STABILITY RESULTS FOR THE IDENTIFICATION OF GENERALIZED IMPEDANCE BOUNDARY COEFFICIENTS

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Talk abstract

We are interested in the identification of a Generalized Impedance Boundary Condition from the far–fields created by one or several incident plane waves at a fixed frequency. We focus on the particular case where this boundary condition is expressed as a second order surface operator: the inverse problem then amounts to retrieve the two functions λ and μ that define this boundary operator. We first derive global Lipschitz stability estimates for the identification of these coefficients. We then establish a new type of stability estimate for the identification of λ and μ when inexact knowledge of the boundary is assumed. Finally, we introduce an optimization method to identify λ and μ using in particular a H^1-type regularization of the gradient. We show some numerical results in two dimensions by assuming either an exact knowledge of the shape of the obstacle or an approximate one.

The forward problem

Statement of the problem

Let D be an open bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) with C^3 boundary, $\Omega_R = B_R \setminus \overline{D}$ where B_R is the ball of radius R and let u^i be a solution to the Helmholtz equation in $\mathbb{R}^d$ for a given wave number k. The scattering problem with generalized impedance boundary conditions (GIBC) amounts to find $u = u^s + u^i$ in $V_R := \{v \in H^1(\Omega_R);\, v_{|\partial D} \in H^1(\partial D)\}$ such that

$$\mathcal{P}(\lambda,\mu,\partial D)\begin{cases} \Delta u + k^2 u = 0 \quad \text{inside } \Omega_R \\ \text{div}_{\partial D}(\mu \nabla_{\partial D} u) + \dfrac{\partial u}{\partial \nu} + \lambda u = 0 \quad \text{on } \partial D \\ \dfrac{\partial u}{\partial r} - S_R(u) = \dfrac{\partial u^i}{\partial r} - S_R(u^i) \ \text{ on } \partial B_R \end{cases}$$

where ν is the outgoing normal on ∂D, $\text{div}_{\partial D}$ is the surface divergence, $\nabla_{\partial D}$ the surface gradient and λ and μ are two complex–valued functions. The so–called Dirichlet–to–Neumann map $S_R : H^{1/2}(\partial B_R) \mapsto H^{-1/2}(\partial B_R)$ is defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g := \partial u^e/\partial r|_{\partial B_R}$ where u^e is the radiating solution of the Helmholtz equation outside B_R and $u^e = g$ on ∂B_R. GIBCs can be interpreted for instance as an asymptotic model for perfect conductors coated with thin layer. The coefficients λ and μ are related to the physical characteristics and the width of the coating (e.g. [8]).

Uniform bound on the solution

If one assumes that λ and μ are $L^\infty(\partial D)$ functions that satisfy

$$\Im m(\lambda) \geq 0\,,\ \Im m(\mu) \leq 0 \quad \text{a.e. on } \partial D$$

and there exists $c > 0$ such that

$$\Re e(\mu) \geq c \qquad \text{a.e. on } \partial D$$

then problem $\mathcal{P}(\lambda,\mu,\partial D)$ has a unique solution. Let K be a compact set of $L^\infty(\partial D)^2$ such that if $(\lambda,\mu) \in K$ then conditions here above are satisfied for $c = c_K$. There exists a constant C_K such that for all $(\lambda,\mu) \in K$ the solution u to $\mathcal{P}(\lambda,\mu,\partial D)$ satisfies the following uniform energy bound

$$\|u\|_{V_R} \leq C_K.$$

This uniform estimate is crucial to prove the stability results that follow.

About stability for the inverse coefficient problem

The inverse coefficient problem

We recall the asymptotic behaviour for the scattered field:

$$u^s(x) \sim \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^\infty(\hat{x}) + O\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty$$

uniformly for all the directions $\hat{x} = x/r \in S^{d-1}$ where S^{d-1} is the unit sphere. Then we define the far–field map

$$T: \quad (\lambda,\mu,\partial D) \rightarrow u^\infty$$

where u^∞ is the far–field associated with the scattered field $u^s = u - u^i$ and u is the unique solution of problem $\mathcal{P}(\lambda,\mu,\partial D)$. The inverse coefficients problem is the following: given an obstacle D, an incident plane wave $u^i = e^{ik\hat{d}\cdot x}$ determined by its incident direction $\hat{d} \in S^{d-1}$ and the generated far–field pattern u^∞ for all $\hat{x} \in S^{d-1}$, retrieve the corresponding impedance coefficients λ and

μ (if they exist). This inverse problem is non linear and severely ill–posed since u^∞ is analytic. Furthermore, uniqueness for the identification of λ and μ from a single incident wave may fail if one makes no additional assumption on the coefficients even in the case where μ is a constant (e.g. [4]). We shall restrict ourselves in the following to the cases where uniqueness holds and prove that some Lipschitz stability estimate is verified when the coefficients belong to some special finite dimensional compact set.

Stability for λ if μ is known

The only stability problem addressed in the literature is $\mu = 0$. On the one hand global log type stability for the identification of Lipschitz continuous λ is established in [12] (see also [9]). On the other hand, a simple adaptation of [13] provides a Lipschitz stability estimate in the case of piecewise constant functions λ. We shall derive a Lipschitz type estimate for λ if μ is a known $C^1(\partial D)$ function and λ belongs to a finite dimensional subspace of $L^\infty(\partial D)$ spanned by a family $(\varphi_n)_{n=1,\cdots,N}$ of linearly independent continuous real–valued functions on ∂D that satisfies the following condition. There exist $C_\varphi > 1$ and $c_\varphi > 0$ such that for each $n \in [1, N]$ there exists a non–empty open set $S_n \subset \partial D$ such that

- $|\varphi_n(x)| \geq C_\varphi \sum_{m \neq n} |\varphi_m(x)|$ for all $x \in S_n$,
- $|\varphi_n(S_n)| \geq c_\varphi$.

Consider a bounded set $\Lambda \subset \text{span}_{\mathbb{C}}\{\varphi_1, \cdots, \varphi_N\} \times C^1(\partial D)$, then the following estimate holds.

Theorem 1. *There exists a constant C_Λ such that for all (λ_1, μ) and (λ_2, μ) in Λ and for $u^{1,\infty} = T(\lambda_1, \mu, \partial D)$ and $u^{2,\infty} = T(\lambda_2, \mu, \partial D)$ we have*

$$\|\lambda_1 - \lambda_2\|_{L^\infty(\partial D)} \leq C_\Lambda \|u^{1,\infty} - u^{2,\infty}\|_{L^2(S^{d-1})}$$

Stability for μ if λ is known

Conversely to the previous part, we shall assume that λ is known. Furthermore we assume that λ and μ belong to a bounded set $\mathcal{B}$ of piecewise constant functions on a given partition of ∂D. Let I be an integer and $(\partial D_i)_{i=1,\cdots,I}$ be I non–overlapping open sets of ∂D such that $\cup_{i=1}^I \overline{\partial D_i} = \partial D$. Then λ and μ are defined with I constants $(\lambda_i)_{i=1,\cdots,I}$ and $(\mu_i)_{i=1,\cdots,I}$ respectively by

$$\lambda(x) = \sum_{i=1}^I \lambda_i \chi_{\partial D_i}(x) \quad \text{and} \quad \mu(x) = \sum_{i=1}^I \mu_i \chi_{\partial D_i}(x)$$

for $x \in \partial D$. We first establish a uniqueness result for μ in this particular case, and adding a geometric assumption on the obstacle combining analyticity and visibility from infinity (see Figure 2 in [4]).

Proposition 1. *Take (λ, μ^1) and (λ, μ^2) in $\mathcal{B}$ and assume that their corresponding far–fields $u^{1,\infty} = T(\lambda, \mu^1, \partial D)$ and $u^{2,\infty} = T(\lambda, \mu^2, \partial D)$ satisfy*

$$u^{1,\infty}(\hat{x}) = u^{2,\infty}(\hat{x}) \qquad \forall \hat{x} \in S^{d-1}.$$

If for all $i = 1, ..., I$, there exists $\widetilde{x}_i \in \partial D_i$ and $\eta_i > 0$ such that $\widetilde{\partial D}_i = \partial D_i \cap B(\widetilde{x}_i, \eta_i)$ is

- *for $d = 2$ either a segment or a portion of a circle,*
- *for $d = 3$ either a portion of a plane, or a portion of a cylinder, or a portion of a sphere,*

and the sets $\{x + \gamma\nu(x),\, x \in \widetilde{\partial D}_i,\, \gamma > 0\}$ are included in Ω, then $\mu_1 = \mu_2$.

Moreover, a Lipschitz stability estimate holds.

Theorem 2. *Under the same assumptions as in Proposition 1, there exists a constant $C_{\mathcal{B}}$ such that for all (λ, μ^1) and (λ, μ^2) in $\mathcal{B}$ and for $u^{1,\infty} = T(\lambda, \mu_1, \partial D)$ and $u^{2,\infty} = T(\lambda, \mu_2, \partial D)$*

$$\|\mu^1 - \mu^2\|_{L^\infty(\partial D)} \leq C_{\mathcal{B}} \|u^{1,\infty} - u^{2,\infty}\|_{L^2(S^{d-1})}.$$

In the case where λ and μ are unknown the reader will find local stability results in [4] but to our knowledge the problem of global stability is still open.

Stability in the case of inexact knowledge of the obstacle.

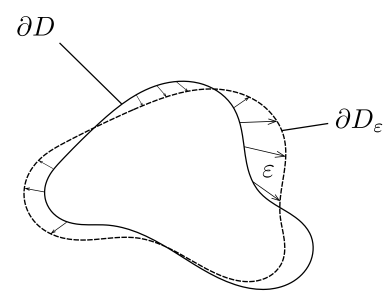

Figure 1: *Illustration of the "perturbed" geometry.*

Here, we are interested in the stability of the reconstruction of λ and μ when exact knowledge of the geometry ∂D is not available. More precisely, let ∂D_ε be a known approximation of ∂D determined by

$$\partial D_\varepsilon := f_\varepsilon(\partial D)$$

where $f_\varepsilon := Id + \varepsilon$ for some function $\varepsilon \in (C^1(\mathbb{R}^d))^d$ such that $\|\varepsilon\|_{(W^{1,\infty}(\mathbb{R}^d))^d} < 1$ (see Figure 1). f_ε is then a C^1–diffeomorphism. We can consider a new problem: given some far–field $T(\lambda_0, \mu_0, \partial D)$, can we reconstruct a good approximation of the impedances λ_0 and μ_0 minimizing the cost function

$$F_\varepsilon(\lambda, \mu) = \|T(\lambda, \mu, \partial D_\varepsilon) - T(\lambda_0, \mu_0, \partial D)\|^2_{L^2(S^{d-1})} \ ?$$

This kind of consideration arises for example if hybrid methods for the reconstruction of the impedance coefficients and the geometry are used (e.g. [10] or [11]) or if the obstacle has been reconstructed using a sampling method (e.g. [6] or [7]) . The answer is positive under the following assumption $(\mathcal{H})$

> There exists a compact $K \subset (L^\infty(\partial D))^2$ and a constant C such that for all $(\lambda, \mu) \in K$ and $(\widetilde{\lambda}, \widetilde{\mu}) \in K$ we have
>
> $$\|\lambda - \widetilde{\lambda}\|_{L^\infty(\partial D)} + \|\mu - \widetilde{\mu}\|_{L^\infty(\partial D)} \leq C\|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|_{L^2(S^{d-1})} .$$

This assumption was the subject of the previous section where we established uniform stability for some particular coefficient problems. We refer to [4] for examples where the constant C depends on λ and μ. In any case,

Theorem 3. *Under assumption* $(\mathcal{H})$ *there exist* ε_0 *and* C *such that for all* $(\lambda, \mu) \in K$, $\|\varepsilon\| \leq \varepsilon_0$ *and* $(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon) \in K$ *that satisfy*

$$\|T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\|_{L^2(S^{d-1})} \leq \delta \quad \text{for } \delta > 0$$

we have

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\|_{L^\infty(\partial D)} + \|\mu_\varepsilon \circ f_\varepsilon - \mu\|_{L^\infty(\partial D)} \leq C(\delta + \|\varepsilon\|).$$

The proof of this Theorem relies on two different properties: continuity of the far–field with respect to the obstacle, uniformly with respect to the impedance coefficients and stability for the inverse coefficient problem for a known obstacle (i.e. assumption $(\mathcal{H})$). It may be adapted to any inverse problem that enjoy this two properties.

Numerical experiments

We illustrate this last result by numerically solving the inverse problem using a non–linear least square method in dimension 2. First of all we give a numerical reconstruction for an exact geometry and then for an inexact geometry. In both cases we enlighten the obstacle with several incident plane waves characterized by their incident direction d_j and the associated far–fields are known on a portion of the unit circle S_j.

The case of an exact geometry

In this part we assume that we exactly know the geometry ∂D (see Figure 3) and we want to retrieve some coefficients (λ_0, μ_0). The method we present hereafter in-

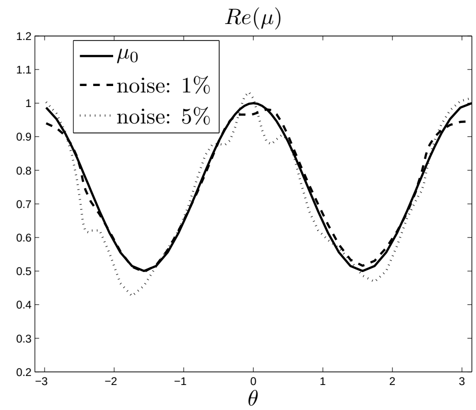

Figure 2: *Reconstruction of* $\mu_0 = 0.5(1 + cos^2(\theta))$, $\mu_{init} = 0.7$ *with* $\lambda = 0$.

volves the minimization of the following non–linear cost function

$$F(\lambda, \mu) = \frac{1}{2}\sum_{j=1}^{N} \|T(\lambda, \mu, \partial D, d_j) - u^\infty_{obs}(\cdot, d_j)\|^2_{L^2(S_j)}$$

with respect to λ and μ where the $u^\infty_{obs}(\cdot, d_j) := T(\lambda_0, \mu_0, \partial D, d_j)$ are the data. At each time step n of the algorithm we update the values of λ_n and μ_n by

$$\begin{cases} \lambda_{n+1} = \lambda_n - \delta\lambda_n \\ \mu_{n+1} = \mu_n - \delta\mu_n \end{cases}$$

where $\delta\lambda_n$ and $\delta\mu_n$ are obtained by a $H^1(\partial D)$ regularization of the Fréchet derivative of F with respect to the variable λ and μ respectively. We present on Figure 2 a reconstruction of the impedance function μ assuming that $\lambda = 0$ is known for several levels of noise on the data. We considered 10 incident plane waves with incidence directions uniformly distributed between $-\pi/2$ and $\pi/2$ with an observation aperture of $\pi/5$. This choice of the incident direction is motivated by the next example. The reader may find a precise description of the algorithm and other numerical reconstructions in [5].

The case of an inexact geometry

To illustrate the stability result with respect to the obstacle stated in Theorem 3 we construct numerically $u^{\infty}_{obs}(\cdot, d_i) = T(\lambda_0, \mu_0, \partial D, d_i)$ for a given geometry ∂D and we minimize the "perturbed" cost function

$$F_{\varepsilon}(\lambda, \mu) = \frac{1}{2} \sum_{i=0}^{N} \|T(\lambda, \mu, \partial D_{\varepsilon}, d_i) - u^{\infty}_{obs}(\cdot, d_i)\|^2_{L^2(S_i)}$$

for a known perturbation ∂D_{ε} (see Figure 3). The perturbation of the geometry is denoted γ and defined by

$$\gamma := \frac{\varepsilon_0}{\text{diam}(D)}.$$

It is reasonable to consider we do not enlighten the obstacle in the direction of the non–convexity (since we have poor knowledge of such area). We still consider 10 incident waves with incident directions uniformly distributed in $[-\pi/2, \pi/2]$. On Figure 4 we can see that the reconstruction remains quite accurate in the enlighten area in the case $\gamma = 3\%$ even if we put $\sigma = 5\%$ of noise on the measurements.

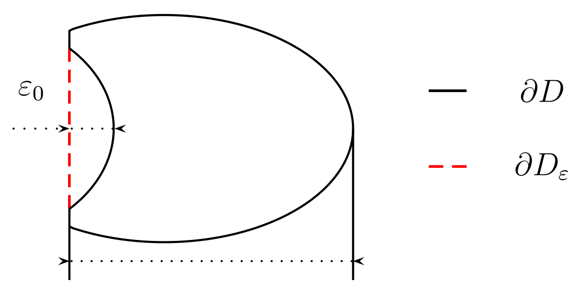

Figure 3: *Perturbed geometry.*

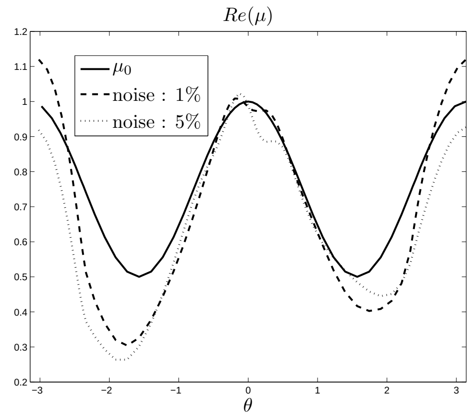

Figure 4: *Perturbed obstacle (see Figure 3),* $\mu_{init} = 0.7$, $\lambda = 0$, $\gamma = 3\%$.

Acknowledgement

The work of Nicolas Chaulet is supported by a DGA (Délégation Générale pour l'Armement) grant.

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