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A way to use Perfectly Matched Layers in the presence of backward guided elastic waves

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The numerical simulation of the interaction of a guided elastic wave with a defect has several applications, particularly in the context of ultrasonic non destructive testing. A simple way to reduce the finite element mesh to a small section of the waveguide enclosing the defect is to extend this domain with the now famous perfectly matched layers (PML), which are supposed to avoid spurious reflections due to the artificial boundaries. Unfortunately it has been shown (see for instance [3]) that this approach fails in the presence of backward modes, which indeed may occur in elastic waveguides for some ranges of frequencies. We show here that a clever use of the method allows to retrieve the expected solution at low additional cost.

The diffraction problem in a waveguide. Let $\Omega_0 = S \times \mathbb{R}$ denote the domain occupied by an elastic waveguide invariant along the axial direction z, $S \subset \mathbb{R}^2$ being the bounded cross-section of the guide. It is also assumed that the guide has a free surface. At a given frequency, the modes of the waveguide are the variable separation solutions to the homogeneous equations of the problem, of the form

$$\mathbf{u}(x_S, z) = \mathbf{u}_n(x_S)e^{i\beta_n z},$$

where $\mathbf{u}$ denotes the displacement field, x_S is the transverse variable and β_n is the axial wavenumber of the considered mode. A mode is said to be propagative if $\beta_n \in \mathbb{R}$ and evanescent if $\mathrm{Im}(\beta_n) \neq 0$. A propagative mode is rightgoing (resp. leftgoing) if its group velocity $\frac{\partial\omega}{\partial\beta}$ is positive (resp. negative) and an evanescent mode is rightgoing (resp. leftgoing) if $\mathrm{Im}(\beta_n) > 0$ (resp. $\mathrm{Im}(\beta_n) < 0$). Using Auld's convention, the rightgoing modes are indexed by $n \in \mathbb{N}^*$, the leftgoing modes then being given by

$$\mathbf{u}(x_S, z) = \mathbf{u}_{-n}(x_S)e^{-i\beta_n z},\ n \in \mathbb{N}^*.$$

Consider now a local perturbation of this waveguide (a crack for instance), Ω denoting the perturbed propagation domain, and a rightgoing propagative mode coming from $z = -\infty$, $\mathbf{u}_{inc}(x_S, z) = \mathbf{u}_{n_0}(x_S)\, e^{i\beta_{n_0} z}$, as an incident field. Then, the total displacement field $\mathbf{u}$ satisfies the following problem

$$\begin{aligned} &\mathrm{div}(\sigma(\mathbf{u})) + \rho\omega^2\mathbf{u} = 0 \text{ in } \Omega, \\ &\sigma(\mathbf{u}) \cdot n = 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

where $\sigma(\mathbf{u})$ denotes the stress tensor, ρ is the mass density, ω is the pulsation and n is the unit outward normal vector to the boundary $\partial\Omega$. The radiation condition implies that the diffracted field $\mathbf{u}_{dif} = \mathbf{u} - \mathbf{u}_{inc}$ is outgoing, which means that it is a superposition of outgoing modes at some distance from the perturbation:

$$\mathbf{u}_{dif}(x_S, z) = \sum_{n\in\mathbb{N}^*} a_n^{\pm}\mathbf{u}_{\pm n}(x_S)\, e^{\pm i\beta_n z} \quad \text{for } \pm z > d. \tag{2}$$

The perfectly matched layers. A way to solve numerically this diffraction problem is to modify the equations in the so-called layers, that is outside of a bounded domain Ω_b containing the perturbation, by making the following substitution in the equations $\frac{\partial}{\partial z} \longrightarrow \alpha \frac{\partial}{\partial z}$, where α is a complex number or, more generally, a complex-valued function. The axial wavenumbers of the modes are modified accordingly and condition (2) becomes

$$\mathbf{u}_{dif,\alpha}(x_S, z) = \sum_{n \in \mathbb{N}^*} a_n^{\pm} \mathbf{u}_{\pm n}(x_S)\, e^{\pm \mathrm{i} \frac{\beta_n}{\alpha} z} \quad \text{for } \pm z > d \tag{3}$$

If $\mathrm{Im}(\beta_n/\alpha) > 0$ for all $n \in \mathbb{N}^*$, $\mathbf{u}_{dif,\alpha}$ is exponentially decreasing and a truncation of the computational domain is possible [2]. In the absence of backward modes, real wavenumbers β_n (associated to propagative modes) are always positive and the condition $\mathrm{Im}(\beta_n/\alpha) > 0$ is achieved by taking $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Im}(\alpha) < 0$. But if backward modes exist (let us assume there are N of them, ordered from 1 to N, so that $\beta_n < 0$ for $n = 1, 2, \ldots, N$), such a choice will not select the correct outgoing solution of the diffraction problem (1). The computed field $\tilde{\mathbf{u}}$ will then satisfy the following condition

$$\tilde{\mathbf{u}}_{dif,\alpha}(x_S, z) = \sum_{n=1}^{N} b_n^{\pm} \mathbf{u}_{\mp n}(x_S)\, e^{\mp \mathrm{i} \frac{\beta_n}{\alpha} z} + \sum_{n > N} b_n^{\pm} \mathbf{u}_{\pm n}(x_S)\, e^{\pm \mathrm{i} \frac{\beta_n}{\alpha} z} \quad \text{for } \pm z > d, \tag{4}$$

instead of (3).

In order to overcome this difficulty, we aim at computing the difference $\mathbf{w}$ between the correct solution $\mathbf{u}$ and the computed one $\tilde{\mathbf{u}}$. The main result is that $\mathbf{w}$ has the following expression

$$\mathbf{w} = \sum_{n=1}^{N} c_n^{+} \mathbf{w}_n^{+} + \sum_{n=1}^{N} c_n^{-} \mathbf{w}_n^{-},$$

where

- each of the fields $\mathbf{w}_n^{\pm}$ is the solution to an auxiliary diffraction problem associated to the n^{th} $\pm$ backward mode, which is solved by using PML,
- the scalars $c_n^{\pm}$ are obtained by inverting a $2N \times 2N$ linear system; the coefficients of this system are the modal amplitudes of the diffracted fields associated to the $\mathbf{w}_n^{\pm}$ and are computed by making use of Fraser's biorthogonality relation [1].

Let us emphasize that solving these auxiliary problems does not require any new computation or assembly of finite element matrices since they are the same as in the original PML problem, used to compute $\tilde{\mathbf{u}}$. The additional cost for the correction of the computed solution is then restrained.

References

[1] *Transparent boundary conditions for the harmonic diffraction problem in an elastic waveguide*, V. Baronian, A.-S. Bonnet-Ben Dhia, and E. Lunéville, JCAM, doi:10.1016/j.cam.2009.08.045, 2009.

[2] *Perfectly matched layers for the convected Helmholtz equation*, E. Bécache, A.-S. Bonnet-Ben Dhia, and G. Legendre, SIAM J. Numer. Anal., 42, pp. 409–433, 2004.

[3] *Guided elastic waves and perfectly matched layers*, E. A. Skelton, S. D.M. Adams, and R. V. Craster, Wave Motion 44, pp. 573-592, 2007.