Tutorial on Control and State Constrained Optimal Control Problems –
Part 2: Mixed Control-State Constraints

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SADCO Summer School
Imperial College London, September 5, 2011
Outline

1. Optimal Control Problems with Control and State Constraints
2. Numerical Method: Discretize and Optimize
3. Theory of Optimal Control Problems with Mixed Control-State Constraints
4. Example: Rayleigh Problem with Different Constraints and Objectives
5. Example: Optimal Exploitation of Renewable Resources
Optimal Control Problem

**State** \( x(t) \in \mathbb{R}^n \), **control** \( u(t) \in \mathbb{R}^m \).

**Dynamics and Boundary Conditions**

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, t_f], \\
x(t) &= \bar{x}_0, \quad \psi(x(t_f)) = 0 \quad (\psi: \mathbb{R}^n \to \mathbb{R}^r).
\end{align*}
\]

**Control and State Constraints**

\[
\begin{align*}
c(x(t), u(t)) &\leq 0, \quad 0 \leq t \leq t_f, \quad (c: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k) \\
s(x(t)) &\leq 0, \quad 0 \leq t \leq t_f, \quad (s: \mathbb{R}^n \to \mathbb{R}^l)
\end{align*}
\]

**Minimize**

\[
J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t)) \, dt
\]
Discretization

For simplicity consider a MAYER–type problem with cost functional

\[ J(u, x) = g(x(t_f)) \, . \]

This can be achieved by considering the additional state variable \( x_0 \) with

\[ \dot{x}_0 = f_0(x, u), \quad x_0(0) = 0. \]

Then we have

\[ x_0(t_f) = \int_0^{t_f} f_0(t, x(t), u(t)) \, . \]

Choose an integer \( N \in \mathbb{N} \), a stepsize \( h \) and grid points \( t_i \):

\[ h = \frac{t_f}{N}, \quad t_i := ih, \quad (i = 0, 1, \ldots, N). \]

Approximation of control and state at grid points:

\[ u(t_i) \approx u_i \in \mathbb{R}^m, \quad x(t_i) \approx x_i \in \mathbb{R}^n \quad (i = 0, \ldots, N) \]
Large-scale NLP using EULER's method

Minimize

\[ J(u, x) = g(x_N) \]

subject to

\[ x_{i+1} = x_i + h \cdot f(t_i, x_i, u_i), \quad i = 0, \ldots, N - 1, \]
\[ x_0 = \bar{x}_0, \quad \psi(x_N) = 0, \]
\[ c(x_i, u_i) \leq 0, \quad i = 0, \ldots, N, \]
\[ s(x_i) \leq 0, \quad i = 0, \ldots, N, \]

Optimization variable for full discretization:

\[ z := (u_0, x_1, u_1, x_2, \ldots, u_{N-1}, x_N, u_N) \in \mathbb{R}^{N(m+n)+m} \]
NLP Solvers

- AMPL: Programming language (Fourer, Gay, Kernighan)
- IPOPT: Interior point method (Andreas Wächter)
- LOQO: Interior point method (Vanderbei et al.)
- Other NLP solvers embedded in AMPL: cf. NEOS server
- NUDOCCCS: optimal control package (Christof Büskens)
- WORHP: SQP solver (Christof Büskens, Matthias Gerdts)
- Special feature: solvers provide LAGRANGE-multipliers as approximations of the adjoint variables.
Optimal Control Problem with Control-State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$.
All functions are assumed to be sufficiently smooth.

Dynamics and Boundary Conditions

$$\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \quad \text{a.e. } t \in [0, t_f], \\
x(0) &= x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k, \\
(0 &= \varphi(x(0), x(t_f)) \quad \text{mixed boundary conditions})
\end{align*}$$

Mixed Control-State Constraints

$$\alpha \leq c(x(t), u(t)) \leq \beta, \quad t \in [0, t_f], \quad c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$$
Control bounds $\alpha \leq u(t) \leq \beta$ are included by $c(x, u) = u$.

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), u) \, dt$$
Hamiltonian

Hamiltonian

\[ H(x, \lambda, u) = \lambda_0 f(x, u) + \lambda f(x, u) \quad \lambda \in \mathbb{R}^n \text{ (row vector)} \]

Augmented Hamiltonian

\[ \mathcal{H}(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu c(x, u) \]
\[ = \lambda_0 f(x, u) + \lambda f(x, u) + \mu c(x, u), \quad \mu \in \mathbb{R}. \]

Let \((u, x) \in \mathcal{L}^\infty([0, T], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, T], \mathbb{R}^n)\) be a locally optimal pair of functions.

Regularity assumption

\[ c_u(x(t), u(t)) \neq 0 \quad \forall \ t \in J_a \]

\[ J_a := \{ t \in [0, t_f] \mid c(x(t), u(t)) = \alpha \text{ or } = \beta \}\]
Minimum Principle of Pontryagin et al. and Hestenes

Let \((u, x) \in L^\infty([0, t_f], \mathbb{R}^m) \times W^{1,\infty}([0, t_f], \mathbb{R}^n)\) be a locally optimal pair of functions that satisfies the regularity assumption. Then there exist

- an adjoint (costate) function \(\lambda \in W^{1,\infty}([0, t_f], \mathbb{R}^n)\) and a scalar \(\lambda_0 \geq 0\),
- a multiplier function \(\mu \in L^\infty([0, t_f], \mathbb{R})\),
- and a multiplier \(\rho \in \mathbb{R}^r\) associated to the boundary condition \(\psi(x(t_f)) = 0\)

that satisfy the following conditions for a.a. \(t \in [0, t_f]\), where the argument \(t\) denotes evaluation along the trajectory \((x(t), u(t), \lambda(t))\):
Minimum Principle of Pontryagin et al. and Hestenes

(i) Adjoint ODE and transversality condition:
\[
\dot{\lambda}(t) = -\mathcal{H}_x(t) = -(\lambda_0 f_0 + \lambda f)_x(t) - \mu(t) c_x(t),
\]
\[
\lambda(t_f) = (\lambda_0 g + \rho \psi)_x(x(t_f)),
\]

(iiia) Minimum Condition for Hamiltonian:
\[
H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq c(x(t), u) \leq \beta \}
\]

(iiib) Local Minimum Condition for Augmented Hamiltonian:
\[
0 = \mathcal{H}_u(t) = (\lambda_0 f_0 + \lambda f)_u(t) + \mu(t) c_u(t)
\]

(iii) Sign of multiplier $\mu$ and complementarity condition:
\[
\mu(t) \leq 0, \text{ if } c(x(t), u(t)) = \alpha; \quad \mu(t) \geq 0, \text{ if } c(x(t), u(t)) = \beta,
\]
\[
\mu(t) = 0, \text{ if } \alpha < c(x(t), u(t)) < \beta.
\]
Evaluation of the Minimum Principle: boundary arc

**Boundary arc:** Let \([t_1, t_2]\), \(0 \leq t_1 < t_2 < t_f\), be an interval with

\[
c(x(t), u(t)) = \alpha \quad \text{or} \quad c(x(t), u(t)) = \beta \quad \forall \; t_1 \leq t \leq t_2.
\]

For simplicity assume a scalar control, i.e., \(m = 1\). Due to the regularity condition \(c_u(x(t), u(t)) \neq 0\) there exists a smooth function \(u_b(x)\) satisfying

\[
c(x, u_b(x)) \equiv \alpha \; (\equiv \beta) \quad \forall \; x \text{ in a neighborhood of the trajectory}.
\]

The control \(u_b(x)\) is called the **boundary control** and yields the **optimal control** by the relation

\[
u(t) = u_b(x(t)).
\]

It follows from the local minimum condition \(0 = H_u = H_u + \mu c_u\) that the multiplier \(\mu\) is given by

\[
\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) / c_u(x, u_b(x)).
\]
Case I: Regular Hamiltonian, $u$ is continuous

CASE I: Consider optimal control problems which satisfy the Assumption: The Hamiltonian $H(x, \lambda, u)$ is regular, i.e., it admits a unique minimum $u$. The strict Legendre condition holds:

$$H_{uu}(t) > 0 \quad \forall \ t \in [0, t_f].$$

(a) Then there exists a "free control" $u = u_{\text{free}}(x, \lambda)$ satisfying

$$H_u(x, \lambda, u_{\text{free}}(x, \lambda)) \equiv 0.$$

(b) The optimal control $u(t)$ is continuous in $[0, t_f]$.

Claim (b) follows from the continuity and regularity of $H$.

The continuity of the control implies junctions conditions at junction points $t_k$ ($k = 1, 2$) with the boundary:

$$u_{\text{free}}(x(t_k), \lambda(t_k)) = u_b(x(t_k)), \quad \mu(t_k) = 0 \quad (k = 1, 2).$$
Rayleigh Problem with Quadratic Control

The Rayleigh problem is a variant of the van der Pol Oscillator, where $x_1$ denotes the electric current.

Control problem for the Rayleigh Equation

Minimize $J(x, u) = \int_0^{t_f} (u^2 + x_1^2) \, dt$ \hspace{1cm} (tf = 4.5)

subject to

\[\begin{align*}
\dot{x}_1 &= x_2, & x_1(0) &= -5, \\
\dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5.
\end{align*}\]

Three types of constraints:

Case (a) : no control constraints.
Case (b) : control constraint $-1 \leq u(t) \leq 1$.
Case (c) : mixed control-state constraint

$$\alpha \leq u(t) + x_1(t)/6 \leq 0, \hspace{0.5cm} \alpha = -1, -2$$
Case I (a) : Rayleigh problem, no constraint

Normal Hamiltonian:

$$H(x, \lambda, u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2 (-x_1 + x_2(1.4 - 0.14x_2^2) + 4u)$$

Adjoint Equations:

$$\dot{\lambda}_1 = -H_{x_1} = -2x_1 + \lambda_2 \quad \lambda_1(t_f) = 0,$$
$$\dot{\lambda}_2 = -H_{x_2} = -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) \quad \lambda_2(t_f) = 0,$$

Minimum condition:

$$0 = H_u = 2u + 4\lambda_2 \quad \Rightarrow \quad u = u_{free}(x, \lambda) = -2\lambda_2.$$  

Shooting method for solving the boundary value problem for $(x, \lambda)$:

Determine unknown shooting vector $s = \lambda(0) \in \mathbb{R}^2$ that satisfies the terminal condition $\lambda(t_f) = 0$ : use Newton’s method.
Case I: Rayleigh problem without constraints

Note: Hamiltonian is regular, control $u(t)$ is continuous (analytic).
Case I (b) : Rayleigh problem, constraint $-1 \leq u(t) \leq 1$

Hamiltonian $H$ and adjoint equations are as in Case (a). The free control is given by $u_{\text{free}}(x, \lambda) = -2\lambda_2$.

Structure of optimal control:

$$
u(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq t_1 \\
-2\lambda_2(t) & \text{for } t_1 \leq t \leq t_2 \\
-1 & \text{for } t_2 \leq t \leq t_3 \\
-2\lambda_2(t) & \text{for } t_3 \leq t \leq t_f 
\end{cases}$$

Junction conditions: Continuity of the control implies

$$u(t_k) = -2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3.$$ 

Shooting method for solving the boundary value problem for $(x, \lambda)$: Determine shooting vector $s = (\lambda(0), t_1, t_2, t_3) \in \mathbb{R}^{2+3}$ that satisfies 2 terminal conditions $\lambda(t_f) = 0$ and 3 junction conditions.
Rayleigh problem with control constraint $|u(t)| \leq 1$

**State variables** $x_1, x_2$

**Phase portrait** $(x_1, x_2)$

**Optimal control** $u$

**Adjoint variables** $\lambda_1, \lambda_2$

**Note:** Hamiltonian is regular, control $u(t)$ is continuous.

**Junction conditions:**

$-2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3$
Case I (c) : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$

Augmented (normal) Hamiltonian:

$$\mathcal{H}(x, \lambda, \mu, u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2 (-x_1 + x_2(1.4 - 0.14x_2^2) + 4u) + \mu(u + x_1/6)$$

Adjoint Equations:

$$\dot{\lambda}_1 = -\mathcal{H}_{x_1} = -2x_1 + \lambda_2 - \mu/6, \quad \lambda_1(t_f) = 0,$$
$$\dot{\lambda}_2 = -\mathcal{H}_{x_2} = -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) \quad \lambda_2(t_f) = 0,$$

Free control : $u_{\text{free}}(x, \lambda) = -2\lambda_2$.

Boundary control : $u_b(x) = \alpha - x_1/6$ for $\alpha \in \{-1, 0\}$.

Multiplier :

$$\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) / c_u(x, u_b(x)) = 2u_b(x) + 4\lambda_2.$$
Case I (c): Rayleigh problem, \(-1 \leq u + x_1/6 \leq 0\)

Hamiltonian is regular, control \(u(t)\) is continuous.

Junction conditions: \(-2\lambda(t_k) = \alpha - x_1(t_k)/6, \alpha \in \{0, -1\}\).
Case I (c) : Rayleigh problem, structure of optimal control for mixed constraint $-1 \leq u + x_1/6 \leq 0$

$$u(t) = \begin{cases} 
-x_1/6 & \text{for } 0 \leq t \leq t_1 \\
-2\lambda_2(t) & \text{for } t_1 \leq t \leq t_2 \\
-1 - x_1/6 & \text{for } t_2 \leq t \leq t_3 \\
-2\lambda_2(t) & \text{for } t_3 \leq t \leq t_4 \\
-x_1/6 & \text{for } t_4 \leq t \leq t_5 \\
-2\lambda_2(t) & \text{for } t_5 \leq t \leq t_f 
\end{cases}$$
Case II: control $u$ appears linearly

**CASE II**: Control appears linearly in the cost functional, dynamics and mixed control-state constraint. Let $u$ be scalar.

### Dynamics and Boundary Conditions

\[
\dot{x}(t) = f_1(x(t)) + f_2(x(t)) \cdot u(t), \text{ a.e. } t \in [0, t_f], \\
x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,
\]

### Mixed Control-State Constraints

\[
\alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u(t) \leq \beta \quad \forall t \in [0, t_f]. \quad c_1, c_2 : \mathbb{R}^n \to \mathbb{R}
\]

### Minimize

\[
J(u, x) = g(x(t_f)) + \int_0^{t_f} (f_{01}(x(t)) + f_{02}(x(t)) \cdot u(t) \, dt
\]
Case II: Hamiltonian and switching function

Normal Hamiltonian

\[ H(x, \lambda, u) = f_{01}(x) + \lambda f_1(x) + [f_{02}(x) + \lambda f_2(x)] \cdot u. \]

Augmented Hamiltonian

\[ H(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu (c_1(x) + c_2(x) \cdot u) \]

The optimal control \( u(t) \) solves the minimization problem

\[ \min \{ H(x(t), \lambda(t), u) | \alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u \leq \beta \} \]

Define the switching function

\[ \sigma(x, \lambda) = H_u(x, \lambda, u) = f_{02}(x) + \lambda f_2(x), \quad \sigma(t) = \sigma(x(t), \lambda(t)). \]
Case II: Hamiltonian

The minimum condition is equivalent to the minimization problem

\[
\min \{ \sigma(t) \cdot u \mid \alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u \leq \beta \}
\]

We deduce the control law

\[
c_1(x(t)) + c_2(x(t)) \cdot u(t) = \begin{cases} 
\alpha, & \text{if } \sigma(t) \cdot c_2(x(t)) > 0 \\
\beta, & \text{if } \sigma(t) \cdot c_2(x(t)) < 0 \\
\text{undetermined}, & \text{if } \sigma(t) \equiv 0
\end{cases}
\]

The control \( u \) is called bang-bang in an interval \( I \subset [0, t_f] \), if \( \sigma(t) \cdot c_2(x(t)) \neq 0 \) for all \( t \in I \). The control \( u \) is called singular in an interval \( I_{\text{sing}} \subset [0, t_f] \), if \( \sigma(t) \cdot c_2(x(t)) \equiv 0 \) for all \( t \in I_{\text{sing}} \).

For the control constraint \( \alpha \leq u(t) \leq \beta \) with \( c_1(x) = 0, \ c_2(x) = 1 \) we get the classical control law

\[
u(t) = \begin{cases} 
\alpha, & \text{if } \sigma(t) > 0 \\
\beta, & \text{if } \sigma(t) < 0 \\
\text{undetermined}, & \text{if } \sigma(t) \equiv 0
\end{cases}
\]
Bang-Bang and Singular Controls

\[ \sigma(t) \]

\[ u(t) \]

\[ u_{max} \]

\[ u_{min} \]

\[ \sigma < 0 \]

\[ \sigma = 0 \]

\[ \sigma > 0 \]

\[ u = u_{max} \]

\[ u = u_{min} \]

\[ u_{sing} \]
Case II: Rayleigh problem with $-1 \leq u(t) \leq 1$

Rayleigh problem with control appearing linearly

Minimize \[ J(x, u) = \int_0^{t_f} (x_1^2 + x_2^2) \, dt \quad (t_f = 4.5) \]
subject to
\[
\begin{align*}
\dot{x}_1 &= x_2, & x_1(0) &= -5, \\
\dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5, \\
-1 \leq u(t) &\leq 1.
\end{align*}
\]

Adjoint Equations:
\[
\begin{align*}
\dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2, & \lambda_1(t_f) &= 0, \\
\dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2), & \lambda_2(t_f) &= 0,
\end{align*}
\]

The switching function $\sigma(t) = H_u(t) = 4\lambda_2(t)$ gives the control law

\[
u(t) = -\text{sign} \left( \lambda_2(t) \right)\]
Case II: Rayleigh problem, $-1 \leq u(t) \leq 1$

Control $u(t)$ is bang-bang-singular.

Switching conditions: $\lambda(t_1) = 0$, $\lambda_2(t) \equiv 0 \forall t \in [t_2, t_f]$. 
Case II: Rayleigh problem, $\alpha \leq u + x_1/6 \leq 0$

Minimize $J(x, u) = \int_0^{t_f} (x_1^2 + x_2^2) \, dt \quad (t_f = 4.5)$

subject to

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 + x_2 (1.4 - 0.14x_2^2) + 4u,$$

and the mixed control-state constraint

$$\alpha \leq u(t) + x_1(t)/6 \leq 0 \quad \forall \ 0 \leq t \leq t_f.$$

Adjoint Equations:

$$\dot{\lambda}_1 = -H_{x_1} = -2x_1 + \lambda_2 - \mu/6,$$
$$\dot{\lambda}_2 = -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2 (1.4 - 0.42x_2^2),$$

$\lambda_1(t_f) = 0,$
$\lambda_2(t_f) = 0.$
Control law for $\alpha \leq u + x_1/6 \leq 0$

The switching function is $\sigma(t) = H_u(t) = 4\lambda_2(t)$. In view of $c_2(x) \equiv 1$ we have the control law

$$u + x_1/6 = \begin{cases} 
\alpha < 0 & \text{, if } \lambda_2(t) > 0 \\
0 & \text{, if } \lambda_2(t) < 0 \\
\text{undetermined} & \text{, if } \lambda_2(t) \equiv 0
\end{cases}$$
Case II: Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$

Note: constraint $u(t) + x_1(t)/6$ is "bang-bang".
Case II: Rayleigh problem, \(-2 \leq u + x_1/6 \leq 0\)

Note: constraint \(u(t) + x_1(t)/6\) is "bang-singular-bang-singular".
**Optimal Fishing, Clark, Clarke, Munro**

**Colin W. Clark, Frank H. Clarke, Gordon R. Munro:**

**State variables and control variables:**

- \( x(t) \) : population biomass at time \( t \in [0, t_f] \),
  - renewable resource, e.g., fish,

- \( K(t) \) : amount of capital invested in the fishery,
  - e.g., number of ”standardized” fishing vessels available,

- \( E(t) \) : fishing effort (control), \( h(t) = E(t)x(t) \) is harvest rate,

- \( I(t) \) : investment rate (control),
Optimal Fishing: optimal control model

Dynamics in $[0, t_f]$ (here: $a = 1$, $b = 5$, $\gamma = 0$)

\[
\dot{x}(t) = a \cdot x(t) \cdot (1 - x(t)/b) - E(t) \cdot x(t), \quad x(0) = x_0,
\]

\[
\dot{K}(t) = I(t) - \gamma \cdot K(t), \quad K(0) = K_0.
\]

Mixed Control-State Constraint and Control Constraint

\[
0 \leq E(t) \leq K(t), \quad 0 \leq I(t) \leq I_{\text{max}}, \quad t \in [0, t_f],
\]

Maximize benefit (parameters: $r = 0.05$, $c_E = 2$, $c_I = 1.1$)

\[
J(u, x) = \int_0^{t_f} \exp(-r \cdot t)( p \cdot E(t) \cdot x(t) - c_E \cdot E(t) - c_I \cdot I(t)) \, dt
\]
Optimal Fishing: $x_0 = 0.5$, $K_0 = 0.2$, $I_{\text{max}} = 0.5$

controls $E$, $I$ and state variables $0.5 \cdot x$, $K$
Optimal Fishing: $x_0 = 0.5$, $K_0 = 0.6$, $l_{max} = 0.5$

Fishing rate: $E(t) = 0$, $E(t) = \text{singular}$, $E(t) = K(t)$.

Investment rate: $I(t) = 0$, $I(t) = l_{max}$, $I(t) = 0$. 
Optimal Fishing: \( x_0 = 0.2, K_0 = 0.1, l_{max} = 0.1 \)

Fishing rate \( E(t) = 0, E(t) = K(t), E(t) \) singular, \( E(t) = K(t) \).

Investment rate \( 2 \) arcs with \( l(t) = l_{max} \).
Optimal Fishing: $x_0 = 1.0$, $K_0 = 0.5$, $I_{max} = 3$

Fishing rate: $E(t) = 0$, $E(t)$ singular $E(t) = K(t)$,

Investment rate: 1 "impulse" with $I(t) = I_{max}$.