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Path-dependent SDEs with jumps and irregular drift: well-posedness and Dirichlet properties

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Abstract. We discuss a concept of path-dependent SDE with distributional drift with possible jumps. We interpret it via a suitable martingale problem, for which we provide existence and uniqueness. The corresponding solutions are expected to be Dirichlet processes, nevertheless we give examples of solutions which do not fulfill this property. In the second part of the paper we indeed state and prove significant new results on the class of Dirichlet processes.

Résumé. Nous introduisons un concept d'EDS dépendant de la trajectoire avec drift distributionnel et avec sauts. On s'attend que les solutions correspondantes soient des processus de Dirichlet; néanmoins nous exhibons des exemples de solutions ne vérifiant pas cette propriété. Dans la seconde partie de l'article nous prouvons par ailleurs de nouveaux résultats significatifs sur la classe des processus de Dirichlet.

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1. Introduction

In this paper we discuss path-dependent stochastic differential equations with possible distributional drift and jumps of the type

$$dX_s = (\beta'(X_s) + H(s, X^s))ds + \sigma(X_s)dW_s^X + k(x) \star (\mu^X - \nu \circ X) + (x - k(x)) \star \mu^X. \tag{1.1}$$

Here $k : \mathbb{R} \to \mathbb{R}$ is a bounded function such that $k(x) = x$ in a neighborhood of 0, $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function depending on k, $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous function, not vanishing at zero. $H : D_-(0,T) \to B(0,T)$ is a bounded and Borel measurable map, where $D_-(0,T)$ (resp. $B(0,T)$) will denote the space of real càglàd (resp. bounded Borel) functions on $[0, T]$. $\mu^X(ds\,dx)$ is the integer valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ corresponding to the jump measure of X and $(\nu \circ X)(ds\,dx) := Q(X_{s-}, dx)ds$, where $Q(\cdot, dx)$ is a transition kernel from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with $Q(y, \{0\}) = 0$, such that, for some $\alpha \in [0, 1]$, $y \mapsto \int_{\mathbb{R}} (1 \wedge |x|^{1+\alpha})\, Q(y, dx)$ is bounded. A solution of (1.1) is a couple $(X, \mathbb{P})$ under which $(\nu \circ X)$ is the compensator of μ^X, W^X is a Brownian motion and X satisfies (1.1). Those solutions will be shown to be not necessarily Dirichlet processes. One of the aim of the paper is indeed to focus on some pathological aspects of Dirichlet processes.

The Markovian case (with $H = 0$) with continuous paths has now a relatively long history. Diffusions in the generalized sense were first considered in the case when the solution is still a semimartingale, beginning with [18]. Later on, many authors considered special cases of SDEs with generalized coefficients. It is difficult to quote them all, see for the first contributions [10], [11], [5] and [17] for a large bibliography in the semimartingale framework. In [10] and [11], the authors studied time-independent one-dimensional SDEs of the form

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + \beta'(X_t)\mathrm{d}t, \quad t \in [0, T], \tag{1.2}$$

whose solutions are possibly non-semimartingale processes, where σ is a strictly positive continuous function and β' is the derivative of a real-valued continuous function. The only supplementary assumption was the existence of the function

$\Sigma(x) = 2\int_0^x \frac{\beta'}{\sigma^2}(y)dy$, $x \in \mathbb{R}$, considered as a suitable limit via regularizations. Those authors considered solutions in law via the use of a suitable martingale problem. The SDE (1.2) was also investigated by [5], where the authors provided a well-stated framework when σ and β are γ-Hölder continuous, $\gamma > \frac{1}{2}$. In [19], the authors have also shown that in some cases strong solutions exist and pathwise uniqueness holds. More recently, in the time-dependent framework (but still one-dimensional), a significant contribution was done by [8]. As far as the multidimensional case is concerned, some important steps were done in [9] and more recently in [6], when the diffusion matrix is the identity and β' is a time-dependent drift in some proper negative Sobolev space. In the non-Markovian case, at our knowledge, the only contribution, i.e. [17], refers to the continuous case.

We can find recent significant literature in the Markovian case with Lévy α-stable noise, including the multidimensional case. The first contribution in this direction was one-dimensional and made by [1]. Further work was done by [16], [7], and [15], the latter even beyond the so-called Young regime. In these works the Brownian motion is replaced by a Lévy α-stable process, which produces the regularization by noise.

Our work includes a non-Markovian drift H. Nevertheless, even when $H = 0$, i.e. in the Markovian case, we go in a different direction with respect to the present literature. The Markovian component of the generator in our case involves local and non-local components. Our equation is driven by a compensated random measure and the regularizing noise is still the Brownian motion. At our knowledge, our work is the first one in the path-dependent case. Our one-dimensional techniques can be adapted to the multidimensional case by the use of the Zvonkin transformation, see e.g. [9]. We have chosen however to be the most general as possibile in the dimension one: in higher dimension the assumptions that one needs are less general.

SDEs with distributional drift of the type (1.1) will be interpreted via a suitable martingale problem with respect to the integro-differential operator $\mathcal{L}$ defined in (3.5), see Definition 3.1. This consists in describing the stochastic behaviour of $f(X)$ under some probability $\mathbb{P}$, when f belongs to the domain $\mathcal{D}_{\mathcal{L}}$ defined in (3.1). In particular, for every $f \in \mathcal{D}_{\mathcal{L}}$, $f(X)$ is a special semimartingale. $(X, \mathbb{P})$ will be a solution of the aforementioned martingale problem. X is in general a finite quadratic variation process (i.e. $[X, X]$ exists) but not necessarily a Dirichlet process (i.e. the sum of a martingale and a zero quadratic variation process), see Remark 6.1. In turn, it will be shown to be a weak Dirichlet process. We recall that, given a filtration $\mathbb{F}$, an $\mathbb{F}$-weak Dirichlet process is a process of the type $X = M + \Gamma$, where M is an $\mathbb{F}$-local martingale and Γ is an $\mathbb{F}$-orthogonal process vanishing at zero.

Making use of the techniques in [4], equation (1.1) can be rigorously expressed as

$$
\begin{aligned}
X = x_0 + \int_0^{\cdot} \sigma(X_s) dW_s^X + \int_{]0,\cdot]\times\mathbb{R}} k(x)\,(\mu^X(ds\,dx) - Q(X_{s-}, dx)ds) + \lim_{n\to\infty} \int_0^{\cdot} L f_n(X_s) ds \\
+ \int_{]0,\cdot]\times\mathbb{R}} (x - k(x))\mu^X(ds\,dx), \qquad (1.3)
\end{aligned}
$$

for every sequence $(f_n) \subseteq \mathcal{D}_{\mathcal{L}}$ such that $f_n \underset{n\to\infty}{\to} Id$ in C^1, where L is the differential operator introduced in (2.5) restricted to $\mathcal{D}_{\mathcal{L}}$. The limit appearing in (1.3) holds in the u.c.p. sense.

We now recall the main results of the paper. In Section 3 we provide a suitable definition for the aforementioned martingale problem, see Definition 3.1, and state some significant stochastic analysis properties of a solution. In particular in Proposition 3.2 we show that, whenever the drift is a function, a solution $(X, \mathbb{P})$ of the classical martingale problem is a solution to a Stroock-Varadhan martingale problem with jumps where the space of test functions is constituted by C^2 bounded functions. In Section 3.2, we make use of a proper bijective function $h \in \mathcal{D}_L$ introduced in Proposition 2.1: Theorem 3.1 states that $(X, \mathbb{P})$ is a solution to the martingale problem if and only if $(h(X), \mathbb{P})$ is a semimartingale with given characteristics. This is a fundamental tool in order to show existence and uniqueness. In Proposition 3.3 we prove that every solution X is a finite quadratic variation weak Dirichlet process. Section 4 is devoted to well-posedness and continuity properties for the martingale problem. Existence and uniqueness is given in Proposition 4.1 in the Markovian case and in Theorem 4.1 in the non-Markovian case. In Proposition 5.1 we study the continuity of the map $\mathcal{L}$, that is exploited in the companion paper [4]. Finally, in Section 6 we insist on the fact that the process X is not necessarily a Dirichlet process. Moreover, we illustrate some new properties related to Dirichlet processes and some pathological aspects. In Appendix A we justify some technical results, in Appendix B, we discuss the stability of finite quadratic variation processes and in Appendix C we recall some basic properties of semimartingales with jumps.

2. Basic notions

2.1. Preliminaries and notations

C^0 (resp. C_b^0) will denote the space of continuous functions (resp. continuous and bounded functions) on $\mathbb{R}$ equipped with the topology of uniform convergence on each compact (resp. equipped with the topology of uniform convergence). C^1

(resp. C^2) will be the space of continuously differentiable (twice continuously differentiable) functions $u:\mathbb{R}\to\mathbb{R}$. They are equipped with the topology of the uniform convergence on compact intervals of the functions and the corresponding derivatives. C^1_b (resp. C^2_b) is the (topological) intersection of C^1 and C^0_b (resp. C^2 and C^0_b). $D(\mathbb{R}_+)$ will denote the space of real càdlàg functions on $\mathbb{R}_+$. We will also indicate by $||\cdot||_\infty$ the essential supremum norm and by $||\cdot||_{var}$ the total variation norm.

Let $T>0$ be a finite horizon. In the following $D(0,T)$ (resp. $D_-(0,T)$, $C(0,T)$, $B(0,T)$) will denote the space of real càdlàg (resp. càglàd, continuous, bounded Borel) functions on $[0,T]$. Those spaces are equipped with the uniform convergence topology. Given $\eta\in D_-(0,T)$ we will use the notation

$$\eta^t(s):=\begin{cases}\eta(s) & \text{if } s<t\\ \eta(t) & \text{if } s\geq t.\end{cases}$$

For $\eta\in D(0,T)$ we write $\eta^-(t)=\eta(t-)$.

We will denote by $\check{\Omega}$ the canonical space, namely the space $D(0,T)$. We will denote by $\check{X}$ the canonical process defined by $\check{X}_t(\check{\omega})=\check{\omega}(t)$, where $\check{\omega}$ is a generic element of $\check{\Omega}$. We also set $\check{\mathcal{F}}=\sigma(\check{X})$. Given a topological space E, in the sequel $\mathcal{B}(E)$ will denote the Borel σ-field associated with E.

A stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ is fixed throughout the section. We will suppose that $\mathbb{F}$ satisfies the usual conditions. Related to $\mathbb{F}$, $\mathcal{P}$ (resp. $\tilde{\mathcal{P}}:=\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$) will denote the predictable σ-field on $\Omega\times[0,T]$ (resp. on $\tilde{\Omega}:=\Omega\times[0,T]\times\mathbb{R}$).

A process X indexed by $\mathbb{R}_+$ will be said to be with integrable variation if the expectation of its total variation is finite. $\mathcal{A}$ (resp. $\mathcal{A}_{\rm loc}$) will denote the collection of all adapted processes with integrable variation (resp. with locally integrable variation), and $\mathcal{A}^+$ (resp $\mathcal{A}^+_{\rm loc}$) the collection of all adapted integrable increasing (resp. adapted locally integrable) processes. The significance of locally is the usual one which refers to localization by stopping times, see e.g. (0.39) of [13].

The concept of random measure will be extensively used throughout the paper. For a detailed discussion on this topic and the unexplained notations see Chapter I and Chapter II, Section 1, in [14], Chapter III in [13], and Chapter XI, Section 1, in [12]. In particular, if μ is a random measure on $[0,T]\times\mathbb{R}$, for any measurable real function H defined on $\Omega\times[0,T]$, one denotes $H\star\mu_t:=\int_{]0,\,t]\times\mathbb{R}}H(\cdot,s,x)\,\mu(\cdot,ds\,dx)$, when the stochastic integral in the right-hand side is defined (with possible infinite values).

We recall that a transition kernel $Q(a,db)$ of a measurable space $(A,\mathcal{A})$ into another measurable space $(B,\mathcal{B})$ is a family $\{Q(a,\cdot):a\in A\}$ of positive measures on $(B,\mathcal{B})$, such that $Q(\cdot,C)$ is $\mathcal{A}$-measurable for each $C\in\mathcal{B}$, see for instance in Section 1.1, Chapter I of [14].

Let X be an adapted (càdlàg) process, so that $X:\Omega\to\check{\Omega}$. We set the corresponding jump measure μ^X by

$$\mu^X(dt\,dx)=\sum_{s\leq T}\mathbb{1}_{\{\Delta X_s\neq 0\}}\,\delta_{(s,\Delta X_s)}(dt\,dx). \tag{2.1}$$

We denote by ν^X the compensator of μ^X, see [14] (Theorem 1.8, Chapter II). From now on for such a process X, $(\mathcal{F}^X_t)$ will denote the corresponding canonical filtration, which will be omitted when self-explanatory.

2.2. *Recalls on generators with distributional drift*

Let $\sigma,\beta\in C^0$ such that $\sigma>0$. We consider formally the PDE operator of the type

$$L\psi=\frac{1}{2}\sigma^2\psi''+\beta'\psi' \tag{2.2}$$

in the sense introduced by [10, 11]. Below we recall some basic analysis tools coming essentially from Section 2 in [10].

Definition 2.1. For a mollifier ρ in the space of Schwartz functions with $\int_{\mathbb{R}}\rho(x)dx=1$, we set

$$\rho_{\frac{1}{n}}(x):=n\,\phi(nx),\quad \beta'_n:=\beta'*\rho_{\frac{1}{n}},\quad \sigma_n:=\sigma*\rho_{\frac{1}{n}},\quad L_n\psi:=\frac{1}{2}\sigma_n^2\psi''+\beta'_n\psi'.$$

Remark 2.1. A priori σ_n,β'_n and L_n depend on the mollifier ρ.

In the sequel we will make use of the standing assumption below.

Hypothesis 2.1. *We assume the existence of the function*

$$\Sigma(x):=\lim_{n\to\infty}2\int_0^x\frac{\beta'_n}{\sigma_n^2}(y)dy \tag{2.3}$$

in C^0, independently from the mollifier.

Hypothesis 2.2. *The function Σ in (2.3) is lower bounded, and $\int_{-\infty}^{0} e^{-\Sigma(x)}dx = \int_0^{+\infty} e^{-\Sigma(x)}dx = +\infty$.*

The following definition and proposition are given in [10], see respectively Proposition 2.3 and the Definition in Section 2.

Definition 2.2. Set

$$\mathcal{D}_L := \{f \in C^1 : \ f'e^{\Sigma} \in C^1\}. \tag{2.4}$$

For any $f \in \mathcal{D}_L$, we introduce

$$Lf = \frac{\sigma^2}{2}(e^{\Sigma} f')' e^{-\Sigma}. \tag{2.5}$$

This defines without ambiguity $L : \mathcal{D}_L \subset C^1 \to C^0$, and shows that $f \mapsto Lf$ is a continuous map with respect to the graph topology of L, i.e., $Lf_n \to Lf$ in $\mathcal{D}_L$ if and only if $f_n \to f$ in C^1 and $Lf_n \to Lf$ in C^0.

Remark 2.2.
(i) Setting $\psi = f \in C^1$ in (2.5), which does not necessarily belong to $\mathcal{D}_L$ in Definition 2.2, we formally find the expression (2.2).
(ii) If $f \in \mathcal{D}_L$, (2.5) is a rigorous representation of (2.2).

Proposition 2.1. *Hypothesis 2.1 is equivalent to ask that there is a solution $h \in \mathcal{D}_L$ to $Lh = 0$ such that $h(0) = 0$ and*

$$h'(x) := e^{-\Sigma(x)}, \quad x \in \mathbb{R}. \tag{2.6}$$

In particular, $h'(0) = 1$, and h' is strictly positive so that h is bijective and the inverse function $h^{-1} : \mathbb{R} \to \mathbb{R}$ is well-defined and continuous.

Remark 2.3. $\mathcal{D}_L$ is a topological subspace of C^1, equipped with the graph topology of L. Notice that in general the space of smooth functions with compact support is not included in $\mathcal{D}_L$.

Definition 2.3. We denote by L^0 the classical PDE operator $L^0\psi(y) = \frac{\sigma_0^2}{2}\psi''(y)$ with

$$\sigma_0(y) = (\sigma h')(h^{-1}(y)). \tag{2.7}$$

We recall the following facts, that are collected in Lemma 2.9, and in Propositions 2.10 and 2.13 in [10].

Proposition 2.2. *Assume Hypotheses 2.1 and 2.2. The following holds.*

(a) *$\mathcal{D}_L$ is dense in C^1.*
(b) *For any $f \in \mathcal{D}_L$ we have $f^2 \in \mathcal{D}_L$, and*

$$Lf^2 = 2fLf + (f'\sigma)^2. \tag{2.8}$$

In particular, $h^2 \in \mathcal{D}_L$ and $Lh^2 = (h'\sigma)^2$.
(c) *$\mathcal{D}_{L^0} = C^2$.*
(d) *$\phi \in \mathcal{D}_{L^0}$ if and only if $\phi \circ h \in \mathcal{D}_L$. Moreover, $L(\phi \circ h) = (L^0\phi) \circ h$ for every $\phi \in C^2$.*

We will also need the following assumption referred to some $\alpha \in [0,1]$. $C^{1+\alpha}_{\text{loc}}$ denotes the set of functions belonging to C^1 whose derivative belongs to C^{α}_{loc}. If $\alpha \in (0,1)$, C^{α}_{loc} denotes the space of locally α-Hölder continuous functions, i.e. the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that, for every $M > 0$, if $|y| \le M$, $|z| \le M$, there exists C_M such that $|f(y) - f(z)| \le C_M|y - z|^{\alpha}$. C^0_{loc} (resp. C^1_{loc}, C^2_{loc}) denotes by convention C^0 (resp. C^1, C^2).

Hypothesis 2.3. *The function Σ introduced in (2.3) belongs to C^{α}_{loc}.*

Remark 2.4. Hypotheses 2.1, 2.2 and 2.3 imply that the function h defined in Proposition 2.1 belongs to $C^{1+\alpha}_{\text{loc}}$ and that h' is bounded.

3. The martingale problem

3.1. Formulation of the martingale problem and related properties

From here on we fix a truncation function $k \in \mathcal{K}$, where as usual $\mathcal{K} := \{k : \mathbb{R} \to \mathbb{R} \text{ bounded: } k(x) = x \text{ in a neighborhood of } 0\}$. Let L be a given operator of the form (2.2) depending on some given functions σ and β. Assume the validity of Hypotheses 2.1 and 2.2, and let h be the function introduced in Proposition 2.1 related to L.

We will consider transition kernels $Q(\cdot, dx)$ from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with $Q(y, \{0\}) = 0$, satisfying the following condition.

Hypothesis 3.1. *For some $\alpha \in [0, 1]$,*

$$y \mapsto \int_{\mathbb{R}} (1 \wedge |x|^{1+\alpha})\, Q(y, dx) \quad \text{is bounded.}$$

Remark 3.1. Let $\mu^X(ds\,dx)$ be the jump measure of a Lévy γ-stable process with $\gamma = (0,2)$. Then $\nu^X(ds\,dx) = Q(y,dx)ds$ with $Q(y,dx) = Q_0(dx) = |x|^{-1-\gamma}dx$. In this case, Hypothesis 3.1 is verified with $\alpha > \gamma - 1$. For instance, if $\gamma \in (0,1)$ then α can be chosen to be zero.

Remark 3.2. Hypothesis 3.1 means that, for some $\alpha \in [0, 1]$, the measure-valued $y \mapsto (1 \wedge |x|^{1+\alpha})\, Q(y, dx)$ is bounded in the total variation norm.

We consider the topological intersection

$$\mathcal{D}_{\mathcal{L}} := \mathcal{D}_L \cap C^{1+\alpha}_{\text{loc}} \cap C^0_b. \tag{3.1}$$

In particular, $C^{1+\alpha}_{\text{loc}} \cap C^0_b$ is a complete metric space equipped with the family of norms $(||f'||_{\alpha,R} + ||f||_\infty)_{R \in \mathbb{N}^*}$, where

$$||g||_{\alpha,R} := \sup_{x,y: x \neq y, |x| \leq R, |y| \leq R} \frac{|g(y) - g(x)|}{|y-x|^\alpha} + \sup_{x:|x| \leq R} |g(x)|. \tag{3.2}$$

Proposition 3.1. *The set $\mathcal{D}_{\mathcal{L}}$ in (3.1) is dense in C^1.*

Proof. Define the unit partition $\chi : \mathbb{R} \to \mathbb{R}$ as the smooth function

$$\chi(a) := \begin{cases} 1 & \text{if } a \leq -1 \\ 0 & \text{if } a \geq 0, \end{cases} \tag{3.3}$$

and such that $\chi(a) \in [0, 1]$ for $a \in (-1, 0)$. Set

$$\chi_N(x) := \chi(|x| - N - 1), \quad x \in \mathbb{R}. \tag{3.4}$$

Notice that $\chi_N(x)$ is a smooth function and

$$\chi_N(x) = \begin{cases} 1 & \text{if } |x| \leq N \\ 0 & \text{if } |x| \geq N+1 \\ \in [0, 1] & \text{otherwise.} \end{cases}$$

Let $(\rho_{\frac{1}{N}})$ be a sequence of mollifiers with compact support converging to the delta measure. Let $f \in C^1$, and define an approximating sequence (f_N) of f by setting $f_N(0) = f(0)$ and

$$f'_N := e^{-\Sigma}(f' e^{\Sigma} \chi_N) * \rho_{\frac{1}{N}}.$$

Notice that f_N is continuous and bounded, being f'_N with compact support. By Remark 2.4, since $e^{-\Sigma} \in C^\alpha_{\text{loc}}$, we get that $f'_N \in C^\alpha_{\text{loc}}$ and $f_N \in C^{1+\alpha}_{\text{loc}}$. Moreover, $f_N \in \mathcal{D}_L$ since $f'_N e^{\Sigma} \in C^1$. Finally, f_N converges to f in C^1 since f'_N converges to f' uniformly on compact sets. □

Consider a functional H defined on $D_-(0,T)$ satisfying the following.

Hypothesis 3.2.
1. *$H : D_-(0,T) \to B(0,T)$ is bounded and Borel measurable.*
2. *H fulfills the non-anticipating property, i.e., for every $\eta \in D_-(0,T)$, $H(\eta)(t) = H(\eta^t)(t)$, $t \in [0,T]$.*

For every $f \in \mathcal{D}_{\mathcal{L}}$ in (3.1), we set $\mathcal{L}f : D_-(0,T) \to B(0,T)$ as

$$(\mathcal{L}f)(\eta)(t) := Lf(\eta(t)) + \sigma(\eta(t))H(\eta)(t)f'(\eta(t)) + \int_{\mathbb{R}} (f(\eta(t)+x) - f(\eta(t)) - k(x)\, f'(\eta(t)))Q(\eta(t), dx), \tag{3.5}$$

with L the operator defined in (2.5).

From here on, for every $\Phi : D_-(0,T) \to B(0,T)$, we will denote $\Phi(s,\eta) := \Phi(\eta)(s)$, $\eta \in D_-(0,T)$, $s \in [0,T]$.

Definition 3.1. We say that $(X,\mathbb{P})$ fulfills the (time-homogeneous) martingale problem with respect to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}$ in (3.5) and $x_0 \in \mathbb{R}$, if for any $f \in \mathcal{D}_{\mathcal{L}}$, the process

$$M^f := f(X_\cdot) - f(x_0) - \int_0^\cdot (\mathcal{L}f)(s, X^-) ds \tag{3.6}$$

is an $(\mathcal{F}_t^X)$-local martingale under $\mathbb{P}$.

Remark 3.3. Hypothesis 3.1 implies that $y \mapsto \int_{\mathbb{R}} (1 \wedge |x|^2)\, Q(y, dx)$ is bounded. In particular the pair $(X,\mathbb{P})$ in Definition 3.1 satisfies

$$\sum_{s \leq \cdot} |\Delta X_s|^2 < \infty \ \text{ a.s.,} \tag{3.7}$$

see Proposition C.1 in [4].

Remark 3.4. Let $k \in \mathcal{K}$ be a generic truncation function. While $\mathcal{D}_{\mathcal{L}}$ in (3.1) does not depend on k, clearly $\mathcal{L}$ defined in (3.5) a priori depends on k, namely $\mathcal{L} = \mathcal{L}^k$. In order to formulate a coherent definition, we should allow β also depending on k, as we will explain below. This in particular forces $L = L^k$ to depend on k as well.

Indeed, let $\tilde{k} \in \mathcal{K}$. By (3.5), for every $\eta \in D_-(0,T)$, we have

$$(\mathcal{L}^k f)(\eta)(t) - (\mathcal{L}^{\tilde{k}} f)(\eta)(t) = L^k f(\eta(t)) - L^{\tilde{k}} f(\eta(t)) + f'(\eta(t))) \int_{\mathbb{R}} (\tilde{k}(x) - k(x))\, Q(\eta(t), dx).$$

Let $(X,\mathbb{P})$ fulfilling the martingale problem with respect to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}^k$ in (3.5) and $x_0 \in \mathbb{R}$. Then $(X,\mathbb{P})$ fulfills the martingale problem with respect to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}^{\tilde{k}}$ in and x_0, if and only if

$$\int_0^\cdot (L^k f(X_{s-}) - L^{\tilde{k}} f(X_{s-})) ds = \int_0^\cdot f'(X_{s-}) \int_{\mathbb{R}} (k(x) - \tilde{k}(x))\, Q(X_{s-}, dx) ds.$$

This condition is verified if

$$\beta^k(X_{s-}) - \beta^{\tilde{k}}(X_{s-}) = \int_{\mathbb{R}} (k(x) - \tilde{k}(x))\, Q(X_{s-}, dx) ds.$$

With this choice, L^k coincides with $L^{\tilde{k}}$ and consequently $\mathcal{L}^k$ coincides $\mathcal{L}^{\tilde{k}}$.

When β' is a continuous function, we recover the classical martingale problem in the sense of Jacod-Shiryaev, see Proposition 3.2 below. In the following $s(\mathcal{H}, X|\mathbb{P}_{\mathcal{H}}; B, C, \nu)$ denotes the set of all solutions $\mathbb{P}$ related to a given probability $\mathbb{P}_{\mathcal{H}}$ and characteristics (B, C, ν), see Definition C.1.

Proposition 3.2. *Let b, σ be continuous functions, and set*

$$B_t = \int_0^t (b(\check{X}_s) + \sigma(\check{X}_s) H(s, \check{X}^-)) ds, \quad C_t = \int_0^t \sigma^2(\check{X}_s) ds, \quad \nu(ds\, dx) = Q(\check{X}_s, dx) ds, \tag{3.8}$$

with Q satisfying Hypothesis 3.1 with $\alpha = 1$, and H satisfying Hypothesis 3.2. Let L be the operator of the form (2.2) with $\beta' := b$. Set $\mathcal{H} = \{A \in \mathcal{F} : \exists A_0 \in \mathcal{B}(\mathbb{R})$ such that $A = \{\omega \in \Omega : \omega(0) \in A_0\}\}$ and $\mathbb{P}_{\mathcal{H}}$ corresponds to δ_{x_0} in the sense that, for any $A \in \mathcal{F}$, $\mathbb{P}_{\mathcal{H}}(A) = \delta_{x_0}(A_0)$ with $A_0 = \{\omega(0) \in \mathbb{R} :\ \omega \in A\}$.

Then $\mathbb{P}$ belongs to $s(\mathcal{H}, X|\mathbb{P}_{\mathcal{H}}; B, C, \nu)$ if and only if $(X,\mathbb{P})$ is a solution to the martingale problem in Definition 3.1 related to $\mathcal{D}_{\mathcal{L}} = C_b^2$, $x_0 = \check{X}_0$ and $\mathcal{L}$ in (3.5).

Proof. By Theorem C.1 together with Definition C.1, $\mathbb{P}$ belongs to $s(\mathcal{H}, X|\mathbb{P}_{\mathcal{H}}; B, C, \nu)$ if and only if, for any $f \in C_b^2$,

$$f(\check{X}_t) - f(\check{X}_0) - \int_0^t \int_{\mathbb{R}} [f(\check{X}_{s-} + x) - f(\check{X}_{s-}) - k(x) f'(\check{X}_{s-})]\, \nu(ds\, dx)$$
$$- \int_0^t \left[(b(\check{X}_s) + \sigma(\check{X}_s) H(s, \check{X}^-)) f'(\check{X}_s) + \frac{1}{2} \sigma^2(\check{X}_s) f''(\check{X}_s) \right] ds$$

is a $\mathbb{P}$-local martingale. This agrees in particular with Definition 3.1 related to $\mathcal{L}$ in (3.5), $\mathcal{D}_{\mathcal{L}} = C_b^2$ and to $x_0 = \check{X}_0$, where L is the operator of the form (2.2) with $\beta' := b$. □

3.2. About equivalent formulations for the martingale problem

We provide an equivalent martingale formulation for $Y = h(X)$, with h the function introduced in Proposition 2.1. This principle can be extended to general bijective C^1-type transformations. For any $y \in \mathbb{R}$, introduce

$$F(y,A) := \int_{\mathbb{R}} \mathbb{1}_A\left(h(h^{-1}(y)+w) - h(h^{-1}(y))\right) Q(h^{-1}(y),dw), \quad A \subseteq \mathbb{R}, \tag{3.9}$$

$$b(y) := (h' \circ h^{-1})(y) \int_{\mathbb{R}} [(h^{-1})'(y)\, k(z) - k(h^{-1}(y+z) - h^{-1}(y))]\, F(y,dz). \tag{3.10}$$

For any $\phi \in C_b^2$, we also define

$$\bar{L}\phi := L^0\phi + b\,\phi', \tag{3.11}$$

$$\bar{H}(t,\eta) := H(t, h^{-1}(\eta)), \tag{3.12}$$

$$\begin{aligned}(\bar{\mathcal{L}}\phi)(t,\eta) &:= \bar{L}\phi(\eta(t)) + \sigma_0(\eta(t))\bar{H}(t,\eta)\phi'(\eta(t)) \\ &\quad + \int_{\mathbb{R}} (\phi(\eta_t + z) - \phi(\eta_t) - k(z)\,\phi'(\eta(t))\,)F(\eta(t),dz), \quad \eta \in D_-(0,T),\end{aligned} \tag{3.13}$$

with L^0 the operator in Definition 2.3 and σ_0 in (2.7).

Remark 3.5. Let $Q(\cdot,dx)$ be a transition kernel satisfying Hypothesis 3.1 for some $\alpha \in [0,1]$, and H be a functional satisfying Hypothesis 3.2. $(Y,\mathbb{P})$ fulfills the martingale problem in Definition 3.1 with respect to C_b^2, $\bar{\mathcal{L}}$ in (3.13) and $y_0 \in \mathbb{R}$ if and only if, for any $\tilde{f} \in C_b^2$,

$$\begin{aligned}&\tilde{f}(Y_t) - \tilde{f}(y_0) - \int_0^{\cdot} \bar{L}\tilde{f}(Y_s)ds - \int_0^t \sigma_0(Y_s)\bar{H}(s,Y^-)\tilde{f}'(Y_s)ds \\ &- \int_0^t \int_{\mathbb{R}} (\tilde{f}(Y_{s-}+z) - \tilde{f}(Y_{s-}) - k(z)\,\tilde{f}'(Y_{s-})\,)F(Y_{s-},dz)ds\end{aligned} \tag{3.14}$$

is an $(\mathcal{F}_t^Y)$-local martingale under $\mathbb{P}$.

For every $x \in \mathbb{R}$, we define $\mathcal{H}_x : w \mapsto h(x+w) - h(x)$ and its inverse function $\mathcal{H}_x^{-1} : w \mapsto h^{-1}(h(x)+w) - x$.

Remark 3.6. $F(h(x),\cdot)$ is the push forward of $Q(x,\cdot)$ via $\mathcal{H}_x^{-1}$, so that $Q(x,\cdot)$ is the push forward of $F(h(x),\cdot)$ through $\mathcal{H}_x$.

Theorem 3.1. *Let $\alpha \in [0,1]$. Assume Hypotheses 2.1, 2.2 and 2.3 with respect to α. Let $Q(\cdot,dx)$ be a transition kernel satisfying Hypothesis 3.1 with respect to α, and H be a functional verifying Hypothesis 3.2. Then $(X,\mathbb{P})$ fulfills the martingale problem in Definition 3.1 with respect to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}$ in (3.5) and $x_0 \in \mathbb{R}$ if and only if $(Y = h(X),\mathbb{P})$ fulfills the martingale problem in Definition 3.1 with respect to C_b^2, $\bar{\mathcal{L}}$ in (3.13) and $h(x_0)$.*

Proof. $(\Rightarrow)$ Let $\tilde{f} \in C_b^2$ and set $f := \tilde{f} \circ h$. Recalling that $h \in C_{\text{loc}}^{1+\alpha}$, we have $f \in \mathcal{D}_{\mathcal{L}}$ by Proposition 2.2-d). $(X,\mathbb{P})$ fulfills the martingale problem in Definition 3.1 with respect to $\mathcal{D}_{\mathcal{L}}$, $\mathcal{L}$ and x_0 if and only if, for any $f \in \mathcal{D}_{\mathcal{L}}$,

$$f(X_t) - f(x_0) - \int_0^t (\mathcal{L}f)(s,X^-)ds$$

is an $(\mathcal{F}_t^X)$-local martingale under $\mathbb{P}$. Setting $y_0 = h^{-1}(x_0)$, this yields that

$$\tilde{f}(Y_t) - \tilde{f}(y_0) - \int_0^t (\mathcal{L}f)(s, h^{-1}(Y^-))ds$$

is an $(\mathcal{F}_t^X)$-local martingale under $\mathbb{P}$, therefore also an $(\mathcal{F}_t^Y)$-local martingale, since X and Y have the same canonical filtration. Using the form of $\mathcal{L}$ in (3.5) and Proposition 2.2-d), we get that

$$\tilde{f}(Y_t) - \tilde{f}(y_0) - \int_0^t L^0\tilde{f}(Y_s)ds - \int_0^t \sigma(Y_s)H(s,h^{-1}(Y^-))(h' \circ h^{-1})(Y_s)\tilde{f}'(Y_s)ds$$

$$(3.15) \qquad -\int_0^t \int_{\mathbb{R}} [f(X_{s-}+w) - f(X_{s-}) - k(w)\, f'(X_{s-})] Q(X_{s-}, dw) ds$$

is an $(\mathcal{F}_t^Y)$-local martingale under $\mathbb{P}$. From Remark 3.6, we have

$$Q(x, A) = \int_{\mathbb{R}} \mathbb{1}_A\, (\mathcal{H}_x^{-1}(z)) F(h(x), dz).$$

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} [f(X_{s-}+w) - f(X_{s-}) - k(w)\, f'(X_{s-})]\, Q(X_{s-}, dw) \\
&= \int_{\mathbb{R}} [f(X_{s-} + \mathcal{H}_{X_{s-}}^{-1}(z)) - f(X_{s-}) - k(\mathcal{H}_{X_{s-}}^{-1}(z))\, f'(X_{s-})] F(Y_{s-}, dz) \\
&= \int_{\mathbb{R}} [f(h^{-1}(Y_{s-}) + h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-})) - f(h^{-1}(Y_{s-})) \\
&\qquad - k(h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-}))\, f'(h^{-1}(Y_{s-}))]\, F(Y_{s-}, dz) \\
(3.16) \quad &= \int_{\mathbb{R}} [\tilde{f}(Y_{s-}+z) - \tilde{f}(Y_{s-}) - k(h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-}))\, \tilde{f}'(Y_{s-})(h' \circ h^{-1})(Y_{s-})] F(Y_{s-}, dz).
\end{aligned}$$

Plugging (3.16) into (3.15) we get that

$$\begin{aligned}
&\tilde{f}(Y_t) - \tilde{f}(y_0) - \int_0^t L^0 \tilde{f}(Y_s) ds - \int_0^t \sigma(Y_s) H(s, h^{-1}(Y^-))(h' \circ h^{-1})(Y_s) \tilde{f}'(Y_s) ds \\
(3.17) \quad &- \int_0^t \int_{\mathbb{R}} [\tilde{f}(Y_{s-}+z) - \tilde{f}(Y_{s-}) - k(h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-}))(h' \circ h^{-1})(Y_{s-})\, \tilde{f}'(Y_{s-})] F(Y_{s-}, dz) ds
\end{aligned}$$

is an $(\mathcal{F}_t^Y)$-local martingale under $\mathbb{P}$. Formula (3.17) can be equivalently rewritten as

$$\begin{aligned}
&\tilde{f}(Y_t) - \tilde{f}(y_0) - \int_0^t L^0 \tilde{f}(Y_s) ds - \int_0^t \sigma_0(Y_s) H(s, h^{-1}(Y^-)) \tilde{f}'(Y_s) ds \\
&- \int_0^t \int_{\mathbb{R}} [\tilde{f}(Y_{s-}+z) - \tilde{f}(Y_{s-}) - k(z) \tilde{f}'(Y_{s-})] F(Y_{s-}, dz)\, ds \\
&- \int_0^t \tilde{f}'(Y_{s-})(h' \circ h^{-1})(Y_{s-}) \int_{\mathbb{R}} [k(z)\, (h^{-1})'(Y_{s-}) - k(h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-}))] F(Y_{s-}, dz) ds,
\end{aligned}$$

which provides formula (3.14) with the operators $\bar{L}$ and $\bar{H}$ given respectively by (3.11) an (3.12). This finally shows that $(Y, \mathbb{P})$ fulfills the martingale problem in Definition 3.1 related to C_b^2, $\bar{\mathcal{L}}$ in (3.13) and $h(x_0)$.

($\Leftarrow$) Let $f \in \mathcal{D}_{\mathcal{L}}$ and set $\phi = f \circ h^{-1}$. By Proposition 2.2-d) $\phi \in C_b^2$. Then, by assumption,

$$\begin{aligned}
&\phi(Y_t) - \phi(h(x_0)) - \int_0^t L^0 \phi(Y_s) ds - \int_0^t \sigma(Y_s) H(s, h^{-1}(Y^-))(h' \circ h^{-1})(Y_s) \phi'(Y_s) ds \\
&- \int_0^t \int_{\mathbb{R}} (\phi(Y_{s-}+z) - \phi(Y_{s-}) - k(z)\, \phi'(Y_{s-})) F(Y_s, dz) ds \\
&+ \int_0^t \int_{\mathbb{R}} \phi'(Y_s)(h' \circ h^{-1})(Y_s)[k(h^{-1}(Y_s+z) - h^{-1}(Y_s)) - (h^{-1})'(Y_s)\, k(z)] F(Y_s, dz) ds
\end{aligned}$$

is an $(\mathcal{F}_t^Y)$-local martingale under $\mathbb{P}$, that in turn gives that

$$\begin{aligned}
&f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds - \int_0^t \sigma(X_s) H(s, X^-) f'(X_s) ds \\
(3.18) \quad &- \int_0^t \int_{\mathbb{R}} [\phi(Y_{s-}+z) - \phi(Y_{s-}) - \phi'(Y_{s-})(h' \circ h^{-1})(Y_{s-}) k(h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-}))]\, F(Y_{s-}, dz) ds
\end{aligned}$$

is an $(\mathcal{F}_t^X)$-local martingale under $\mathbb{P}$. At this point, using (3.9), we get

$$\int_{\mathbb{R}} [\phi(Y_{s-}+z) - \phi(Y_{s-}) - \phi'(Y_{s-})(h'\circ h^{-1})(Y_{s-})\, k(h^{-1}(Y_{s-}+z) - h^{-1}(Y_{s-}))]\, F(Y_{s-}, dz)$$

$$= \int_{\mathbb{R}} [f(X_{s-}+w) - f(X_{s-}) - f'(X_{s-})\, k(w)]\, Q(X_{s-}, dw). \tag{3.19}$$

Plugging (3.19) into (3.18) we get the result. □

3.3. Weak Dirichlet property

The notion of characteristics of weak Dirichlet processes was introduced in Section 3.3 in [4], extending the classical one for semimartingales, see Appendix C. We will denote by X^c the unique continuous local martingale component of X, see Proposition 3.2 in [4].

Below, $\check{Y}$ replaces $\check{X}$ in the role of canonical process.

Proposition 3.3. *Let $\alpha \in [0,1]$. Assume Hypotheses 2.1, 2.2 and 2.3 with respect to α. Let $Q(\cdot, dx)$ be a transition kernel satisfying Hypotheses 3.1 with respect to α, and H be a functional satisfying Hypothesis 3.2. If $(X, \mathbb{P})$ is a solution to the martingale problem in Definition 3.1 related to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}$ in (3.5) and $x_0 \in \mathbb{R}$, then the following holds.*

1. *$Y = h(X)$ is a semimartingale with characteristics $B = \int_0^\cdot (b(\check{Y}_s) + \sigma_0(\check{Y}_s)\bar{H}(s, \check{Y}^-))ds$, $C = \int_0^\cdot c(\check{Y}_s)ds$, $\tilde{\nu}(ds\, dz) = F(\check{Y}_s, dz)ds$, where σ_0, b and $\bar{H}$ are defined respectively in (2.7), (3.10) and (3.12), $F(y, dz)$ is the measure introduced in (3.9), and $c(y) := \sigma_0^2(y)$.*
2. *X is a weak Dirichlet process of finite quadratic variation with characteristic $\nu(ds\, dw) = Q(\check{X}_{s-}, dw)ds$.*
3. *$\langle X^c, X^c\rangle = \int_0^\cdot \sigma^2(X_s)ds$.*

Proof. 1. It is a direct consequence of Theorems 3.1 and C.1.

2. By definition $X = h^{-1}(Y)$, with $h^{-1} \in C^1$. By item 1., Y is a semimartingale, so it is a weak Dirichlet process of finite quadratic variation. In particular, X has finite quadratic variation, see Lemma B.1-1. Moreover, we can apply Theorem 3.36 in [4] to $h^{-1}(Y)$, so that X turns out to be a weak Dirichlet process. Finally, by item 1. and (3.9),

$$\begin{aligned}
\tilde{\nu}(ds, A) &= F(h(\check{X}_{s-}), A)ds \\
&= \int_{\mathbb{R}} \mathbb{1}_A\,(h(h^{-1}(h(\check{X}_{s-})) + w) - h(h^{-1}(h(\check{X}_{s-}))))\, Q(h^{-1}(h(\check{X}_{s-})), dw)ds \\
&= \int_{\mathbb{R}} \mathbb{1}_A\,(h(\check{X}_{s-}+w) - h(\check{X}_{s-}))\, Q(\check{X}_{s-}, dw)ds.
\end{aligned}$$

Then, by Remark 3.41 in [4] with $v(t,y) = h^{-1}(y)$, $\nu^Y = \nu$ and $\nu^X = \tilde{\nu}$, the characteristic ν of X is given by

$$\begin{aligned}
\nu(A, ds) &= \int_{\mathbb{R}} \mathbb{1}_A\,(h^{-1}(h(\check{X}_{s-}) + z) - \check{X}_{s-})\, \tilde{\nu}(ds,\, dz) \\
&= \int_{\mathbb{R}} \mathbb{1}_A\,(h^{-1}(h(\check{X}_{s-}) + h(\check{X}_{s-}+w) - h(\check{X}_{s-})) - \check{X}_{s-})\, Q(\check{X}_{s-}, dw)ds \\
&= \int_{\mathbb{R}} \mathbb{1}_A\,(h^{-1}(h(\check{X}_{s-}+w)) - \check{X}_{s-})\, Q(\check{X}_{s-}, dw)ds \\
&= \int_{\mathbb{R}} \mathbb{1}_A(w)\, Q(\check{X}_{s-}, dw)ds.
\end{aligned}$$

3. From item 1,

$$C \circ Y = \int_0^\cdot (\sigma^2 h')(h^{-1}(h(X_s)))ds = \int_0^\cdot |h'(X_s)|^2 \sigma^2(X_s)ds.$$

On the other hand, by formula (3.45) in Remark 3.42-(i) in [4],

$$C \circ Y = \int_0^\cdot |h'(X_s)|^2 d\langle X^c, X^c\rangle_s,$$

and the conclusion follows. □

Remark 3.7. If $(X,\mathbb{P})$ is a solution to the martingale problem in Definition 3.1 related to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}$ in (3.5), and $x_0 \in \mathbb{R}$, then it is not necessarily a Dirichlet process.

Consider for instance the case $X = W + S$ with W a Brownian motion and S an γ-stable Lévy process with $\gamma \in (0,1)$. This can be seen as a trivial solution of our martingale problem with $\sigma \equiv 1$ and $Q(y,dx) = Q_0(dx) = |x|^{-1-\gamma}dx$. We remark that X is a Dirichlet process if and only if S is a Dirichlet process. Assume ab absurdo that S is a Dirichlet process. Since S is also a semimartingale, then S is special semimartingale, see Lemma 6.1 and Proposition 5.14 in [3]. However, $x\mathbb{1}_{|x|>1} \star Q_0 = +\infty$, and therefore it cannot be a special semimartingale, see Proposition 2.29, Chapter II, in [14]. Notice that, in the case $\gamma \in [1,2)$, S instead is a special semimartingale because $x\mathbb{1}_{|x|>1} \star Q_0 < +\infty$.

We will state and prove new results on Dirichlet processes in Section 6.

4. Well-posedness of the martingale problem

In order to formulate the well-posedness of the martingale problem we will make use of the following hypothesis about some transition kernel $Q(\cdot, dx)$.

Hypothesis 4.1. *For some* $\alpha \in [0,\, 1]$,

$$y \mapsto (1 \wedge |x|^{1+\alpha})\, Q(y,dx) \quad \text{is continuous in the total variation topology.} \tag{4.1}$$

Remark 4.1. According to Remark 3.1, Hypothesis 4.1 is trivially verified in the case of $Q(y,dx) = Q_0(dx) = |x|^{-1-\gamma}dx$ if $\alpha > \gamma - 1$, being the measure-valued function (4.1) constant.

Remark 4.2.
(i) If Hypothesis 4.1 holds true for some $\alpha \in [0,\, 1]$, then $y \mapsto (1 \wedge |x|^2)\, Q(y,dx)$ is continuous in the total variation topology.
(ii) Item (i) in turn implies that $y \mapsto \int_B (1 \wedge |x|^2)\, Q(y,dx)$ is continuous for all $B \in \mathcal{B}(\mathbb{R})$.

We consider again the functions Σ and h introduced respectively in (2.3) and in Proposition 2.1. We will make the following additional assumption.

Hypothesis 4.2. *Σ is bounded and is α-Hölder continuous in the whole space for some $\alpha \in [0,\, 1]$ (where 0-Hölder continuous means uniformly continuous).*

Remark 4.3.
(i) Under Hypothesis 4.2, h' is upper and lower bounded as well.
(ii) Hypothesis 4.2 implies Hypotheses 2.2 and 2.3.
(iii) For some $\alpha \in (0,\, 1)$, Hypothesis 4.2 is equivalent to ask that Σ belongs to the Besov space $\mathcal{C}^\alpha$, see e.g. Section 2.7 in [2].

We start by considering the Markovian case.

Proposition 4.1. *Let $\alpha \in [0,1]$. Let L be an operator of the form (2.2) with σ bounded. Assume Hypotheses 2.1 and 4.2 with respect to α. Let $Q(\cdot,dx)$ be a transition kernel satisfying Hypotheses 3.1, and 4.1 with respect to α. Then the martingale problem in Definition 3.1 related to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}$ in (3.5) with $H \equiv 0$ and $x_0 \in \mathbb{R}$ admits existence and uniqueness.*

Proof. By Theorem 3.1, existence and uniqueness of the martingale problem in Definition 3.1 with respect to $\mathcal{D}_{\mathcal{L}}$, $\mathcal{L}$ with $H \equiv 0$ and x_0 is equivalent to existence and uniqueness of the martingale problem in Definition 3.1 related to C^2_b $\bar{\mathcal{L}}$ in (3.13) with $\bar{H} \equiv 0$ and $h(x_0)$. On the other hand, by Theorem C.1, $Y = h(X)$ is a solution to the latter martingale problem if and only if it a semimartingale with local characteristics $B = \int_0^{\cdot} b(\check{Y}_s)ds$, $C = \int_0^{\cdot} c(\check{Y}_s)ds$, $\tilde{\nu}(ds\,dz) = F(\check{Y}_s, dz)ds$, with, for every $y \in \mathbb{R}$,

$$F(y,A) := \int_{\mathbb{R}} \mathbb{1}_A\left(h(h^{-1}(y)+w) - h(h^{-1}(y))\right) Q(h^{-1}(y),dw), \quad A \subseteq \mathbb{R},$$

$$b(y) := (h' \circ h^{-1})(y) \int_{\mathbb{R}} [(h^{-1})'(y)\, k(z) - k(h^{-1}(y+z) - h^{-1}(y))]\, F(y,dz),$$

$$c(y) := (\sigma h')^2(h^{-1}(y)).$$

The result will then follow by using Theorem C.2, provided we verify Hypothesis C.1 for b, c and $F(\cdot,dz)$, i.e. that

(i) b is bounded;
(ii) c is bounded, continuous, and not vanishing at zero;
(iii) the function $y \mapsto \int_B (1 \wedge |z|^2) F(y, dz)$ is bounded and continuous for all $B \in \mathcal{B}(\mathbb{R})$.

We start by item (ii). Recall that by Remark 2.4 and Proposition 2.1, we can take $h \in C^1$, h' being bounded and h^{-1} being continuous. Since σ is continuous, this implies that the function c is continuous as well. Moreover, since σ is bounded, c is also bounded. Finally, c is not vanishing at zero by formula (2.6) and the fact that σ is never zero.

We then prove that

(iii)' the function $y \mapsto (1 \wedge |z|^{1+\alpha}) F(y, dz)$ is bounded and continuous in the total variation norm.

In particular, this would imply item (iii), see Remark 4.2. We have

$$\begin{aligned}(1 \wedge |z|^{1+\alpha})\, F(y, dz) &= (1 \wedge |h(h^{-1}(y) + w) - h(h^{-1}(y))|^{1+\alpha})\, Q(h^{-1}(y), dw)\\ &= (1 \wedge (\psi(y, w)\, |w|)^{1+\alpha})\, Q(h^{-1}(y), dw) := I(y; dw)\end{aligned}$$

with

$$\psi(y, w) := \int_0^1 h'(h^{-1}(y) + aw) da.$$

Since h' is bounded, there is a constant C_1 such that $\psi^{1+\alpha} \le C_1$.

Let us first prove the boundedness of the map $y \mapsto I(y; dw)$. We have $I(y; dw) = I_1(y; dw) + I_2(y; dw)$ with

$$\begin{aligned}I_1(y; dw) &:= \mathbb{1}_{\{0<|w|^{1+\alpha} \le \frac{1}{C_1}\}} (1 \wedge (\psi(y, w)\, |w|)^{1+\alpha})\, Q(h^{-1}(y), dw),\\ I_2(y; dw) &:= \mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}} (1 \wedge (\psi(y, w)\, |w|)^{1+\alpha})\, Q(h^{-1}(y), dw).\end{aligned}$$

For $\ell_1(w) := \mathbb{1}_{\{0<|w|^{1+\alpha} \le \frac{1}{C_1}\}}$ and $\ell_2(w) := \mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}}$, we set

$$\begin{aligned}\tilde{Q}^{\ell_1}(h^{-1}(y), dw) &:= \mathbb{1}_{\{0<|w|^{1+\alpha} \le \frac{1}{C_1}\}} |w|^{1+\alpha} Q(h^{-1}(y), dw),\\ \tilde{Q}^{\ell_2}(h^{-1}(y), dw) &:= \mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}} Q(h^{-1}(y), dw).\end{aligned}$$

For every $y \in \mathbb{R}$,

$$||I_1(y; dw)||_{var} \le C_1 \int_{\mathbb{R}} \mathbb{1}_{\{0<|w|^{1+\alpha} \le \frac{1}{C_1}\}} |w|^{1+\alpha}\, Q(h^{-1}(y), dw) = C_1 \sup_{z \in \mathbb{R}} ||\tilde{Q}^{\ell_1}(z, dw)||_{var}$$

and

$$||I_2(y; dw)||_{var} \le \sup_{z \in \mathbb{R}} ||\tilde{Q}^{\ell_2}(z, dw)||_{var},$$

whereas previous supremum are finite by Lemma A.1.

Let us now prove the continuity of the map $y \mapsto I(y; dw)$. Let (y_n) be a real sequence converging to $y_0 \in \mathbb{R}$. We have

$$I(y_n; dw) - I(y_0; dw) = J_1(y_n, y_0; dw) + J_2(y_n, y_0; dw)$$

with

$$\begin{aligned}J_1(y_n, y_0; dw) &:= (1 \wedge (\psi(y_n, w)\, |w|)^{1+\alpha})\, [Q(h^{-1}(y_n), dw) - Q(h^{-1}(y_0), dw)],\\ J_2(y_n, y_0; dw) &:= \{(1 \wedge (\psi(y_n, w)\, |w|)^{1+\alpha}) - (1 \wedge (\psi(y_0, w)\, |w|)^{1+\alpha})\}\, Q(h^{-1}(y_0), dw).\end{aligned}$$

Concerning J_1, we have $J_1 = J_1' + J_1''$, where

$$\begin{aligned}J_1'(y_n, y_0; dw) &:= \mathbb{1}_{\{0<|w|^{1+\alpha} \le \frac{1}{C_1}\}} (1 \wedge (\psi(y_n, w)\, |w|)^{1+\alpha})\, [Q(h^{-1}(y_n), dw) - Q(h^{-1}(y_0), dw)],\\ J_1''(y_n, y_0; dw) &:= \mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}} (1 \wedge (\psi(y_n, w)\, |w|)^{1+\alpha})\, [Q(h^{-1}(y_n), dw) - Q(h^{-1}(y_0), dw)].\end{aligned}$$

We get

$$J_1'(y_n, y_0; dw) = \mathbb{1}_{\{0<|w|^{1+\alpha}\leq \frac{1}{C_1}\}} \psi^{1+\alpha}(y_n, w)\,|w|^{1+\alpha}\,[Q(h^{-1}(y_n), dw) - Q(h^{-1}(y_0), dw)]$$

so that

$$||J_1'(y_n, y_0; dw)||_{var} \leq C_1 ||\tilde{Q}^{\ell_1}(h^{-1}(y_n), dw) - \tilde{Q}^{\ell_1}(h^{-1}(y_0), dw)||_{var}$$

and analogously

$$\begin{aligned} ||J_1''(y_n, y_0; dw)||_{var} &\leq ||\mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}}[Q(h^{-1}(y_n), dw) - Q(h^{-1}(y_0), dw)]||_{var} \\ &\leq ||\tilde{Q}^{\ell_2}(h^{-1}(y_n), dw) - \tilde{Q}^{\ell_2}(h^{-1}(y_0), dw)||_{var}. \end{aligned}$$

The convergence of both terms follows by Lemma A.1 applied respectively to $\tilde{Q}^{\ell_1}(h^{-1}(y_0), dw)$ and $\tilde{Q}^{\ell_2}(h^{-1}(y_0), dw)$, and taking into account the continuity of h^{-1}.

Regarding J_2 we have $J_2 = J_2' + J_2''$ with

$$\begin{aligned} J_2'(y_n, y_0; dw) &:= \{(1 \wedge (\psi(y_n, w)\,|w|)^{1+\alpha}) - (1 \wedge (\psi(y_0, w)\,|w|)^{1+\alpha})\}\mathbb{1}_{\{0<|w|^{1+\alpha}\leq \frac{1}{C_1}\}}(w)\,Q(h^{-1}(y_0), dw), \\ J_2''(y_n, y_0; dw) &:= \{(1 \wedge (\psi(y_n, w)\,|w|)^{1+\alpha}) - (1 \wedge (\psi(y_0, w)\,|w|)^{1+\alpha})\}\mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}}(w)\,Q(h^{-1}(y_0), dw). \end{aligned}$$

Notice that

$$J_2'(y_n, y_0; dw) = (\psi(y_n, w) - \psi(y_0, w))\mathbb{1}_{\{0<|w|^{1+\alpha}\leq \frac{1}{C_1}\}}(w)|w|^{1+\alpha}\,Q(h^{-1}(y_0), dw),$$

so that

$$||J_2'(y_n, y_0; dw)||_{var} \leq \int_{\mathbb{R}} |\psi(y_n, w) - \psi(y_0, w)|\mathbb{1}_{\{0<|w|^{1+\alpha}\leq \frac{1}{C_1}\}}(w)|w|^{1+\alpha}\,Q(h^{-1}(y_0), dw).$$

On the other hand,

$$||J_2''(y_n, y_0; dw)||_{var} \leq \int_{\mathbb{R}} \{(1 \wedge (\psi(y_n, w)\,|w|)^{1+\alpha}) - (1 \wedge (\psi(y_0, w)\,|w|)^{1+\alpha})\}\mathbb{1}_{\{|w|^{1+\alpha} > \frac{1}{C_1}\}}(w)\,Q(h^{-1}(y_0), dw).$$

Therefore $||J_2'(y_n, y_0; dw)||_{var}$ and $||J_2''(y_n, y_0; dw)||_{var}$ converge to zero by the Lebesgue dominated convergence theorem, taking into account respectively the finiteness $\tilde{Q}^{\ell_1}(h^{-1}(y_0), dw)$ and $\tilde{Q}^{\ell_2}(h^{-1}(y_0), dw)$ due to Lemma A.1, and the continuity of h', h^{-1}. This proves (iii)'.

Finally, let us prove item (i). We first notice that

$$\begin{aligned} b(y) &= (h' \circ h^{-1})(y) \int_{\mathbb{R}} [(h^{-1})'(y)\,k(z) - k(h^{-1}(y+z) - h^{-1}(y))]\,F(y, dz) \\ &= (h' \circ h^{-1})(y) \int_{\mathbb{R}} [(h^{-1})'(y)\,k(z) - k(z\,\bar{\psi}(y, z))]\,F(y, dz), \end{aligned} \tag{4.2}$$

with $\bar{\psi}(y, z) := \int_0^1 (h^{-1})'(y + a\,z)\,da$. Also in this case we can find a constant $\bar{C}_1 \geq 1$ such that $\psi \leq \bar{C}_1$. For some $R \in (0, 1)$, define $\mathcal{B}_R := \{z \in \mathbb{R} :\ |z| \leq R\}$ as the neighborhood of $z = 0$ on which $k(z) = z$. We also introduce $\bar{\mathcal{B}} := \{z \in \mathbb{R} :\ |z| \leq \frac{R}{\bar{C}_1}\} \subset \mathcal{B}_R$. Identity (4.2) reads

$$\begin{aligned} b(y) &= (h' \circ h^{-1})(y) \int_{\mathbb{R}} \Big[\int_0^1 ((h^{-1})'(y) - (h^{-1})'(y + a\,z))\,da\Big] z\,\mathbb{1}_{\bar{\mathcal{B}}}(z)\,F(y, dz) \\ &\quad + (h' \circ h^{-1})(y) \int_{\mathbb{R}} [(h^{-1})'(y)\,k(z) - k(z\,\bar{\psi}(y, z))]\,\mathbb{1}_{\bar{\mathcal{B}}^c}(z)\,F(y, dz). \end{aligned} \tag{4.3}$$

In the sequel we suppose $\alpha \in (0, 1]$, the case $\alpha = 0$ needs some easy adaptation. Concerning the boundedness of b, we first notice that by (2.6) together with Hypothesis 4.2, for every $a \in [0, 1]$,

$$|(h^{-1})'(y) - (h^{-1})'(y + a\,z)|\,\mathbb{1}_{\bar{\mathcal{B}}}(z) = |e^{\Sigma(h^{-1}(y))} - e^{\Sigma(h^{-1}(y+az))}|\mathbb{1}_{\bar{\mathcal{B}}}(z)$$

$$\leq C_2\, e^{||\Sigma||_\infty} |h^{-1}(y) - h^{-1}(y+az)|^\alpha \mathbb{1}_{\bar{\mathcal{B}}}(z)$$
$$\leq C_2\, e^{(1+\alpha||\Sigma||_\infty)} |z|^\alpha \mathbb{1}_{\bar{\mathcal{B}}}(z),$$

where C_2 is a Hölder constant for Σ. Therefore by (4.3)

$$|b(y)| \leq ||h'||_\infty C_2\, e^{(1+\alpha||\Sigma||_\infty)} \int_{\mathbb{R}} |z|^{1+\alpha}\, \mathbb{1}_{\bar{\mathcal{B}}}(z)\, F(y,dz) + ||h'||_\infty ||k||_\infty (1 + ||(h^{-1})'||_\infty) \int_{\mathbb{R}} \mathbb{1}_{\bar{\mathcal{B}}^c}(z)\, F(y,dz),$$

and the conclusion follows by Lemma A.1 applied to $\ell_1(z) = \mathbb{1}_{\mathcal{B}}(z)$ and $\ell_2(z) = \mathbb{1}_{\mathcal{B}^c}(z)$. □

We finally can state the general existence and uniqueness theorem for the possibly path-dependent case.

Theorem 4.1. *Let $\alpha \in [0,1]$. Let L be an operator of the form (2.5) with σ bounded. Assume Hypotheses 2.1 and 4.2 with respect to α. Let $Q(\cdot, dx)$ be a transition kernel satisfying Hypotheses 3.1 and 4.1 with respect to α, and H be a functional satisfying Hypothesis 3.2. Then existence and uniqueness holds for the martingale problem in Definition 3.1 related to $\mathcal{D}_{\mathcal{L}}$ in (3.1), $\mathcal{L}$ in (3.5) and $x_0 \in \mathbb{R}$.*

Proof. *Step 1.* Let X be an $(\mathcal{F}_t^X)$-weak Dirichlet with characteristic $\nu(ds\, dx)$ such that $(1 \wedge |x|^2) \star (\nu \circ X) \in \mathcal{A}_{\text{loc}}$, and with $(\mathcal{F}_t^X)$-continuous local martingale X^c under $\mathbb{P}$ such that $\langle X^c, X^c \rangle = \int_0^\cdot \sigma^2(X_s) ds$. We set

$$W_t := \int_0^t \frac{1}{\sigma(X_s)} dX_s^c, \quad t \in [0,T].$$

Consequently W is an $(\mathcal{F}_t^X)$-local martingale with $\langle W, W \rangle_t = t$, and therefore by Lévy's characterization theorem, W is an $(\mathcal{F}_t^X)$-Brownian motion. Let H be a functional defined on $D_-(0,T)$ satisfying Hypotheses 3.2. We define

$$\tilde{W}_t := W_t - \int_0^t H(s, X^-) ds, \quad t \in [0,T]. \tag{4.4}$$

Then, by the Novikov condition,

$$\kappa_t := \exp\Big\{ \int_0^t H(s, X^-) dW_s - \frac{1}{2} \int_0^t |H(s, X^-)|^2 ds \Big\}, \quad t \in [0,T],$$

is an $(\mathcal{F}_t^X)$-martingale. By Girsanov's theorem, $\tilde{W}$ is an $(\mathcal{F}_t^X)$-Brownian motion under the probability $\mathbb{Q}$ defined by

$$d\mathbb{Q} = \kappa_T d\mathbb{P}. \tag{4.5}$$

Let $f \in \mathcal{D}_{\mathcal{L}}$, and set $\eta_s(x) := f(X_{s-} + x) - f(X_{s-})$ and

$$\xi_s(x) := \eta_s(x) \star (\mu^X - (\nu \circ X)).$$

The process ξ is an $(\mathcal{F}_t^X)$-purely discontinuous local martingale under $\mathbb{P}$, see considerations in Definition 1.27-(ii), Chapter II, in [14]. In particular $\langle \xi, M \rangle = 0$ for every continuous local martingale M. We claim that

$$\xi \text{ remains an } (\mathcal{F}_t^X)\text{-local martingale under } \mathbb{Q}. \tag{4.6}$$

Indeed, set $\tau_n := \inf\{t \in [0,T] :\ |X_{t-}| > n\}$. We recall that the càglàd process (X_{t-}) is locally bounded. Then the process $\xi^n := \xi \mathbb{1}_{[0,\tau_n]}$ is a (square integrable) martingale under $\mathbb{P}$. As a matter of fact, $\eta_s(x) \mathbb{1}_{[0,\tau_n]}(s) \in \mathcal{L}^2(\mu^X)$ (and in particular belongs to $\mathcal{G}^2(\mu^X)$, see the end of Section 2 in [4]) since

$$\eta_s^2(x) \mathbb{1}_{\{|x|>1\}} \leq 4||f||_\infty^2,$$
$$\eta_s^2(x) \mathbb{1}_{[0,\tau_n]}(s) \mathbb{1}_{\{|x|\leq 1\}} \leq ||f'(\cdot) \mathbb{1}_{[-(n+1),n+1]}(\cdot)||_\infty\, x^2 \mathbb{1}_{\{|x|\leq 1\}}.$$

To prove that ξ^n remains an $(\mathcal{F}_t^X)$-martingale under $\mathbb{Q}$, we need to show that, for every $\mathcal{F}_s^X$-measurable random variable F, $\mathbb{E}^{\mathbb{Q}}[(\xi_t^n - \xi_s^n) F] = 0$. Indeed, the left-hand side gives

$$\mathbb{E}^{\mathbb{P}}[\kappa_T (\xi_t^n - \xi_s^n) F] = \mathbb{E}^{\mathbb{P}}[(\kappa_t - \kappa_s)(\xi_t^n - \xi_s^n) F] = \mathbb{E}^{\mathbb{P}}[(\langle \kappa, \xi \rangle_t^n - \langle \kappa, \xi^n \rangle_s) F] = 0,$$

since $\langle \kappa, \xi^n \rangle = 0$, being ξ^n an $(\mathcal{F}_t^X)$-purely discontinuous local martingale. This shows that ξ is an $(\mathcal{F}_t^X)$-local martingale under $\mathbb{Q}$.

Step 2: existence. Let $(X, \mathbb{P})$ be a solution to the martingale problem in Definition 3.1 related to $\mathcal{D}_\mathcal{L}$ in (3.1), $\mathcal{L}$ in (3.5) with $H \equiv 0$ and $x_0 \in \mathbb{R}$. By Proposition 3.3 with $H \equiv 0$, X is an $(\mathcal{F}_t^X)$-weak Dirichlet with characteristic $\nu(ds\,dx) = Q(\check{X}_{s-}, dx)ds$, and with $(\mathcal{F}_t^X)$-continuous local martingale X^c under $\mathbb{P}$ such that $\langle X^c, X^c \rangle = \int_0^\cdot \sigma^2(X_s)ds$. By the uniqueness of the decomposition for special weak Dirichlet processes and Corollary 3.37 in [4], for every $f \in \mathcal{D}_\mathcal{L}$ we have

$$
\begin{aligned}
&f(X_\cdot) - f(x_0) - \int_0^\cdot (\mathcal{L}f)(s, X^-)ds \\
&= \int_0^\cdot (f'\sigma)(X_s)dW_s + \int_0^\cdot (f(X_{s-} + x) - f(X_{s-}))(\mu^X(ds\,dx) - Q(X_{s-}, dx)ds).
\end{aligned}
\tag{4.7}
$$

Plugging in (4.7) the process $\tilde{W}$ defined in (4.4), we get

$$
\begin{aligned}
&f(X_\cdot) - f(x_0) - \int_0^\cdot (\mathcal{L}f)(s, X^-)ds - \int_0^\cdot (f'\sigma)(X_s)H(s, X^-)ds \\
&= \int_0^\cdot (f'\sigma)(X_s)d\tilde{W}_s + \int_0^\cdot (f(X_{s-} + x) - f(X_{s-}))(\mu^X - \nu \circ X)(ds\,dx).
\end{aligned}
$$

Let $\mathbb{Q}$ be the probability constructed in (4.5). By (4.6) in Step 1.

$$
\int_0^\cdot (f'\sigma)(X_s)d\tilde{W}_s + \int_0^\cdot (f(X_{s-} + x) - f(X_{s-}))(\mu^X - \nu \circ X)(ds\,dx)
$$

is an $(\mathcal{F}_t^X)$-local martingale under $\mathbb{Q}$. Therefore, $(X, \mathbb{Q})$ is proved to be a solution to the martingale problem in the statement.

Step 3: uniqueness. Let $(X^i, \mathbb{P}^i)$, $i = 1, 2$, be two solutions of the martingale problem in Definition 3.1 related to $x_0 \in \mathbb{R}$, $\mathcal{D}_\mathcal{L}$ in (3.1), and $\mathcal{L}$ in (3.5). By Proposition 3.3-(2)(3), X^i is an $(\mathcal{F}_t^{X^i})$-weak Dirichlet with characteristic $\nu(ds\,dx) = Q(\check{X}_{s-}, dx)ds$, with $(\mathcal{F}_t^{X^i})$-local martingale $X^{i,c}$ under $\mathbb{P}^i$ such that $\langle X^{i,c}, X^{i,c} \rangle = \int_0^\cdot \sigma^2(X_s^i)ds$. Consequently, by Lévy's characterization theorem,

$$
W^i := \int_0^\cdot \frac{1}{\sigma^2(X_s^i)} dX_s^{i,c}, \quad t \in [0, T],
$$

is an $(\mathcal{F}_t^{X^i})$-Brownian motion. We define the $\mathbb{P}^i$-martingale

$$
\kappa_t^i := \Big\{ - \int_0^t H(s, X^{i-})dW_s - \frac{1}{2}\int_0^t |H(s, X^{i-})|^2 ds \Big\}, \quad t \in [0, T],
$$

and the probability $\mathbb{Q}^i$ such that $d\mathbb{Q}^i = \kappa_T^i d\mathbb{P}^i$. By Girsanov's theorem, under $\mathbb{Q}^i$,

$$
B_t^i := W_t^i + \int_0^t H(s, X^{i-})ds
$$

is a Brownian motion. By formula (4.6) in Step 1 (replacing H with $-H$), $(f(X_{s-}^i + \cdot) - f(X_{s-}^i)) \star (\mu^{X^i} - \nu \circ X^i)$ remains an $(\mathcal{F}_t^{X^i})$-martingale under $\mathbb{Q}^i$.

Therefore, $(X^i, \mathbb{Q}^i)$ solves the martingale problem in Definition 3.1 related to $\mathcal{D}_\mathcal{L}$ in (3.1), $\mathcal{L}$ in (3.5) with $H \equiv 0$ and $x_0 \in \mathbb{R}$.

By the uniqueness of the above mentioned the martingale problem stated in Proposition 4.1, X^i, $i = 1, 2$, under $\mathbb{Q}^i$ have the same law. Hence, for every Borel set $B \in \mathcal{B}(C([0, T]))$, we have

$$
\mathbb{P}^1(X^1 \in B) = \int_\Omega \frac{1}{V_T^1(X^1)} \mathbb{1}_{X^1 \in B} d\mathbb{Q}^1 = \int_\Omega \frac{1}{V_T^2(X^2)} \mathbb{1}_{X^2 \in B} d\mathbb{Q}^2 = \mathbb{P}^2(X^2 \in B).
$$

Therefore, X^1 under $\mathbb{P}^1$ has the same law as X^2 under $\mathbb{P}^2$. Finally, uniqueness holds for the martingale problem in Definition 3.1 related to $\mathcal{D}_\mathcal{L}$ in (3.1), $\mathcal{L}$ in (3.5) and $x_0 \in \mathbb{R}$. $\square$

5. Further continuity properties

We introduce here some continuity properties which are used in the companion paper [4].

Let $C_{BUC}(D_-(0,T);B(0,T))$ be the set of functions $G: D_-(0,T) \to B(0,T)$ bounded and uniformly continuous on closed balls $B_M \subset D_-(0,T)$ of radius M. $C_{BUC}(D_-(0,T);B(0,T))$ is a Fréchet space equipped with the distance generated by the seminorms

$$\sup_{\eta \in B_M} ||G(\eta)||_\infty, \quad M \in \mathbb{N}.$$

For $f \in C^{1+\alpha}_{\mathrm{loc}} \cap C^0_b$, we set

$$F^f(y) := \int_{\mathbb{R}} (f(y+x) - f(y) - k(x)\, f'(y)) Q(y,dx), \quad y \in \mathbb{R}. \tag{5.1}$$

Proposition 5.1. *Let $\alpha \in [0,1]$. Assume Hypotheses 2.1, 2.2 and 2.3 with respect to α. Let H be a functional satisfying Hypothesis 3.2 and $Q(\cdot,dx)$ be a transition kernel satisfying Hypothesis 4.1 with respect to α. Assume moreover that H is uniformly continuous on closed balls. Below we will make use of $\mathcal{D}_\mathcal{L}$ and $\mathcal{L}$ defined in (3.1) and (3.5), respectively. Then the following holds.*

1. *For every $f \in \mathcal{D}_\mathcal{L}$, $\mathcal{L}f \in C_{BUC}(D_-(0,T);B(0,T))$.*
2. *The linear map $\mathcal{L}: \mathcal{D}_\mathcal{L} \to C_{BUC}(D_-(0,T);B(0,T))$ is continuous.*

Proof. Let us start by proving item 1. Let $f \in \mathcal{D}_\mathcal{L}$. Let us first show that $\eta \mapsto J^f(\eta)(t) := F^f(\eta(t))$ belongs to $C_{BUC}(D_-(0,T);B(0,T))$. Let $M > 0$. We show that J^f is bounded and uniformly continuous on $B_M := \{\eta \in D_-(0,T) : ||\eta||_\infty \leq M\}$.

Since F^f is continuous by Lemma A.2, it is a bounded function on bounded intervals. Therefore J^f is bounded, being B_M bounded.

Let $\delta > 0$ and $\eta_1, \eta_2 \in B_M$ such that

$$\sup_{t\in[0,T]} |\eta_1(t) - \eta_2(t)| < \delta.$$

Then, for every $t \in [0,T]$,

$$|F^f(\eta_1(t)) - F^f(\eta_2(t))| \leq \sup_{\substack{y_1,y_2 \\ |y_1|\leq M, |y_2|\leq M, |y_1-y_2|<\delta}} |F^f(y_1) - F^f(y_2)|.$$

This implies that J^f is uniformly continuous on B_M, since F^f is unifomly continuous on compact sets.

The map $\eta \mapsto I^f(\eta)(t) := Lf(\eta(t))$ is bounded and uniformly continuous on B_M because $y \mapsto Lf(y)$ is bounded and uniformly continuous on compact intervals.

It remains to prove that $\eta \mapsto W^f(\eta)(t) := H(\eta)(t)(\sigma f')(\eta(t))$, is bounded and uniformly continuous on B_M. The map $\eta \mapsto (\sigma f')(\eta(\cdot))$ is bounded and uniformly continuous by the same reasons as before, while the map $\eta \mapsto H(\eta)$ is bounded and uniformly continuous by assumption.

Let us now prove item 2. We recall that, for every $f \in \mathcal{D}_\mathcal{L}$,

$$f \mapsto (\mathcal{L}f)(\eta)(t) = Lf(\eta(t)) + \sigma(\eta(t))H(\eta)(t)f'(\eta(t)) + F^f(\eta(t)), \quad \eta \in D_-(0,T), t \in [0,T].$$

Since $\mathcal{L}$ is the sum of three linear operators, it will be enough to study the continuity at zero. We suppose first that $f \mapsto F^f$ is continuous from $\mathcal{D}_\mathcal{L}$ to C^0. This would imply that $f \mapsto J^f(\eta)(t)$ is continuous. Indeed, let $M > 0$ and B_M be the closed ball of $D_-(0,T)$ with radius M. We have

$$\sup_{t\in[0,T],\eta\in B_M} |J^f(\eta)(t)| \leq \sup_{y:|y|\leq M} |F^f(y)|.$$

Let us thus prove that $f \mapsto F^f$ is continuous from $\mathcal{D}_\mathcal{L}$ to C^0. For any $f \in \mathcal{D}_\mathcal{L}$, we decompose $F^f = F^f_1 + F^f_2$, where F^f_1, F^f_2 are the functions introduced in (A.3), namely, for every $y \in \mathbb{R}$,

$$F^f_1(y) = \int_{\mathcal{B}} (f(y+x) - f(y) - k(x)\, f'(y)) Q(y,dx),$$

$$F^f_2(y) = \int_{\mathbb{R}\setminus\mathcal{B}} (f(y+x) - f(y) - k(x)\, f'(y)) Q(y,dx),$$

and $\mathcal{B} = [-R, R]$ is a neighborhood of $x = 0$, such that $k(x) = x$ on $\mathcal{B}$. We have

$$F_1^f(y) = \int_{\mathcal{B}} (f(y+x) - f(y) - x f'(y))Q(y,dx) = \int_{\mathcal{B}} G^f(y,x)\, |x|^{1+\alpha}\, Q(y,dx),$$

with

$$G^f(y,x) := \int_0^1 \frac{f'(y+ax) - f'(y)}{|x|^\alpha}\, da. \tag{5.2}$$

Using that

$$\sup_{y \in K, x \in \mathcal{B}} G^f(y,x) \le ||f'||_{\alpha, M+R}, \tag{5.3}$$

where $|| \cdot ||_{\alpha, M+R}$ was defined in (3.2), we get

$$\sup_{y:|y|\le M} |F_1^f(y)| \le ||f'||_{\alpha,M+R} \sup_{y:|y|\le M} ||\mathbb{1}_{\mathcal{B}}(x)|x|^{1+\alpha} Q(y,dx)||_{var},$$

where previous supremum is finite by Lemma A.1-b) with $\ell_1(x) = \mathbb{1}_{\mathcal{B}}(x)$, taking into account Hypothesis 4.1. Therefore this converges to zero when f converges to zero in $\mathcal{D}_{\mathcal{L}}$. This establishes the continuity of $f \mapsto F_1^f$.

On the other hand, the continuity of $f \mapsto F_2^f$ follows from the inequality

$$\sup_{y:|y|\le M} |F_2^f(y)| \le \Big(2||f||_\infty + ||k||_\infty \sup_{y:|y|\le M} |f'(y)|\Big) \sup_{y:|y|\le M} ||\mathbb{1}_{\mathcal{B}^c}(x) Q(y,dx)||_{var},$$

where previous supremum is finite taking into account again Lemma A.1-b) with $\ell_2 = \mathbb{1}_{\mathcal{B}^c}(x)$, again taking into account Hypothesis 4.1.

We then remark that $f \mapsto I^f(\eta)(t)$ is continuous. As a matter of fact,

$$\sup_{t\in[0,T], \eta \in B_M} |I^f(\eta)(t)| \le \sup_{y:|y|\le M} |Lf(y)|,$$

and this converges to zero when f converges to zero in $\mathcal{D}_{\mathcal{L}}$ (and therefore in $\mathcal{D}_L$), taking into account the continuity of L by Definition 2.2.

Finally, the continuity of $f \mapsto W^f(\eta)(t)$ follows from the boundedness of H, and the fact that, since f converges to zero on $\mathcal{D}_{\mathcal{L}}$, then f' converges to zero uniformly on compacts. □

6. New results on Dirichlet processes

For a weak Dirichlet process X, we will denote by X^c its unique martingale component, see Proposition 3.2 in [4]. We start by stating the following result.

Lemma 6.1. *Let X be a Dirichlet process. Then X is a special weak Dirichlet process, and*

$$[X,X]^c = [X^c, X^c].$$

Remark 6.1. A special semimartingale $Y = M + V$ is a Dirichlet process if and only if V is a continuous process. Indeed, $[V,V] = \sum_{s\le \cdot} |\Delta V_s|^2$.

Proof. It is a consequence of Proposition 5.7 and Corollary 5.8-(ii) in [3], where $M^c = X^c$ by the uniqueness of the decomposition in Proposition 3.2 in [4]. □

We say that ν^X *does not jump* if

$$\nu^X(\{t\} \times B) = 0 \quad \forall t \in [0,T], \quad B \in \mathcal{B}(\mathbb{R}^*). \tag{6.1}$$

Remark 6.2. If (6.1) holds true, then obviously

$$\int_{\mathbb{R}} x\,\nu^X(\{t\}\times dx)=0, \quad t\in[0,T]. \tag{6.2}$$

The converse is not true. Indeed, consider for instance the case $\nu^X(dt\,dx)=Q(dx)d\psi_t$, with ψ an increasing càdlàg discontinuous function and $Q(dx)$ a symmetric measure, i.e. such that $Q(B)=Q(-B)$, $B\in\mathcal{B}(\mathbb{R})$.

Proposition 6.1. *If X is a Dirichlet process, then (6.2) holds true.*

Proof. Suppose that X is a Dirichlet process. Then by Lemma 6.1, X is a special weak Dirichlet process and by Corollary 3.22-(ii) in [4],

$$X=X^c+M^{d,X}+\Gamma^X, \tag{6.3}$$

with X^c the unique continuous martingale part of X, $M^{d,X}=x\star(\mu^X-\nu^X)$ and Γ^Y a predictable and $\mathbb{F}$-orthogonal process. We have therefore

$$\Delta\Gamma_t^X=\int_{\mathbb{R}} x\,\nu^X(\{t\}\times\,dx), \quad t\in[0,T]. \tag{6.4}$$

By uniqueness of decomposition of Dirichlet processes and (6.3), $[\Gamma^X,\Gamma^X]=0$, therefore $\Delta\Gamma_t^X=0$ for all $t\in[0,T]$, and so (6.2) holds true. □

Let $\varphi:\mathbb{R}\to\mathbb{R}$, X be a càdlàg process with jump measure μ^X such that $Y=\varphi(X)$ is a weak Dirichlet process. We recall that Y is a special weak Dirichlet process if and only if there exists a constant $a>0$ such that

$$(\varphi(X_{s-}+x)-\varphi(X_{s-}))\,\mathbb{1}_{\{|x|>a\}}\star\mu^X\in\mathcal{A}^+_{\text{loc}}, \tag{6.5}$$

see Theorem 3.16 in [4].

Remark 6.3. The converse of Proposition 6.1 is not true in general. Indeed, by Remark 3.7, there exist processes X such that ν^X does not jump (therefore satisfying (6.2)) that nevertheless are not Dirichlet processes, because (6.5) with $\varphi\equiv Id$ is not verified.

Suppose X to be a Dirichlet process and $\varphi\in C^1(\mathbb{R})$. Is $Y=\varphi(X)$ necessarily a Dirichlet process?

When X is a continuous Dirichlet process and $\varphi\in C^1$, then Y is a Dirichlet process, see the proof of Proposition 4.6 of [20]. By Lemma 6.1, Y is also a special weak Dirichlet process. Below we discuss the case when X is a discontinuous Dirichlet process.

Theorem 6.1. *Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a C^1 function and X be a Dirichlet process. Then $\varphi(X)$ is a Dirichlet process if and only if (6.5) holds true for some $a>0$ and*

$$\int_{\mathbb{R}}(\varphi(X_{t-}+x)-\varphi(X_{t-}))\,\nu^X(\{t\}\times dx)=0 \quad \forall t\in[0,T]. \tag{6.6}$$

Remark 6.4. If X is continuous, then (6.5) and (6.6) are obviously verified, so we retrieve the result stated in the continuous case.

Proof of Theorem 6.1. Let us set $Y:=\varphi(X)$. We first prove the direct implication. By Lemma 6.1, Y is a special weak Dirichlet process. On the other hand, by Theorem 3.16 in [4], this implies that (6.5) holds for all $a>0$. Finally, (6.6) follows from Proposition 6.1 applied to the process Y, since

$$0=\int_{\mathbb{R}} y\,\nu^Y(\{t\}\times\,dy)=\int_{\mathbb{R}}(\varphi(X_{t-}+x)-\varphi(X_{t-}))\,\nu^X(\{t\}\times dx), \quad t\in[0,T].$$

We prove now the converse implication. Since X is a weak Dirichlet process with finite quadratic variation and taking into account (6.5), we can apply Corollary 3.37 in [4]. Therefore Y is a special weak Dirichlet process with decomposition

$$Y=Y_0+\int_0^\cdot \varphi'(X_s)dX_s^c+M^{d,\varphi}+\Gamma(\varphi), \tag{6.7}$$

with $M^{d,\varphi} = (\varphi(s, X_{s-} + x) - \varphi(s, X_{s-})) \star (\mu^X - \nu^X)$ and $\Gamma(\varphi)$ predictable and $\mathbb{F}$-orthogonal. To show that Y is a Dirichlet process, we need to prove that

$$\Gamma(\varphi) := Y - Y^c - M^{d,\varphi} \tag{6.8}$$

is a zero quadratic variation process. By (6.8), we get

$$[\Gamma(\varphi), \Gamma(\varphi)] = [Y, Y] + [Y^c, Y^c] + [M^{d,\varphi}, M^{d,\varphi}] - 2[Y, Y^c] - 2[Y, M^{d,\varphi}], \tag{6.9}$$

provided the latter covariation exists. In fact we have used that $[Y^c, M^{d,\varphi}] = 0$ being $M^{d,\varphi}$ martingale orthogonal. Since $M^{d,\varphi} + \Gamma(\varphi)$ is orthogonal, $[Y, Y^c] = [Y^c, Y^c]$. By Proposition 5.3 in [3], we get

$$[M^{d,\varphi}, M^{d,\varphi}] = \sum_{s \leq \cdot} |\Delta M^{d,\varphi}_s|^2.$$

Collecting previous considerations, (6.9) reads

$$[\Gamma(\varphi), \Gamma(\varphi)] = [Y, Y] - [Y^c, Y^c] + \sum_{s \leq \cdot} |\Delta M^{d,\varphi}_s|^2 - 2[Y, M^{d,\varphi}]. \tag{6.10}$$

Provided the latter covariation exists, $\Gamma(\varphi)$ is a finite quadratic variation process.

Now, $X = X^c + M^{d,X} + \Gamma^X$, and X is a Dirichlet process. Therefore, by uniqueness of the decomposition of such a process, Γ^X is a zero quadratic variation process and in particular continuous. Therefore we get

$$[X, M^{d,\varphi}] = [X^c + M^{d,X} + \Gamma^X, M^{d,\varphi}] = [M^{d,X}, M^{d,\varphi}] + [\Gamma^X, M^{d,\varphi}],$$

and the latter covariation above vanishes since

$$|[\Gamma^X, M^{d,\varphi}]| \leq \{[\Gamma^X, \Gamma^X][M^{d,\varphi}, M^{d,\varphi}]\}^{1/2} = 0.$$

So

$$[X, M^{d,\varphi}] = \sum_{s \leq \cdot} \Delta M^{d,X}_s \Delta M^{d,\varphi}_s = \sum_{s \leq \cdot} \Delta X_s \Delta M^{d,\varphi}_s$$

by Proposition 5.3 in [3]. It follows that $(X, M^{d,\varphi})$ has all its mutual covariations. Therefore, by Lemma B.1-2,

$$[Y, M^{d,\varphi}] = \int_0^{\cdot} \varphi'(X_{s-}) d[X, M^{d,\varphi}]_s = \sum_{s \leq \cdot} \varphi'(X_{s-}) \Delta X_s \Delta \varphi(X_s) \tag{6.11}$$

so $[Y, M^{d,\varphi}]$ exists and, going back to (6.10), we conclude that $\Gamma(\varphi)$ is a finite quadratic variation process.

In particular, formula (6.11) gives

$$[Y, M^{d,\varphi}]^c = 0. \tag{6.12}$$

Taking the continuous component in the equality (6.10) and formula (6.12), we get

$$[\Gamma(\varphi), \Gamma(\varphi)]^c = [Y, Y]^c - [Y^c, Y^c].$$

By Lemma B.1-1 and Lemma 6.1, we get

$$[Y, Y]^c = \int_0^{\cdot} |\varphi'(X_{s-})|^2 d[X, X]^c_s = \int_0^{\cdot} |\varphi'(X_{s-})|^2 d[X^c, X^c]_s.$$

By Theorem 3.36 in [4],

$$[Y^c, Y^c] = \int_0^{\cdot} |\varphi'(X_{s-})|^2 d[X^c, X^c]_s,$$

which implies $[\Gamma(\varphi), \Gamma(\varphi)]^c = 0$. It remains to prove that $\Delta\Gamma(\varphi) = 0$, since $[\Gamma(\varphi), \Gamma(\varphi)] = [\Gamma(\varphi), \Gamma(\varphi)]^c + \sum_{s \leq \cdot} |\Delta\Gamma(\varphi)|^2$. By (6.8),

$$\Delta\Gamma_s(\varphi) = \int_{\mathbb{R}} (\varphi(X_{s-} + x) - \varphi(X_{s-})) \nu^X(\{s\} \times dx),$$

which is zero by assumption (6.6). $\square$

Remark 6.5. a) If ν^X does not jump then obviously (6.6) holds true. In this case, according to Theorem 6.1, $Y = \varphi(X)$ is a Dirichlet process if and only if (6.5) holds true for some $a > 0$.

b) It is possible to have a Dirichlet process X and a process $Y = \varphi(X)$, with $\varphi \in C^1$, that is not a Dirichlet process. We can indeed show the existence of a martingale X such that $\varphi(X)$ is not even a special weak Dirichlet process: we will show that (6.5) is not verified, and so, by the direct implication of Theorem 6.1, Y cannot be a Dirichlet process.

To this end, let Z be a Cauchy random variable, in particular its density is

$$p(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

We set

$$\tilde{Z} = \sqrt{Z}\mathbb{1}_{\{Z>0\}} + \sqrt{-Z}\mathbb{1}_{\{Z<0\}}.$$

Clearly, $\mathbb{E}[|\tilde{Z}|] < \infty$ and $\mathbb{E}[\tilde{Z}] = 0$. We define now

$$X_t := \begin{cases} 0 \ \text{ if } t \in [0,1[\\ \tilde{Z} \text{ if } t > 1. \end{cases}$$

We consider the filtration $\mathbb{F} = (\mathcal{F}_t)$, with $\mathcal{F}_t$ being the trivial σ-algebra for $t \in [0,1[$ and being $\sigma(Z)$ for $t \geq 1$. It follows that X is a martingale: in fact $X_t \in L^1$ for all $t \geq 0$, and

$$\mathbb{E}[X_t|\mathcal{F}_s] = \begin{cases} 0 = X_s & \text{if } s,t \in [0,1[\\ \mathbb{E}[X_t] = \mathbb{E}[\tilde{Z}] = 0 = X_s & \text{if } s < 1, t > 1 \\ \tilde{Z} = X_s & \text{if } t > s > 1. \end{cases}$$

On the other hand, setting $\varphi(x) = x^2$, we have

$$\varphi(X) = \begin{cases} 0 & \text{if } t \in [0,1[\\ |Z| & \text{if } t \geq 1. \end{cases}$$

Now $Y = \varphi(X)$ is a semimartingale since it is an increasing process, but (6.5) is not verified. As a matter of fact,

$$(\varphi(X_{s-}+x) - \varphi(X_{s-}))\,\mathbb{1}_{\{|x|>1\}} \star \mu_t^X = \sum_{s \leq t} \Delta\varphi(X_s)\mathbb{1}_{\{|\Delta X_s|>1\}} = |Z|\,\mathbb{1}_{\{|Z|>1\}}\mathbb{1}_{\{t\geq 1\}} \notin \mathcal{A}^+_{\text{loc}},$$

since $\mathbb{E}[|Z|] = \infty$.

Appendix A: Some technical results

Lemma A.1. *Let $\ell_1, \ell_2 : \mathbb{R} \to \mathbb{R}$ be Borel measurable and bounded functions such that ℓ_1 has compact support and ℓ_2 has support in $\mathbb{R}^*$. Set*

$$y \mapsto \ell_1(x)|x|^{1+\alpha}\,Q(y,dx) =: \tilde{Q}^{\ell_1}(y,dx), \tag{A.1}$$

$$y \mapsto \ell_2(x)\,Q(y,dx) =: \tilde{Q}^{\ell_2}(y,dx). \tag{A.2}$$

a) If Q satisfies Hypothesis 3.1 for some $\alpha \in [0,1]$, then (A.1)-(A.2) are bounded in the total variation norm.
b) If Q satisfies Hypothesis 4.1 for some $\alpha \in [0,1]$, then (A.1)-(A.2) are continuous in the total variation norm.

Proof. Let $R_1 > 1$ such that $\mathcal{B}_{R_1} := \{x \in \mathbb{R} : |x| \leq R_1\}$ contains the compact support of ℓ_1, and $0 < R_2 \leq 1$ such that $\mathcal{B}^c_{R_2} := \{x \in \mathbb{R} : |x| > R_2\}$ contains the support of ℓ_2. Let $M_{\ell_1} := \sup|\ell_1|$, $M_{\ell_2} := \sup|\ell_2|$.

a) We first prove that $\tilde{Q}^{\ell_1}(\cdot,dx)$ and $\tilde{Q}^{\ell_2}(\cdot,dx)$ in (A.1)-(A.2) are bounded in the total variation norm. For $y \in \mathbb{R}$, we have

$$\begin{aligned} \tilde{Q}^{\ell_1}(y,dx) &= \mathbb{1}_{\{|x|\leq R_1\}}\,\ell_1(x)|x|^{1+\alpha}\,Q(y,dx) \\ &= \mathbb{1}_{\{|x|\leq 1\}}\,\ell_1(x)|x|^{1+\alpha}\,Q(y,dx) + \mathbb{1}_{\{1<|x|\leq R_1\}}\,\ell_1(x)|x|^{1+\alpha}\,Q(y,dx). \end{aligned}$$

We get

$$||\tilde{Q}^{\ell_1}(y,dx)||_{var} = ||\mathbb{1}_{\{|x|\le 1\}}\,\ell_1(x)(|x|^{1+\alpha}\wedge 1)\,Q(y,dx) + \mathbb{1}_{\{1<|x|\le R_1\}}\,\ell_1(x)|x|^{1+\alpha}\,Q(y,dx)||_{var}$$
$$\le M_{\ell_1}[||(|x|^{1+\alpha}\wedge 1)\,Q(y,dx)||_{var} + R_1^{1+\alpha}||\mathbb{1}_{\{1<|x|\le R_1\}}\,Q(y,dx)||_{var}]$$
$$\le M_{\ell_1}(1+R_1^{1+\alpha})||(1\wedge|x|^{1+\alpha})\,Q(y,dx)||_{var}.$$

On the other hand,

$$\tilde{Q}^{\ell_2}(y,dx) = \mathbb{1}_{\{|x|>R_2\}}\ell_2(x)\,Q(y,dx) = \mathbb{1}_{\{R_2<|x|\le 1\}}\ell_2(x)\,Q(y,dx) + \mathbb{1}_{\{|x|>1\}}\ell_2(x)\,Q(y,dx),$$

so that

$$||\tilde{Q}^{\ell_2}(y,dx)||_{var} = ||\mathbb{1}_{\{R_2<|x|\le 1\}}\,\ell_2(x)\,Q(y,dx) + \mathbb{1}_{\{|x|>1}\,\ell_2(x)(|x|^{1+\alpha\}}\wedge 1)\,Q(y,dx)||_{var}$$
$$\le \frac{1}{R_2^{1+\alpha}}||\mathbb{1}_{\{R_2<|x|\le 1\}}\,\ell_2(x)\,(|x|^{1+\alpha}\wedge 1)Q(y,dx)||_{var} + ||\mathbb{1}_{\{|x|>1\}}\,\ell_2(x)(|x|^{1+\alpha}\wedge 1)\,Q(y,dx)||_{var}$$
$$\le M_{\ell_2}\Big(1+\frac{1}{R_2^{1+\alpha}}\Big)||(1\wedge|x|^{1+\alpha})\,Q(y,dx)||_{var}.$$

By Hypothesis 3.1 together with Remark 3.2, this proves that $\tilde{Q}^{\ell_1}(\cdot,dx)$ and $\tilde{Q}^{\ell_2}(\cdot,dx)$ in (A.1)-(A.2) are bounded in the total variation norm.

b) Let us now prove that $y\mapsto\tilde{Q}^{\ell_1}(y,dx)$ and $y\mapsto\tilde{Q}^{\ell_2}(y,dx)$ in (A.1)-(A.2) are continuous in the total variation norm. Let (y_n) be a real sequence converging to y_0. We have

$$\tilde{Q}^{\ell_1}(y_n,dx) - \tilde{Q}^{\ell_1}(y_0,dx) = \mathbb{1}_{\{|x|\le 1\}}\,\ell_1(x)|x|^{1+\alpha}\,[Q(y_n,dx)-Q(y_0,dx)]$$
$$+\,\mathbb{1}_{\{1<|x|\le R_1\}}\,\ell_1(x)|x|^{1+\alpha}\,[Q(y_n,dx)-Q(y_0,dx)],$$

and thus

$$||\tilde{Q}^{\ell_1}(y_n,dx) - \tilde{Q}^{\ell_1}(y_0,dx)||_{var} \le ||\mathbb{1}_{\{|x|\le 1\}}\,\ell_1(x)(1\wedge|x|^{1+\alpha})\,[Q(y_n,dx)-Q(y_0,dx)||_{var}$$
$$+\,||\mathbb{1}_{\{1<|x|\le R_1\}}\,\ell_1(x)|x|^{1+\alpha}\,[Q(y_n,dx)-Q(y_0,dx)]||_{var}$$
$$\le M_{\ell_1}(1+R_1^{1+\alpha})||(1\wedge|x|^{1+\alpha})(Q(y_n,dx)-Q(y_0,dx))||_{var}.$$

On the other hand,

$$\tilde{Q}^{\ell_2}(y_n,dx) - \tilde{Q}^{\ell_2}(y_0,dx) = \mathbb{1}_{\{R_2<|x|\le 1\}}\ell_2(x)\,[Q(y_n,dx)-Q(y_0,dx)]$$
$$+\,\mathbb{1}_{\{|x|>1\}}\ell_2(x)\,[Q(y_n,dx)-Q(y_0,dx)],$$

so that

$$||\tilde{Q}^{\ell_2}(y_n,dx) - \tilde{Q}^{\ell_2}(y_0,dx)||_{var} \le ||\mathbb{1}_{\{R_2<|x|\le 1\}}\,\ell_2(x)\,[Q(y_n,dx)-Q(y_0,dx)]||_{var}$$
$$+\,||\mathbb{1}_{\{|x|>1\}}\,\ell_2(x)(1\wedge|x|^{1+\alpha})\,[Q(y_n,dx)-Q(y_0,dx)]||_{var}$$
$$\le M_{\ell_2}\big(\frac{1}{R_2^{1+\alpha}}+1\big)||(1\wedge|x|^{1+\alpha})(Q(y_n,dx)-Q(y_0,dx))||_{var}.$$

By Hypothesis 4.1, this proves that $y\mapsto\tilde{Q}^{\ell_1}(y,dx)$ and $y\mapsto\tilde{Q}^{\ell_2}(y,dx)$ in (A.1)-(A.2) are continuous in total variation topology. □

Lemma A.2. *Let $Q(\cdot,dx)$ be a transition kernel satisfying Hypothesis 4.1 for some $\alpha\in[0,\,1]$. Then the function $f\mapsto F^f$, $C^{1+\alpha}_{\mathrm{loc}}\cap C^0_b\to\mathbb{R}$, defined in (5.1) is continuous.*

Proof. Let $\mathcal{B} = [-R, R]$ be a neighborhood of $x = 0$, such that $k(x) = x$ on $\mathcal{B}$. Define

$$F^f(y) = \int_{\mathcal{B}} (f(y+x) - f(y) - k(x)\, f'(y))Q(y, dx) + \int_{\mathbb{R}\setminus\mathcal{B}} (f(y+x) - f(y) - k(x)\, f'(y))Q(y, dx) =: F_1^f(y) + F_2^f(y). \tag{A.3}$$

Let (y_n) be a sequence converging to y_0 in $\mathbb{R}$, and let $K = [-M, M]$ be a compact set containing (y_n). We start by noticing that

$$F_1^f(y) = \int_{\mathcal{B}} (f(y+x) - f(y) - x\, f'(y))Q(y, dx) = \int_{\mathcal{B}} G^f(y, x)\, |x|^{1+\alpha} Q(y, dx),$$

with G^f in (5.2). Then

$$\begin{aligned} F_1^f(y_n) - F_1^f(y_0) &= \int_{\mathcal{B}} G^f(y_n, x)\, |x|^{\alpha+1}\, Q(y_n, dx) - \int_{\mathcal{B}} G^f(y_0, x)\, |x|^{\alpha+1}\, Q(y_0, dx) \\ &= \int_{\mathcal{B}} [G^f(y_n, x) - G^f(y_0, x)]\, |x|^{\alpha+1}\, Q(y_0, dx) + \int_{\mathcal{B}} G^f(y_n, x)\, |x|^{\alpha+1}\, [Q(y_0, dx) - Q(y_n, dx)]. \end{aligned} \tag{A.4}$$

We recall (5.3), namely

$$\sup_{y\in K, x\in\mathcal{B}} G^f(y, x) \le ||f'||_{\alpha, M+R}.$$

For every $x \in \mathbb{R}$, $y \mapsto G^f(y, x)$ is continuous by Lebesgue dominated convergence theorem. Then the first term in the right-hand side of (A.4) converges again by the Lebesgue dominated convergence theorem, taking into account that the measure $\tilde{Q}^{\ell_1}(y, dx) := \mathbb{1}_{\mathcal{B}}(x)|x|^{1+\alpha}\, Q(y, dx)$ is finite thanks to Lemma A.1 with $\ell_1(x) = \mathbb{1}_{\mathcal{B}}(x)$. The convergence of the second term in the right-hand side of (A.4) follows by the continuity of $y \mapsto \tilde{Q}^{\ell_1}(y, dx)$ in the total variation topology due to Lemma A.1, noting that

$$\Big| \int_{\mathcal{B}} G^f(y_n, x)\, |x|^{\alpha+1}\, [Q(y_0, dx) - Q(y_n, dx)] \Big| \le ||f'||_{\alpha, M+R}\, ||\tilde{Q}^{\ell_1}(y_0, dx) - \tilde{Q}^{\ell_1}(y_n, dx)||_{var}. \tag{A.5}$$

On the other hand,

$$\begin{aligned} F_2^f(y_n) - F_2^f(y_0) &= \int_{\mathbb{R}\setminus\mathcal{B}} ((f(y_n + x) - f(y_n)) - (f(y_0 + x) - f(y_0))Q(y_0, dx) \\ &\quad + (f'(y_n) - f'(y_0)) \int_{\mathbb{R}\setminus\mathcal{B}} k(x) Q(y_0, dx) \\ &\quad + \int_{\mathbb{R}\setminus\mathcal{B}} (f(y_n + x) - f(y_n) - k(x) f'(y_n))\, [Q(y_0, dx) - Q(y_n, dx)] \\ &=: I'(y_n, y_0) + I''(y_n, y_0) + I'''(y_n, y_0). \end{aligned} \tag{A.6}$$

Since (y_n) lives in the compact K and f is bounded, we have

$$\begin{aligned} I'''(y_n, y_0) &= \int_{\mathbb{R}} (f(y_n + x) - f(y_n) - f'(y_n) k(x)) \mathbb{1}_{\mathcal{B}^c}(x)\, [Q(y_0, dx) - Q(y_n, dx)] \\ &\le C ||\tilde{Q}^{\ell_2}(y_0, dx) - \tilde{Q}^{\ell_2}(y_n, dx)||_{var} \end{aligned}$$

with $\tilde{Q}^{\ell_2}(y, dx) := \mathbb{1}_{\mathcal{B}^c}(x)\, Q(y, dx)$, and $C = 2||f||_\infty + ||k||_\infty \sup_{y\in K} |f'(y)|$. The convergence follows from the continuity of in the total variation topology due to Lemma A.1 applied to $\ell_2(x) = \mathbb{1}_{\mathcal{B}^c}(x)$.

On the other hand, the convergence of $I'(y_n, y_0)$ follows by the Lebesgue dominated convergence theorem, taking into account that the measure $\tilde{Q}^{\ell_2}(y_0, dx)$ is finite due to Lemma A.1, and the fact that f bounded.

Finally, the convergence of $I''(y_n, y_0)$ follows because f' is continuous, taking into account that the measure $k(x)\mathbb{1}_{\mathcal{B}^c}(x) Q(y_0, dx)$ is finite by Lemma A.1 applied to $\ell_2(x) = k(x)\mathbb{1}_{\mathcal{B}^c}(x)$. □

Appendix B: Stability of finite quadratic variation processes

The following result was well understood in the context of Föllmer's discretizations, but was never established in the regularization framework.

Lemma B.1. *1. Let $Y = \varphi(X)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^1 function and X is a càdlàg process of finite quadratic variation. Then*

$$[Y, Y]_t = \int_0^t (\varphi'(X_{s-}^2 d[X, X]_s^c + \sum_{s \le t} (\Delta\varphi(X_s))^2.$$

In particular, Y is also a finite quadratic variation process.

2. Let $Y^1 = \varphi(X^1)$ and $Y^2 = \phi(X^2)$, where φ and ϕ are C^1 functions and X^1, X^2 are càdlàg processes such that (X^1, X^2) has all its mutual covariations. Then

$$[Y^1, Y^2]_t = \int_0^t \varphi'(X_s^1)\phi'(X_{s-}^2) d[X^1, X^2]_s^c + \sum_{s \le t} \Delta\varphi(X_s^1)\, \Delta\phi(X_s^2).$$

Proof. 1. Let $t \in [0, T]$, $\varepsilon \in [0, 1]$. We expand, for $s \in [0, T]$,

$$\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_{s\wedge t}) = I_1^{\varphi}(s, t, \varepsilon)(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}),$$

where

$$I_1^{\varphi}(s, t, \varepsilon) = \int_0^1 \varphi'(X_{s\wedge t} + a(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}))\, da.$$

Consequently,

$$\begin{aligned}\frac{1}{\varepsilon}(\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_{s\wedge t}))^2 &= \frac{1}{\varepsilon}((I_1^{\varphi}(s, t, \varepsilon))^2 - (\varphi'(X_s))^2)(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t})^2 \\ &\quad + \frac{1}{\varepsilon}(\varphi'(X_s))^2(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t})^2.\end{aligned}$$

Integrating from 0 to t, we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_s))^2 ds &= \frac{1}{\varepsilon}\int_0^t ((I_1^{\varphi}(s, t, \varepsilon))^2 - (\varphi'(X_s)^2)(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds \\ &\quad + \frac{1}{\varepsilon}\int_0^t (\varphi'(X_s))^2(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds \\ &= J_1(t, \varepsilon) + J_2(t, \varepsilon). \qquad \text{(B.1)}\end{aligned}$$

We notice that, without restriction of generality, passing to a suitable subsequence, we can suppose (with abuse of notation) that

$$[X, X]^{\varepsilon} := \frac{1}{\varepsilon}\int_0^{\cdot} (X_{(s+\varepsilon)\wedge \cdot} - X_s)^2\, ds \underset{\varepsilon\to 0}{\to} [X, X], \quad \text{uniformly a.s.} \qquad \text{(B.2)}$$

Since X is a finite quadratic variation process, by Lemma A.5 in [3], taking into account Definition A.2 and Corollary A.4-2. in [3], if g is a càdlàg process then

$$\frac{1}{\varepsilon}\int_0^t g_s(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds = \frac{1}{\varepsilon}\int_0^t g_{s-}(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds \underset{\varepsilon\to 0}{\to} \int_0^t g_{s-} d[X, X]_s, \quad \text{u.c.p.} \qquad \text{(B.3)}$$

Therefore, taking $g_s = (\varphi'(X_s))^2$ in (B.3), we get

$$J_2(\cdot, \varepsilon) \underset{\varepsilon\to 0}{\to} \int_0^{\cdot} (\varphi'(X_{s-}))^2 d[X, X]_s, \quad \text{u.c.p.} \qquad \text{(B.4)}$$

Next step consists in proving that

$$J_1(\cdot,\varepsilon) \underset{\varepsilon\to 0}{\to} \sum_{s\le t}\Big[\Big(\int_0^1 \varphi'(X_{s-}+a\Delta X_s)da\Big)^2-(\varphi'(X_{s-}))^2\Big](\Delta X_s)^2, \quad \text{u.c.p.} \tag{B.5}$$

We fix a realization $\omega\in\Omega$. Proceeding as in the proof of Proposition 2.14 in [3], let (t_i) be an enumeration of all the jumps of $X(\omega)$ in $[0,T]$. We have $\sum_i(\Delta X_{t_i}(\omega))^2<\infty$.

Let $\gamma>0$ and $N=N(\gamma)$ such that

$$\sum_{i=N+1}^{\infty}(\Delta X_{t_i}(\omega))^2\le\gamma^2. \tag{B.6}$$

We introduce

$$A(\varepsilon,N)=\bigcup_{i=1}^N]t_i-\varepsilon,t_i], \quad B(\varepsilon,N)=\bigcup_{i=1}^N]t_{i-1},t_i-\varepsilon]=[0,T]\setminus A(\varepsilon,N).$$

We decompose

$$J_1(t,\varepsilon)=\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{A(\varepsilon,N)}(s)J_{10}(s,t,\varepsilon)ds+\int_0^t \mathbb{1}_{B(\varepsilon,N)}(s)J_{10}(s,t,\varepsilon)ds =: J_{1A}(t,\varepsilon,N)+J_{1B}(t,\varepsilon,N), \tag{B.7}$$

where we have denoted

$$J_{10}(s,t,\varepsilon):=(X_{(s+\varepsilon)\wedge t}-X_s)^2((I_1^{\varphi}(s,t,\varepsilon))^2-(\varphi'(X_s)^2).$$

By Lemma 2.11 in [3], it follows that, uniformly in $t\in[0,T]$,

$$J_{1A}(t,\varepsilon,N) \underset{\varepsilon\to 0}{\to} \sum_{i=1}^N \mathbb{1}_{]0,t]}(t_i)(\Delta X_{t_i})^2\Big(\Big(\int_0^1 \varphi'(X_{t_i-}+a\Delta X_{t_i})da\Big)^2-(\varphi'(X_{t_i-}))^2\Big). \tag{B.8}$$

On the other hand,

$$J_{1B}(t,\varepsilon,N)=\sum_{i=1}^N\frac{1}{\varepsilon}\int_0^t (X_{(s+\varepsilon)\wedge t}-X_s)^2 I_{1B}^{\varphi,i}(s,t,\varepsilon)\,ds,$$

where

$$\begin{aligned} I_{1B}^{\varphi,i}(s,t,\varepsilon)&=\mathbb{1}_{]t_{i-1},t_i-\varepsilon]}(s)\Big[\Big(\int_0^1 \varphi'(X_{s\wedge t}+a(X_{(s+\varepsilon)\wedge t}-X_{s\wedge t}))\,da\Big)^2-(\varphi'(X_s))^2\Big]\\ &=\mathbb{1}_{]t_{i-1},t_i-\varepsilon]}(s)\Big[\int_0^1 \varphi'(X_{s\wedge t}+a(X_{(s+\varepsilon)\wedge t}-X_{s\wedge t}))\,da-\varphi'(X_s)\Big]\cdot\\ &\quad\cdot\Big[\int_0^1 \varphi'(X_{s\wedge t}+a(X_{(s+\varepsilon)\wedge t}-X_{s\wedge t}))\,da+\varphi'(X_s)\Big]. \end{aligned}$$

For every $i=1,...,N$, we have

$$|I_{1B}^{\varphi,i}(s,t,\varepsilon)|\le 2\sup_{y\in[X_s,X_{s+\varepsilon}]}|\varphi'(y)|\,\delta\Big(\varphi,\sup_i\sup_{\substack{p,q\in[t_{i-1},t_i]\\|p-q|\le\varepsilon}}|X_p-X_q|\Big).$$

By Lemma 2.12 in [3], there is ε_0 such that, if $\varepsilon<\varepsilon_0$, then

$$|I_{1B}^{\varphi,i}(s,t,\varepsilon)|\le 2\sup_{y\in[-||X||_\infty,||X||_\infty]}|\varphi'(y)|\,\delta(\varphi,3\gamma).$$

Consequently, for $\varepsilon<\varepsilon_0$,

$$\sup_{t\in[0,T]}|J_{1B}(t,\varepsilon,N)|\le 2\sup_{y\in[-||X||_\infty,||X||_\infty]}|\varphi'(y)|\,\delta(\varphi,3\gamma)\sup_{t\in[0,T]}[X,X]_t^{\varepsilon}, \tag{B.9}$$

where the latter supremum is finite by (B.2). Going back to (B.7) we get

$$
\begin{aligned}
&\sup_{t\in[0,T]}\Big|J_1(t,\varepsilon)-\sum_{i=1}^{\infty}\mathbb{1}_{]0,t]}(t_i)(\Delta X_{t_i})^2\Big[\Big(\int_0^1\varphi'(X_{t_i-}+a\Delta X_{t_i})da\Big)^2-(\varphi'(X_{t_i-}))^2\Big]\Big|\\
&\le \sup_{t\in[0,T]}\Big|J_{1A}(t,\varepsilon,N)-\sum_{i=1}^{N}\mathbb{1}_{]0,t]}(t_i)(\Delta X_{t_i})^2\Big[\Big(\int_0^1\varphi'(X_{t_i-}+a\Delta X_{t_i})da\Big)^2-(\varphi'(X_{t_i-}))^2\Big]\Big|\\
&+\Big|\sum_{i=N+1}^{\infty}\mathbb{1}_{]0,T]}(t_i)(\Delta X_{t_i})^2\Big(\int_0^1\varphi'(X_{t_i-}+a\Delta X_{t_i})da\Big)^2\Big|+\sup_{t\in[0,T]}|J_{1B}(t,\varepsilon,N)|.
\end{aligned}
\tag{B.10}
$$

Taking the $\limsup_{\varepsilon\to 0}$ in (B.10), collecting (B.6), (B.8) and (B.9), we get

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\sup_{t\in[0,T]}\Big|J_1(t,\varepsilon)-\sum_{i=1}^{\infty}\mathbb{1}_{]0,t]}(t_i)(\Delta X_{t_i})^2\Big[\Big(\int_0^1\varphi'(X_{t_i-}+a\Delta X_{t_i})da\Big)^2-(\varphi'(X_{t_i-}))^2\Big]\Big|\\
&\le \sum_{i=N+1}^{\infty}\mathbb{1}_{]0,T]}(t_i)(\Delta X_{t_i})^2\sup_{y\in[-||X||_\infty,||X||_\infty]}|\varphi'(y)|+2\sup_{y\in[-||X||_\infty,||X||_\infty]}|\varphi'(y)|\,\delta(\varphi,3\gamma)\sup_{\varepsilon<\varepsilon_0}\sup_{t\in[0,T]}[X,X]^\varepsilon_t\\
&\le\Big[\gamma^2+2\sup_{\varepsilon<\varepsilon_0}\sup_{t\in[0,T]}[X,X]^\varepsilon_t\,\delta(\varphi,3\gamma)\Big]\sup_{y\in[-||X||_\infty,||X||_\infty]}|\varphi'(y)|.
\end{aligned}
$$

Since γ is arbitrary and φ' is uniformly continuous on compact intervals, then (B.5) is proved. By (B.4) and (B.5), and the fact that $[X,X]=[X,X]^c+\sum_{s\le t}(\Delta X_s)^2$, (B.1) yields

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^t(\varphi(X_{(s+\varepsilon)\wedge t})-\varphi(X_s))^2ds \underset{\varepsilon\to 0}{\to}&\int_0^{\cdot}(\varphi'(X_{s-}))^2d[X,X]^c_s+\sum_{s\le t}(\varphi'(X_{s-}))^2(\Delta X_s)^2\\
&+\sum_{s\le t}\Big[\Big(\int_0^1\varphi'(X_{s-}+a\Delta X_s)da\Big)^2-(\varphi'(X_{s-}))^2\Big](\Delta X_s)^2\\
&=\int_0^{\cdot}(\varphi'(X_{s-}))^2d[X,X]^c_s+\sum_{s\le t}\Big(\int_0^1\varphi'(X_{s-}+a\Delta X_s)da\Big)^2(\Delta X_s)^2,\quad\text{u.c.p.}
\end{aligned}
$$

The result follows because

$$
\Delta\varphi(X_s)=\varphi(X_s)-\varphi(X_{s-})=\Delta X_s\int_0^1\varphi'(X_{s-}+a\Delta X_s)da.
$$

2. The result follows from point 1. by polarity arguments. □

Appendix C: Recalls on semimartingales with jumps

We recall that a special semimartingale is a semimartingale X which admits a decomposition $X=N+V$, where N is a local martingale and V is a finite variation and predictable process, see Definition 4.21, Chapter I, in [14]. Fixing $V_0=0$, such a decomposition is unique, and is called canonical decomposition of X, see respectively Proposition 3.16 and Definition 4.22, Chapter I, in [14].

Assume now that X is a semimartingale with jump measure μ^X. Given $k\in\mathcal{K}$, the process $X^k:=X-\sum_{s\le\cdot}[\Delta X_s-k(\Delta X_s)]$ is a special semimartingale with unique decomposition

$$
X^k=X^c+M^{k,d}+B^{k,X},
\tag{C.1}
$$

where $M^{k,d}$ is a purely discontinuous local martingale, X^c is the unique continuous martingale part of X (it coincides with the process X^c introduced in Proposition 3.2 in [4]), and $B^{k,X}$ is a predictable process of bounded variation.

According to Definition 2.6, Chapter II in [14], the characteristics of X associated with $k\in\mathcal{K}$ are then given by the triplet (B^k,C,ν) on $(\check\Omega,\check{\mathcal{F}},\check{\mathbb{F}})$ such that

(i) B^k is $\check{\mathbb{F}}$-predictable, with finite variation on finite intervals, and $B^k_0 = 0$, namely $B^{k,X} = B^k \circ X$ is the process in (C.1);
(ii) C is a continuous process of finite variation with $C_0 = 0$, namely $C^X := C \circ X = \langle X^c, X^c \rangle$;
(iii) ν is an $\check{\mathbb{F}}$-predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$, namely $\nu^X := \nu \circ X$ is the compensator of μ^X.

Theorem C.1 (Theorem 2.42, Chapter II, in [14])**.** *Let X be an adapted càdlàg process. Let B^k be an $\check{\mathbb{F}}$-predictable process, with finite variation on finite intervals, and $B^k_0 = 0$, C be an $\check{\mathbb{F}}$-adapted continuous process of finite variation with $C_0 = 0$, and ν be an $\check{\mathbb{F}}$-predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$. There is equivalence between the two following statements.*

(i) X is a real semimartingale with characteristics (B^k, C, ν).
(ii) For each bounded function f of class C^2, the process

$$f(X_\cdot) - f(X_0) - \frac{1}{2}\int_0^\cdot f''(X_s)\, dC^X_s - \int_0^\cdot f'(X_{s-})\, dB^{k,X}_s$$
$$- \int_0^\cdot \int_{\mathbb{R}} (f(X_{s-} + x) - f(X_{s-}) - k(x)\, f'(X_{s-}))\, \nu^X(ds\, dx)$$

is a local martingale.

Let $(\Omega, \mathcal{F}, \mathbb{F})$ be the canonical filtered space, and X the canonical process. Let moreover $\mathcal{H}$ be another σ-algebra and $\mathbb{P}_{\mathcal{H}}$ be a probability measure on $(\Omega, \mathcal{H})$.

Definition C.1 (Definition 2.4, Chapter III, in [14]). A solution to the martingale problem associated to $(\mathcal{H}, X)$ and $(\mathbb{P}_{\mathcal{H}}; B^k, C, \nu)$ is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that

(i) the restriction $\mathbb{P}|_{\mathcal{H}}$ of $\mathbb{P}$ to $\mathcal{H}$ equals $\mathbb{P}_{\mathcal{H}}$;
(ii) X is a semimartingale on the basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with characteristics (B^k, C, ν).

We denote by $s(\mathcal{H}, X|\mathbb{P}_{\mathcal{H}}; B^k, C, \nu)$ the set of all solutions $\mathbb{P}$.

Definition C.2 (Definition 2.18, Chapter III, in [14]). Let $b^k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Borel, $c : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Borel and nonnegative, and $Q_s(x, dy)$ be a transition kernel from $(\mathbb{R} \times \mathbb{R}_+, \mathcal{B}(\mathbb{R} \times \mathbb{R}_+))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with $Q_s(x, \{0\}) = 0$. X is called a diffusion with jumps on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ related to $(b^k, c, Q_s(x, dy))$ if it is a semimartingale with characteristics

$$B^k_t = \int_0^t b^k(s, \check{X}_s) ds, \quad C_t = \int_0^t c(s, \check{X}_s) ds, \quad \nu(ds\, dx) = Q_s(\check{X}_{s-}, dx) ds. \tag{C.2}$$

Remark C.1. Suppose that X is as in Definition C.2, then $\mathbb{P}$ is a solution to the martingale problem associated to $(\mathcal{H}, X)$ and $(\mathbb{P}_{\mathcal{H}}; B^k, C, \nu)$, with $\mathcal{H} = \sigma(X_0)$ and $\mathbb{P}_{\mathcal{H}}$ the law of X_0.

Hypothesis C.1. *Let $b^k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Borel, $c : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Borel and nonnegative, and $Q_s(x, de)$ be a transition kernel from $(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with $Q_s(x, \{0\}) = 0$. We assume that*

(i) b^k is bounded;
(ii) c is bounded, continuous on $\mathbb{R}_+ \times \mathbb{R}$ and not vanishing at zero;
(iii) the functions $(t, y) \mapsto \int_A (|x|^2 \wedge 1) Q_t(y, dx)$ are bounded and continuous for all $A \in \mathcal{B}(\mathbb{R})$.

Theorem C.2 (Theorem 2.34, Chapter III, in [14])**.** *Let (B^k, C, ν) be of the type in (C.2), and such that $b^k(s, x)$, $c(s, x)$, $Q_s(x, dy)$ satisfy Hypothesis C.1. Then, there is a transition kernel $P_z(d\omega)$ from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ into $(\Omega, \mathcal{F})$ with the following property: for every z, P_z is the unique probability measure under which the canonical process X is a diffusion with jumps, with $P_z(X_0 = z) = 1$ and with characteristics given by (C.2).*

Corollary C.1. *Let $P_z(d\omega)$ be the transition kernel introduced in Theorem C.2. Let $z \in \mathbb{R}$. Then P_z is the unique solution to the martingale problem associated to $(\mathcal{H}, X)$ and $(\mathbb{P}_{\mathcal{H}}; B^k, C, \nu)$, with $\mathcal{H} = \sigma(X_0)$ and $\mathbb{P}_{\mathcal{H}}$ determined by $\mathbb{P}(X_0 \in B) = \delta_z(B)$.*

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