transition conditions** are introduced in the literature depending on the model problem (linear or nonlinear, dynamical or static, dissipative or non-dissipative, etc). Does the vanishing viscosity limit work with other transition conditions?
- Prove that the various definitions of viscosity solutions for HJ on networks are equivalent, at least under some regularity assumptions.
- Establish a connection between HJ equation and conservation laws on networks (some results are available in Imbert-Monneau-Zidani).
- Generalize the classical and the viscosity solution theory to fully nonlinear second order problems on networks.
- Various applications: MFG on networks, homogenization, stability and asymptotic behavior of dynamical systems on networks, etc (some of these problems are discussed in the probabilistic literature).
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Thank You!

Everything should be made as simple as possible, but not simpler (A.Einstein)