Characteristics and Itô's formula for weak Dirichlet processes: an equivalence result

Elena Bandini * Francesco Russo †

Abstract

The main objective consists in generalizing a well-known Itô formula of J. Jacod and A. Shiryaev: given a càdlàg process S, there is an equivalence between the fact that S is a semimartingale with given characteristics (B^k, C, ν) and a Itô formula type expansion of $F(S)$, where F is a bounded function of class C^2. This result connects weak solutions of path-dependent SDEs and related martingale problems. We extend this to the case when S is a weak Dirichlet process. A second aspect of the paper consists in discussing some untreated features of stochastic calculus for finite quadratic variation processes.

Key words: Martingale problem, Itô formula, weak Dirichlet process.
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1 Introduction

The motivating formula is the one of J. Jacod and A. Shiryaev concerning a generic càdlàg process X. Let k be a truncation function cutting large jumps, i.e., a bounded function such that $k(x) = x$ in a neighborhood of zero. This says that X is a semimartingale with characteristics (B^k, C, ν) if and only if for every $F \in C_b^{1,2}([0, T] \times \mathbb{R})$,

$$F(\cdot, X_\cdot) - F(0, X_0) - \int_0^\cdot \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^\cdot \partial^2_{xx} F(s, X_s)\, d(C \circ X)_s - \int_0^\cdot \partial_x F(s, X_{s-})\, d(B^k \circ X)_s$$
$$- \int_{]0,\cdot] \times \mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - k(x)\, \partial_x F(s, X_{s-}))\, (\nu \circ X)(ds\, dx) \tag{1.1}$$

is a local martingale, i.e. a Itô formula expansion. This equivalence was stated in Theorem 2.42, Chapter II, in [12], where F is time homogeneous, but can be easily extended to the non-homogeneous case, see also Appendix A. Decomposition (1.1) can be seen as a martingale problem formulation for a class of function $F \in C_b^{1,2}([0, T] \times \mathbb{R})$ for which $F(\cdot, X)$ is a prescribed special semimartingale.

Given a filtration $(\mathcal{F}_t)$, a (càdlàg) $(\mathcal{F}_t)$-weak Dirichlet process is defined as the sum of a local $(\mathcal{F}_t)$-martingale M and an $(\mathcal{F}_t)$-*martingale orthogonal* process A which means that $[A, N] = 0$, where N is a generic continuous $(\mathcal{F}_t)$-martingale. Typical examples of martingale orthogonal processes are by definition purely discontinuous martingales and bounded variation processes, see Proposition 2.14 in [1]. In particular, if X is an $(\mathcal{F}_t)$-semimartingale, then X is an $(\mathcal{F}_t)$-weak Dirichlet process. If one forces M to be continuous, then the decomposition $X = M + A$

*Università di Bologna, Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40126 Bologna, Italy; e-mail: elena.bandini7@unibo.it.

†ENSTA Paris, Unité de Mathématiques Appliquées, Institut Polytechnique de Paris, 828, boulevard des Maréchaux, F-91120 Palaiseau, France; e-mail: francesco.russo@ensta-paris.fr.

(fixing $A_0 = 0$) is unique, see Proposition 3.2 in [4]. In this case the continuous local martingale component M is denoted by X^c. If A is predictable then it is called $(\mathcal{F}_t)$-*special weak Dirichlet process*. The filtration $(\mathcal{F}_t)$ will be often omitted when it is self-explanatory or it is the canonical filtration $(\mathcal{F}_t^X)$ associated with the process X.

The concept of continuous weak Dirichlet process was introduced in [8], and further analyzed and extended to the multidimensional case in [10]; a first application to stochastic control was performed in [9]. In particular therein the authors investigated the stability of weak Dirichlet processes through $C^{0,1}$-transformations. Indeed, if X is an $\mathbb{R}^d$-valued weak Dirichlet process admitting all its mutual covariations, and $F \in C^{0,1}([0,T] \times \mathbb{R}^d)$, then $F(\cdot, X)$ is still a weak Dirichlet process. In the jump case, [6] introduced the notion of special Dirichlet process generalizing the notion of special semimartingale. In that framework the general notion of weak Dirichlet process appeared in [1] and was applied to the BSDEs theory in [2]. The aforementioned stability property has been extended in [4] to the case when X is a jump process. A generalization of the $C^{0,1}$-stability in the continuous framework, but u being a path-dependent functional, has been performed in [5] with application to finance. A survey on weak Dirichlet processes is provided in Chapter 15 of [15].

The notion of characteristics, well known for semimartingales, has been extended in [4] to a generic weak Dirichlet process as follows. Cutting large jumps via a truncation function k, the process

$$X_t - \sum_{s \le t} (X_s - k(\Delta X_s)), \quad t \ge 0, \tag{1.2}$$

is a special weak Dirichlet process. If $(\mathcal{F}_t) = (\mathcal{F}_t^X)$, then the process (1.2) can be decomposed as

$$B^k \circ X + X^c + k(x) \star (\mu^X - \nu \circ X),$$

with $[X^c, X^c] = C \circ X$ and ν a random measure with $\nu \circ X$ the compensator of the jump counting measure μ^X. Therefore, X fulfills the equation

$$X = X^c + k(x) \star (\mu^X - \nu \circ X) + B^k \circ X + (x - k(x)) \star \mu^X. \tag{1.3}$$

Our main Theorem 4.2 provides a generalization of the Jacod-Shiryaev equivalence theorem (see (1.1)) when X is not necessarily a semimartingale but a weak Dirichlet process, see Definition 2.3. Namely it states that a finite quadratic variation process X is a weak Dirichlet process with local characteristics (B^k, C, ν) if and only if, for each bounded function F of class $C^{1,2}([0,T] \times \mathbb{R})$, $\int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s$ is a martingale orthogonal process, and the process

$$\begin{aligned} &F(\cdot, X_\cdot) - F(0, X_0) - \int_0^\cdot \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^\cdot \partial_{xx}^2 F(s, X_s)\, (d(C \circ X)_s + d[B^k \circ X, B^k \circ X]_s^c) \\ &- \int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s \\ &- \int_{]0,\,\cdot] \times \mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - k(x)\, \partial_x F(s, X_{s-}))\, (\nu \circ X)(ds\, dx) \end{aligned} \tag{1.4}$$

is an $(\mathcal{F}_t^X)$-local martingale, where $\int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s$ is the forward integral in the sense of [14] (see Definition 2.1), see [1], and also [7]. For proving (1.4), a fundamental tool of independent interest is the following (see Theorem 3.2): if X is an $(\mathcal{F}_t^X)$-weak Dirichlet process with finite quadratic variation with decomposition (1.3), then $B^k \circ X$ is a finite quadratic variation process, with

$$[B^k \circ X, B^k \circ X]_t = [X, X]_t^c - [X^c, X^c]_t + \sum_{s \le t} \Big| \int_{\mathbb{R}} k(x) (\nu \circ X)(\{s\} \times dx) \Big|^2. \tag{1.5}$$

This in particular extends the results when X is semimartingale or a Dirichlet process: in those cases, $[B^k \circ X, B^k \circ X]_t = \sum_{s \le t} |\Delta(B^k \circ X)|^2 = \sum_{s \le t} \left| \int_{\mathbb{R}} k(x)(\nu \circ X)(\{s\} \times dx) \right|^2$ and therefore $[X, X]^c_t = [X^c, X^c]_t$, see Remark 3.4.

An important aspect of Theorem 4.2 is that it prolongates the classical equivalence between weak solutions of stochastic differential equations and martingale problems in the framework of continuous Markov processes, see [16].

Formulation (1.4) fits the one of Theorem 4.3 in [4], where we specify $F(\cdot, X)$ for F belonging to some domain $\mathcal{D}_{\mathcal{S}}$ which is a subspace of $C^{0,1}([0,T] \times \mathbb{R})$. In the present case, $\mathcal{D}_{\mathcal{S}}$ is the space of bounded functions of class $C^{1,2}([0,T] \times \mathbb{R})$, and the processes $F(\cdot, X)$ are no longer special semimartingales but special weak Dirichlet processes. In [4] we were specially interested in domains for which the processes $F(\cdot, X)$ were still semimartingales, see Definition 4.12 in [4]; this included the framework of X being a solution of a stochastic differential equation with jump and singular (distributional) drift, see Theorem 4.1 in [3].

A second aspect of the paper consists in discussing some properties of the covariation processes. We give an explicit expression for the covariation of two càdlàg finite quadratic variation processes, provided that the continuous component of the quadratic variation of one of them vanishes, see Lemma 2.6. We also provide a stability result for C^1 transformations of finite quadratic variation processes, see Lemma 2.7.

The paper is organized as follows. Section 2 is devoted to some useful results concerning stochastic calculus via regularization for jump processes, including the related properties to the covariance. In Section 3 we formulate some basic recalls on weak Dirichlet processes. Section 4 is devoted to the statement and the proof of the main Theorem 4.2.

2 Elements of stochastic calculus via regularization for jump processes

2.1 Preliminaries

In the whole article, we are given a fixed maturity $T > 0$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We will consider the space of functions $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto u(t,x)$, which are of class $C^{0,1}$ or $C^{1,2}$. $C^{0,1}_b$ (resp. $C^{1,2}_b$) stands for the class of bounded functions which belong to $C^{0,1}$ (resp. $C^{1,2}$). $C^{0,1}$ is equipped with the topology of uniform convergence on each compact of u and $\partial_x u$. C^0 (resp. C^0_b) will denote the space of continuous functions (resp. continuous and bounded functions) on $\mathbb{R}$ equipped with the topology of uniform convergence on each compact (resp. equipped with the topology of uniform convergence), while C^1 (resp. C^2) will be the space of continuously differentiable (twice continuously differentiable) functions $u : \mathbb{R} \to \mathbb{R}$. C^1_b (resp. C^2_b) stands for the class of bounded functions which belong to C^1 (resp. C^2).

The concept of random measure will be extensively used throughout the paper. For a detailed discussion on this topic and the unexplained notations, we refer to Chapter I and Chapter II, Section 1, in [12], Chapter III in [13], and Chapter XI, Section 1, in [11]. In particular, if μ is a random measure on $[0,\, T] \times \mathbb{R}$, for any measurable real function H defined on $\Omega \times [0,\, T]$, one denotes

$$H \star \mu_t := \int_{]0,\, t] \times \mathbb{R}} H(\cdot, s, x)\, \mu(\cdot, ds\, dx),$$

when the stochastic integral in the right-hand side is defined (with possible infinite values).

Let X be an adapted càdlàg process. We denote by ΔX, with $\Delta X_t = X_t - X_{t-}$, the corresponding jump process. We set the corresponding jump measure μ^X by

$$\mu^X(dt\,dx) = \sum_{s>0} \mathbb{1}_{\{\Delta X_s \neq 0\}}\, \delta_{(s,\Delta X_s)}(dt\,dx). \tag{2.1}$$

In this case, $H \star \mu^X_t = \sum_{0<s\leq t} H(\cdot, s, \Delta X_s)$. We denote by $\nu^X = \nu^{X,\mathbb{P}}$ the compensator of μ^X, see [12], Theorem 1.8, Chapter II. The dependence on $\mathbb{P}$ will be omitted when self-explanatory.

Definition 2.1. *Let X be a càdlàg process and Y be a process belonging to $L^1([0,\,T])$ a.s.*
$\int_{]0,\,\cdot]} Y_s\, d^- X_s$ denotes the forward integral of Y with respect to X, i.e., the u.c.p. limit, whenever it exists, of

$$\int_{]0,\,t]} Y(s)\, \frac{X((s+\varepsilon)\wedge t) - X(s)}{\varepsilon}\, ds, \quad t \in [0,T].$$

Remark 2.2. Let (Y,Z), (Y',Z') be two pair of processes and $\Omega_0 \subset \Omega$ be an event such that

$$Y_t \mathbb{1}_{\Omega_0} = Y'_t \mathbb{1}_{\Omega_0}, \quad Z_t \mathbb{1}_{\Omega_0} = Z'_t \mathbb{1}_{\Omega_0}, \quad \forall t \in [0,T],$$

where the equality holds in the sense of indistinguishability. Then

$$\int_0^{\cdot} Y_s d^- Z_s = \int_0^{\cdot} Y'_s d^- Z'_s$$

in the sense that, if an integral exists, then the other exists and they are equal.

Definition 2.3. *For two càdlàg processes X and Y, we define the covariation of X and Y, denoted $[X,Y]$, as the u.c.p. limit (if it exists) of*

$$[X,Y]^{\varepsilon}(t) := \int_{]0,\,t]} \frac{(X((s+\varepsilon)\wedge t) - X(s))(Y((s+\varepsilon)\wedge t) - Y(s))}{\varepsilon}\, ds, \quad t \in [0,T]. \tag{2.2}$$

A càdlàg process X will be called a finite quadratic variation process whenever $[X,X]$ exists.

By Lemma 2.10 in [1], we know that

$$[X,X] = [X,X]^c + \sum_{s\leq \cdot} |\Delta X_s|^2, \tag{2.3}$$

where $[X,X]^c$ is the continuous component of $[X,X]$.

Remark 2.4. By Proposition 1.1 in [14], if X,Y are two càdlàg semimartingales and H is a càdlàg adapted process we have the following.

(i) $[X,Y]$ exists and it is the usual bracket.

(ii) $\int_{]0,\,\cdot]} H\, d^- X$ is the usual stochastic integral $\int_{]0,\,\cdot]} H_{s-} dX_s$.

2.2 New technical results

Proposition 2.5. *Let $\varphi : [0,T]\times\mathbb{R}\times\Omega \to \mathbb{R}$ be a measurable function such that*

$$|\varphi(s,x)| \star \mu^X_T = \sum_{s\leq T} |\varphi(s,\Delta X_s)| < +\infty \quad \text{a.s.}$$

Set $\tilde{A}_t := \varphi(s,x)\star\mu^X_t$. Then, for every càdlàg process H,

$$\int_{]0,\cdot]} H_s d^- \tilde{A}_s = H_{s-}\, \varphi(s,x) \star \mu^X.$$

Proof. By Remark 2.4-(ii),

$$\int_{]0,\cdot]} H_s d^- \tilde{A}_s = \int_{]0,\cdot]} H_{s-}\, d\tilde{A}_s.$$

Indeed, the right-hand side is a Lebesgue integral, so no condition of adaptability on H is required. On the other hand,

$$\int_{]0,\cdot]} H_{s-}\, d\tilde{A}_s = \sum_{s\leq \cdot} H_{s-}\, \Delta\tilde{A}_s = \sum_{s\leq \cdot} H_{s-}\, \varphi(s,\Delta X_s) = H_{s-}\, \varphi(s,x) \star \mu^X.$$

□

Lemma 2.6. *Let Y, Z be two càdlàg finite quadratic variation processes. Suppose that $[Y,Y]^c = 0$. Then*

$$[Y,Z] = \sum_{t\leq \cdot} \Delta Y_t \Delta Z_t.$$

Proof. Given a sequence (ε_n) converging to zero, we need to show the existence of (n_k) so that

$$[Y,Z]^{\varepsilon_{n_k}} \underset{k\to+\infty}{\longrightarrow} \sum_{t\leq \cdot} \Delta Y_t\, \Delta Z_t, \quad \text{a.s. uniformly.}$$

By extraction of subsequences, we can suppose that for almost all ω, there is a subsequence of (ε_n), still denoted by the same letter, such that

$$[Y,Y]^{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} [Y,Y] = \sum_{t\leq \cdot} |\Delta Y_t|^2, \quad \text{uniformly,}$$
$$[Z,Z]^{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} [Z,Z], \quad \text{uniformly.}$$

Let us thus fix a realization of $\omega \in \Omega$. Let $(t_i)_{i\geq 1}$ be the sequence including the jumps of $Y(\omega)$ and $Z(\omega)$, obviously in $]0,T]$. In the sequel we will omit the dependence on ω.

By (2.3), we can take $\gamma > 0$ and $N = N(\gamma)$ such that

$$\sum_{i=N+1}^{\infty} |\Delta Z_{t_i}|^2 + \sum_{i=N+1}^{\infty} |\Delta Y_{t_i}|^2 \leq \gamma.$$

We proceed similarly as for the proof of Proposition 2.14 in [1]. By renumerating increasingly the set $(t_i)_{i=1}^N$ and setting $t_0 = 0$, we define

$$A(\varepsilon,N) = \bigcup_{i=1}^{N}]t_i - \varepsilon, t_i], \tag{2.4}$$

$$B(\varepsilon,N) = \bigcup_{i=1}^{N}]t_{i-1}, t_i - \varepsilon] = [0,T] \setminus A(\varepsilon,N), \tag{2.5}$$

with $\varepsilon < \inf_{i=1,\dots,N} |t_i - t_{i-1}|$. We decompose

$$\frac{1}{\varepsilon}\int_{]0,s]} (Y_{(t+\varepsilon)\wedge s} - Y_t)(Z_{(t+\varepsilon)\wedge s} - Z_t)dt - \sum_{t\leq s} \Delta Y_t \Delta Z_t \tag{2.6}$$

into

$$I_A^{Y,Z}(\varepsilon,N,s) + I_{B_1}^{Y,Z}(\varepsilon,N,s) + I_{B_2}^{Y,Z}(N,s), \tag{2.7}$$

where

$$I_A^{Y,Z}(\varepsilon, N, s) = \frac{1}{\varepsilon}\int_{]0,s]\cap A(\varepsilon,N)} (Y_{(t+\varepsilon)\wedge s} - Y_t)(Z_{(t+\varepsilon)\wedge s} - Z_t)dt - \sum_{i=1}^{N} \mathbb{1}_{]0,s]}(t_i)\Delta Y_{t_i}\Delta Z_{t_i},$$
$$I_{B1}^{Y,Z}(\varepsilon, N, s) = \frac{1}{\varepsilon}\int_{]0,s]\cap B(\varepsilon,N)} (Y_{(t+\varepsilon)\wedge s} - Y_t)(Z_{(t+\varepsilon)\wedge s} - Z_t)dt,$$
$$I_{B2}^{Y,Z}(N, s) = -\sum_{i=N+1}^{\infty} \mathbb{1}_{]0,s]}(t_i)\Delta Y_{t_i}\Delta Z_{t_i}.$$

In order to prove that

$$I_A^{Y,Z}(\varepsilon_n, N, \cdot) \underset{n\to+\infty}{\to} 0 \quad \text{uniformly,} \tag{2.8}$$

by bilinearity it is enough to show that $I_A^{\eta,\eta}(\varepsilon_n, N, \cdot)$ goes to zero, uniformly, for $\eta = Y, Z, Y+Z$. Now, by Lemma 2.11 of [1],

$$I_A^{\eta,\eta}(\varepsilon_n, N, s) + \sum_{i=1}^{N} \mathbb{1}_{]0,s]}(t_i)(\Delta\eta_{t_i})^2 = \sum_{i=1}^{N} \frac{1}{\varepsilon}\int_{t_i-\varepsilon}^{t_i} \mathbb{1}_{]0,\,s]}(t)\,\phi(\eta_{(t+\varepsilon)\wedge s}, \eta_t)\,dt$$
$$\overset{\varepsilon\to 0}{\longrightarrow} \sum_{i=1}^{N} \mathbb{1}_{]0,\,s]}(t_i)\,\phi(\eta_{t_i}, \eta_{t_i-}), \quad \text{uniformly in} \quad s \in [0,\,T],$$

with $\phi(x_1, x_2) = (x_1 - x_2)^2$. This implies that $I_A^{\eta,\eta}(\varepsilon_n, N, \cdot)$ goes to zero, uniformly.

Since $[Y, Y]^c = 0$, taking into account (2.3) we have

$$\frac{1}{\varepsilon}\int_{]0,s]} (Y_{(t+\varepsilon_n)\wedge s} - Y_t)^2 dt - \sum_{t\le s} |\Delta Y_t|^2 \underset{n\to+\infty}{\longrightarrow} 0, \quad \text{uniformly in} \quad s \in [0,\,T]. \tag{2.9}$$

On the other hand,

$$|I_{B2}^{Y,Y}(N, s)| \le \sum_{i=N+1}^{\infty} |\Delta Y_{t_i}|^2 \le \gamma. \tag{2.10}$$

Collecting (2.9), (2.8) with $Z = Y$ and (2.10), it follows from (2.6) and (2.7) both for $Z = Y$ that

$$\limsup_{n\to+\infty} |I_{B1}^{Y,Y}(\varepsilon_n, N, s)| \le \gamma. \tag{2.11}$$

We come back to the estimate of (2.6). The absolute value of (2.6), with $\varepsilon = \varepsilon_n$, is bounded by

$$|I_A^{Y,Z}(\varepsilon_n, N, s)| + |I_{B_1}^{Y,Z}(\varepsilon_n, N, s)| + |I_{B_2}^{Y,Z}(N, s)|$$
$$\le |I_A^{Y,Z}(\varepsilon_n, N, s)| + \sqrt{I_{B1}^{Y,Y}(\varepsilon_n, N, s)}\sqrt{I_{B1}^{Z,Z}(\varepsilon_n, N, s)} + \sqrt{\sum_{i=N+1}^{\infty} |\Delta Y_{t_i}|^2}\sqrt{\sum_{i=N+1}^{\infty} |\Delta Z_{t_i}|^2}$$
$$\le |I_A^{Y,Z}(\varepsilon_n, N, s)| + \sqrt{I_{B1}^{Y,Y}(\varepsilon_n, N, s)}\,\sqrt{[Z, Z]_T^{\varepsilon_n}} + \gamma. \tag{2.12}$$

Taking the $\limsup_{n\to\infty}$ in (2.12), taking into account (2.8) and (2.11), we get

$$\limsup_{n\to\infty} (|I_A^{Y,Z}(\varepsilon_n, N, s)| + |I_{B_1}^{Y,Z}(\varepsilon_n, N, s)| + |I_{B_2}^{Y,Z}(N, s)|) \le \sqrt{\gamma}\,\sqrt{[Z, Z]_T} + \gamma.$$

Since γ is arbitrarily chosen, we have shown that (2.6) converges uniformly to zero. This implies that (2.6) converges u.c.p. to zero. □

The following result of stability of finite quadratic variation processes was well understood in the context of Föllmer's discretizations, but was never established in the regularization framework.

Lemma 2.7. *1. Let $Y = \varphi(X)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^1 function and X is a càdlàg process of finite quadratic variation. Then*

$$[Y, Y]_t = \int_0^t (\varphi'(X_{s-})^2 d[X, X]^c_s + \sum_{s \le t} (\Delta\varphi(X_s))^2, t \in [0, T].$$

In particular, Y is also a finite quadratic variation process.

2. Let $Y^1 = \varphi(X^1)$ and $Y^2 = \phi(X^2)$, where φ and ϕ are C^1 functions and X^1, X^2 are càdlàg processes such that (X^1, X^2) has all its mutual covariations. Then

$$[Y^1, Y^2]_t = \int_0^t \varphi'(X^1_s)\phi'(X^2_{s-}) d[X^1, X^2]^c_s + \sum_{s \le t} \Delta\varphi(X^1_s)\, \Delta\phi(X^2_s), t \in [0, T]$$

Proof. 1. Let $t \in [0, T]$ and $\varepsilon \in [0, 1]$. We expand, for $s \in [0, T]$,

$$\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_{s\wedge t}) = I^\varphi_1(s, t, \varepsilon)(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}),$$

where

$$I^\varphi_1(s, t, \varepsilon) = \int_0^1 \varphi'(X_{s\wedge t} + a(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}))\, da.$$

Consequently,

$$\begin{aligned}\frac{1}{\varepsilon}(\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_{s\wedge t}))^2 &= \frac{1}{\varepsilon}((I^\varphi_1(s, t, \varepsilon))^2 - (\varphi'(X_s))^2)(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t})^2 \\ &+ \frac{1}{\varepsilon}(\varphi'(X_s))^2(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t})^2.\end{aligned}$$

Integrating from 0 to t, we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_s))^2 ds &= \frac{1}{\varepsilon}\int_0^t ((I^\varphi_1(s, t, \varepsilon))^2 - (\varphi'(X_s)^2)(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds \\ &+ \frac{1}{\varepsilon}\int_0^t (\varphi'(X_s))^2(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds \\ &=: J_1(t, \varepsilon) + J_2(t, \varepsilon).\end{aligned} \tag{2.13}$$

We notice that, without restriction of generality, passing to a suitable subsequence, we can suppose (with abuse of notation) that

$$[X, X]^\varepsilon := \frac{1}{\varepsilon}\int_0^\cdot (X_{(s+\varepsilon)\wedge\cdot} - X_s)^2\, ds \underset{\varepsilon\to 0}{\to} [X, X], \quad \text{uniformly a.s.} \tag{2.14}$$

Since X is a finite quadratic variation process, by Lemma A.5 in [1], taking into account Definition A.2 and Corollary A.4-2. in [1], if g is a càdlàg process then

$$\frac{1}{\varepsilon}\int_0^t g_s(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds = \frac{1}{\varepsilon}\int_0^t g_{s-}(X_{(s+\varepsilon)\wedge t} - X_s)^2 ds \underset{\varepsilon\to 0}{\to} \int_0^t g_{s-} d[X, X]_s, \quad \text{u.c.p.} \tag{2.15}$$

Therefore, taking $g_s = (\varphi'(X_s))^2$ in (2.15), we get

$$J_2(\cdot,\varepsilon) \underset{\varepsilon\to 0}{\to} \int_0^{\cdot} (\varphi'(X_{s-}))^2 d[X,X]_s, \quad \text{u.c.p.} \tag{2.16}$$

Next step consists in proving that

$$J_1(\cdot,\varepsilon) \underset{\varepsilon\to 0}{\to} \sum_{s\leq \cdot} \Big[\Big(\int_0^1 \varphi'(X_{s-} + a\Delta X_s)da\Big)^2 - (\varphi'(X_{s-}))^2\Big](\Delta X_s)^2, \quad \text{u.c.p.} \tag{2.17}$$

We fix a realization $\omega \in \Omega$. Proceeding as in the proof of Proposition 2.14 in [1], let (t_i) be an enumeration of all the jumps of $X(\omega)$ in $[0,\, T]$. We have $\sum_i (\Delta X_{t_i}(\omega))^2 < \infty$. Let $\gamma > 0$ and $N = N(\gamma)$ such that

$$\sum_{i=N+1}^{\infty} (\Delta X_{t_i}(\omega))^2 \leq \gamma^2. \tag{2.18}$$

We decompose

$$\begin{aligned} J_1(t,\varepsilon) &= \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{A(\varepsilon,N)}(s) J_{10}(s,t,\varepsilon)ds + \int_0^t \mathbb{1}_{B(\varepsilon,N)}(s) J_{10}(s,t,\varepsilon)ds \\ &=: J_{1A}(t,\varepsilon,N) + J_{1B}(t,\varepsilon,N), \end{aligned} \tag{2.19}$$

where we have denoted

$$J_{10}(s,t,\varepsilon) := (X_{(s+\varepsilon)\wedge t} - X_s)^2((I_1^{\varphi}(s,t,\varepsilon))^2 - (\varphi'(X_s)^2),$$

and the sets $A(\varepsilon,N)$ and $B(\varepsilon,N)$ are the ones introduced in (2.4)-(2.5). By Lemma 2.11 in [1], it follows that, uniformly in $t \in [0,\, T]$,

$$J_{1A}(t,\varepsilon,N) \underset{\varepsilon\to 0}{\to} \sum_{i=1}^{N} \mathbb{1}_{]0,\,t]}(t_i)(\Delta X_{t_i})^2 \Big(\Big(\int_0^1 \varphi'(X_{t_i-} + a\Delta X_{t_i})da\Big)^2 - (\varphi'(X_{t_i-}))^2\Big). \tag{2.20}$$

On the other hand,

$$J_{1B}(t,\varepsilon,N) = \sum_{i=1}^{N} \frac{1}{\varepsilon}\int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)^2 I_{1B}^{\varphi,i}(s,t,\varepsilon)\, ds,$$

where

$$\begin{aligned} I_{1B}^{\varphi,i}(s,t,\varepsilon) &= \mathbb{1}_{]t_{i-1},t_i-\varepsilon[}(s)\Big[\Big(\int_0^1 \varphi'(X_{s\wedge t} + a(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}))\, da\Big)^2 - (\varphi'(X_s))^2\Big] \\ &= \mathbb{1}_{]t_{i-1},t_i-\varepsilon[}(s)\Big[\int_0^1 \varphi'(X_{s\wedge t} + a(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}))\, da - \varphi'(X_s)\Big]\cdot \\ &\quad \cdot \Big[\int_0^1 \varphi'(X_{s\wedge t} + a(X_{(s+\varepsilon)\wedge t} - X_{s\wedge t}))\, da + \varphi'(X_s)\Big]. \end{aligned}$$

For every $i = 1, ..., N$, we have

$$|I_{1B}^{\varphi,i}(s,t,\varepsilon)| \leq 2 \sup_{y\in[X_s,X_{s+\varepsilon}]} |\varphi'(y)|\, \delta\Big(\varphi, \sup_i \sup_{\substack{p,q\in]t_{i-1},t_i[\\ |p-q|\leq\varepsilon}} |X_p - X_q|\Big).$$

We notice that there is ε_0 such that, if $\varepsilon < \varepsilon_0$, $\sup_{\substack{p,q\in]t_{i-1},t_i[\\ |p-q|\le\varepsilon}} |X_p - X_q| \le 3\gamma$, where we have applied Lemma 2.12 in [1] to the prolongation by continuity of X to the extremities restricted to $]t_{i-1}, t_i[$. Therefore, for $\varepsilon < \varepsilon_0$,

$$|I_{1B}^{\varphi,i}(s,t,\varepsilon)| \le 2 \sup_{y\in[-||X||_\infty, ||X||_\infty]} |\varphi'(y)|\, \delta(\varphi,\, 3\gamma),$$

and consequently,

$$\sup_{t\in[0,T]} |J_{1B}(t,\varepsilon,N)| \le 2\delta(\varphi,\, 3\gamma) \sup_{t\in[0,T]} [X,X]^\varepsilon_t \sup_{y\in[-||X||_\infty, ||X||_\infty]} |\varphi'(y)|, \tag{2.21}$$

where the latter supremum is finite by (2.14). Going back to (2.19) we get

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big| J_1(t,\varepsilon) - \sum_{i=1}^{\infty} \mathbb{1}_{]0,\,t]}(t_i)(\Delta X_{t_i})^2 \Big[\Big(\int_0^1 \varphi'(X_{t_i-} + a\Delta X_{t_i})da\Big)^2 - (\varphi'(X_{t_i-}))^2\Big]\Big| \\
&\le \sup_{t\in[0,T]} \Big| J_{1A}(t,\varepsilon,N) - \sum_{i=1}^{N} \mathbb{1}_{]0,\,t]}(t_i)(\Delta X_{t_i})^2 \Big[\Big(\int_0^1 \varphi'(X_{t_i-} + a\Delta X_{t_i})da\Big)^2 - (\varphi'(X_{t_i-}))^2\Big]\Big| \\
&+ \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\,T]}(t_i)(\Delta X_{t_i})^2 \Big[\Big(\int_0^1 \varphi'(X_{t_i-} + a\Delta X_{t_i})da\Big)^2 - (\varphi'(X_{t_i-}))^2\Big] \\
&+ \sup_{t\in[0,T]} |J_{1B}(t,\varepsilon,N)|.
\end{aligned} \tag{2.22}$$

Taking the $\limsup_{\varepsilon\to 0}$ in (2.22), collecting (2.20) and (2.21), we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \sup_{t\in[0,T]} \Big| J_1(t,\varepsilon) - \sum_{i=1}^{\infty} \mathbb{1}_{]0,\,t]}(t_i)(\Delta X_{t_i})^2 \Big[\Big(\int_0^1 \varphi'(X_{t_i-} + a\Delta X_{t_i})da\Big)^2 - (\varphi'(X_{t_i-}))^2\Big]\Big| \\
&\le 2 \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\,T]}(t_i)(\Delta X_{t_i})^2 \sup_{y\in[-||X||_\infty, ||X||_\infty]} |\varphi'(y)|^2 \\
&+ 2 \sup_{\varepsilon<\varepsilon_0} \sup_{t\in[0,T]} [X,X]^\varepsilon_t\, \delta(\varphi,\, 3\gamma) \sup_{y\in[-||X||_\infty, ||X||_\infty]} |\varphi'(y)| \\
&\le 2\gamma^2 \sup_{y\in[-||X||_\infty, ||X||_\infty]} |\varphi'(y)|^2 + 2 \sup_{\varepsilon<\varepsilon_0} \sup_{t\in[0,T]} [X,X]^\varepsilon_t\, \delta(\varphi,\, 3\gamma) \sup_{y\in[-||X||_\infty, ||X||_\infty]} |\varphi'(y)|,
\end{aligned}$$

where in the latter inequality we have used (2.18). Since γ is arbitrary and φ' is uniformly continuous on compact intervals, then

$$J_1(\cdot,\varepsilon) \underset{\varepsilon\to 0}{\rightarrow} \sum_{s\le\cdot} \Big[\Big(\int_0^1 \varphi'(X_{s-} + a\Delta X_s)da\Big)^2 - (\varphi'(X_{s-}))^2\Big](\Delta X_s)^2,$$

uniformly in t for the fixed ω. In particular, this implies (2.17).

By (2.16) and (2.17), and the fact that $[X,X] = [X,X]^c + \sum_{s\le t}(\Delta X_s)^2$, (2.13) yields

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t (\varphi(X_{(s+\varepsilon)\wedge t}) - \varphi(X_s))^2 ds \\
&\underset{\varepsilon\to 0}{\longrightarrow} \int_0^\cdot (\varphi'(X_{s-}))^2 d[X,X]^c_s + \sum_{s\le t} (\varphi'(X_{s-}))^2(\Delta X_s)^2
\end{aligned}$$

$$+\sum_{s\le t}\Big[\Big(\int_0^1 \varphi'(X_{s-}+a\Delta X_s)da\Big)^2-(\varphi'(X_{s-}))^2\Big](\Delta X_s)^2$$
$$=\int_0^\cdot (\varphi'(X_{s-}))^2 d[X,X]^c_s+\sum_{s\le t}\Big(\int_0^1 \varphi'(X_{s-}+a\Delta X_s)da\Big)^2(\Delta X_s)^2, \quad \text{u.c.p.}$$

The result follows because

$$\Delta\varphi(X_s)=\varphi(X_s)-\varphi(X_{s-})=\Delta X_s\int_0^1 \varphi'(X_{s-}+a\Delta X_s)da.$$

2. The result follows from point 1 by polarity arguments. □

3 Back to weak Dirichlet processes

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration fulfilling the usual conditions on $(\Omega,\mathcal{F},\mathbb{P})$. Let $X=(X_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation. Cutting large jumps via a truncation function k, by Corollary 3.15 in [4], the process

$$X_t-\sum_{s\le t}(X_s-k(\Delta X_s))=X_t-(x-k(x))\star\mu^X_t, \quad t\in[0,T], \tag{3.1}$$

is a special weak Dirichlet process, and therefore it admits a unique decomposition

$$X-(x-k(x))\star\mu^X=B^{k,X}+M^{k,X}$$

with $M^{k,X}$ an $(\mathcal{F}_t)$-martingale and $B^{k,X}$ an $(\mathcal{F}_t)$-martingale orthogonal and predictable process. By Corollary 3.22-(i) in [4], this decomposition has the form

$$X-(x-k(x))\star\mu^X=B^{k,X}+X^c+k(x)\star(\mu^X-\nu^X),$$

where X^c is the unique local martingale component of X, ν^X is the compensator of μ^X. In particular, $X=M+A$ with

$$M:=X^c+k(x)\star(\mu^X-\nu^X), \tag{3.2}$$
$$A:=B^{k,X}+(x-k(x))\star\mu^X. \tag{3.3}$$

Remark 3.1. The unique decomposition of the weak Dirichlet process X, provided in Proposition 3.2 in [4], is therefore

$$X=X^c+\Gamma \tag{3.4}$$

with

$$\Gamma=k(x)\star(\mu^X-\nu^X)+B^{k,X}+(x-k(x))\star\mu^X.$$

We also define $C^X:=[X^c,X^c]$. The triplet $(B^{k,X},C^X,\nu^X)$ will be associated to the characteristics of X when it is an $(\mathcal{F}^X_t)$-weak Dirichlet process, see Section 4 below.

By Remark 3.1 we easily see that

$$\Delta B^{k,X}_t=\int_{\mathbb{R}} k(x)\nu^X(\{t\}\times dx). \tag{3.5}$$

This property is classical for semimartingales, see formula (2.14), Section II, in [12]. We recall that when X is a semimartingale, $B^{k,X}$ has bounded variation. This is no longer the case in the present framework. Nevertheless, we have the following.

Theorem 3.2. *Let X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation with decomposition (3.4). Then $B^{k,X}$ is a finite quadratic variation process, with*

$$[B^{k,X}, B^{k,X}] = [X,X]^c_t - [X^c, X^c] + \sum_{s\leq\cdot} \Big| \int_{\mathbb{R}} k(x)\nu^X(\{s\}\times dx)\Big|^2. \tag{3.6}$$

Proof. Recalling that $X = M + A$ with M and A provided respectively by (3.2) and (3.3), we have

$$B^{k,X} = X - X^c - k(x) \star (\mu^X - \nu^X) - (x - k(x)) \star \mu^X.$$

Since bounded variation processes and purely discontinuous martingales are martingale orthogonal processes, it follows that

$$\begin{aligned}
[B^{k,X}, B^{k,X}]_t &= [X,X]_t + [X^c, X^c]_t + [k(x)\star(\mu^X-\nu^X), k(x)\star(\mu^X-\nu^X)]_t \\
&\quad + [(x-k(x))\star\mu^X, (x-k(x))\star\mu^X]_t - 2[X, X^c]_t - 2[X, k(x)\star(\mu^X-\nu^X)]_t \\
&\quad - 2[X, (x-k(x))\star\mu^X]_t + 2[k(x)\star(\mu^X-\nu^X), (x-k(x))\star\mu^X]_t,
\end{aligned} \tag{3.7}$$

provided the right-hand side is well-defined. Now we notice that, by (3.4), $[X, X^c] = [X^c, X^c]$. Moreover, since $(x - k(x)) \star \mu^X$ is a bounded variation process, by Proposition 2.14 in [1],

$$\begin{aligned}
[(x-k(x))\star\mu^X, (x-k(x))\star\mu^X]_t &= \sum_{s\leq t} |\Delta X_s - k(\Delta X_s)|^2, \\
[X, (x-k(x))\star\mu^X]_t &= \sum_{s\leq t} \Delta X(\Delta X_s - k(\Delta X_s)), \\
[k(x)\star(\mu^X-\nu^X), (x-k(x))\star\mu^X]_t &= \sum_{s\leq t} (\Delta X_s - k(\Delta X_s)) \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx).
\end{aligned}$$

On the other hand, being $k(x) \star (\mu^X - \nu^X)$ a purely discontinuous martingale, by Proposition 5.3 in [1],

$$[k(x)\star(\mu^X-\nu^X), k(x)\star(\mu^X-\nu^X)]_t = \sum_{s\leq t} \Big| \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx)\Big|^2.$$

Finally, by Lemma 2.6 with $Y = k(x) \star (\mu^X - \nu^X)$ and $Z = X$,

$$[X, k(x)\star(\mu^X-\nu^X)]_t = \sum_{s\leq t} \Delta X_s \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx).$$

Plugging previous terms in (3.7) we get

$$\begin{aligned}
[B^{k,X}, B^{k,X}]_t &= [X,X]_t - [X^c, X^c]_t + \sum_{s\leq t} \Big| \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx)\Big|^2 \\
&\quad + \sum_{s\leq t} |\Delta X_s - k(\Delta X_s)|^2 - 2\sum_{s\leq t} \Delta X_s \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx) \\
&\quad - 2\sum_{s\leq t} \Delta X(\Delta X_s - k(\Delta X_s)) + 2\sum_{s\leq t} (\Delta X_s - k(\Delta X_s)) \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx) \\
&= [X,X]_t - [X^c, X^c]_t + \sum_{s\leq t} \Big| \int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{s\}\times dx)\Big|^2
\end{aligned}$$

$$+\sum_{s\le t}|\Delta X_s-k(\Delta X_s)|^2-2\sum_{s\le t}\Delta X(\Delta X_s-k(\Delta X_s))$$
$$-2\sum_{s\le t}k(\Delta X_s)\int_{\mathbb{R}}k(x)(\mu^X-\nu^X)(\{s\}\times dx), \tag{3.8}$$

which implies in particular that $[B^{k,X},B^{k,X}]_t$ is finite. Now, recalling that $[X,X]_t=[X,X]_t^c+\sum_{s\le t}|\Delta X_s|^2$, and noticing that

$$\sum_{s\le t}|\Delta X_s-k(\Delta X_s)|^2=\sum_{s\le t}|\Delta X_s|^2+\sum_{s\le t}|k(\Delta X_s)|^2-2\sum_{s\le t}\Delta X_s\,k(\Delta X_s),$$
$$-2\sum_{s\le t}\Delta X(\Delta X_s-k(\Delta X_s))=-2\sum_{s\le t}|\Delta X_s|^2+2\sum_{s\le t}\Delta X_s\,k(\Delta X_s),$$

formula (3.8) reads

$$[B^{k,X},B^{k,X}]_t=[X,X]_t-[X^c,X^c]_t-\sum_{s\le t}|\Delta X_s|^2+\sum_{s\le t}|k(\Delta X_s)|^2 \tag{3.9}$$
$$+\sum_{s\le t}\Big(\Big|\int_{\mathbb{R}}k(x)(\mu^X-\nu^X)(\{s\}\times dx)\Big|^2-2\,k(\Delta X_s)\int_{\mathbb{R}}k(x)(\mu^X-\nu^X)(\{s\}\times dx)\Big).$$

Finally, we notice that

$$\Big|\int_{\mathbb{R}}k(x)(\mu^X-\nu^X)(\{s\}\times dx)\Big|^2-2k(\Delta X_s)\int_{\mathbb{R}}k(x)(\mu^X-\nu^X)(\{s\}\times dx)$$
$$=|k(\Delta X_s)|^2+\Big|\int_{\mathbb{R}}k(x)\nu^X(\{s\}\times dx)\Big|^2-2\,k(\Delta X_s)\int_{\mathbb{R}}k(x)\nu^X(\{s\}\times dx)$$
$$-2|k(\Delta X_s)|^2+2k(\Delta X_s)\int_{\mathbb{R}}k(x)\nu^X(\{s\}\times dx)$$
$$=\Big|\int_{\mathbb{R}}k(x)\nu^X(\{s\}\times dx)\Big|^2-|k(\Delta X_s)|^2. \tag{3.10}$$

Plugging (3.10) in (3.9) (noticing that $[X,X]_t-\sum_{s\le t}|\Delta X_s|^2=[X,X]_t^c$) we get (3.6). □

Corollary 3.3. *Let X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation with decomposition (3.4). Then*

$$[B^{k,X},B^{k,X}]=[X,X]^c-[X^c,X^c]+\sum_{s\le\cdot}|\Delta B_s^{k,X}|^2, \tag{3.11}$$

or equivalently,

$$[X,X]^c=[X^c,X^c]+[B^{k,X},B^{k,X}]^c. \tag{3.12}$$

Proof. Taking into account (3.5), formula (3.6) of Theorem 3.2 can be rewritten as (3.11), which is in turn equivalent to (3.12). □

Remark 3.4. If X is a semimartingale (resp. a Dirichlet process) by Corollary 3.3 we recover the result

$$[X,X]^c=[X^c,X^c], \tag{3.13}$$

proved in Proposition 3.5 in [4] (resp. in Proposition 6.2 in [3]). In fact we have the following.

(i) Let X be a semimartingale. Then $B^{k,X}$ is a finite variation process, so by Proposition 3.14 in [1]

$$[B^{k,X}, B^{k,X}]_t = \sum_{s \le t} |\Delta B_s^{k,X}|^2,$$

and therefore by (3.11) we recover (3.13).

(ii) Let X be a Dirichlet process. Then it is a special weak Dirichlet process, and by (3.2)-(3.3), taking into account Corollary 3.22-(ii) in [4], it admits the unique decomposition

$$X = X^c + x \star (\mu^X - \nu^X) + (x - k(x)) \star \nu^X + B^{k,X}.$$

Let $Y = (x - k(x)) \star \nu^X$. Since X is a Dirichlet process,

$$[Y + B^{k,X}, Y + B^{k,X}] = 0,$$

so that

$$[B^{k,X}, B^{k,X}] = -[Y, Y] - 2[Y, B^{k,X}] = -\sum_{s \le \cdot} |\Delta Y_s|^2 - 2[Y, B^{k,X}].$$

On the other hand, by Lemma 2.4

$$[Y, B^{k,X}] = \sum_{s \le \cdot} \Delta Y_s \Delta B_s^{k,X},$$

so that

$$[B^{k,X}, B^{k,X}]_t = -\sum_{s \le t} |\Delta Y_s|^2 - 2 \sum_{s \le t} \Delta Y_s \Delta B_s^{k,X}.$$

We get that $[B^{k,X}, B^{k,X}]_t^c = 0$, so by (3.12) we recover (3.13).

We state here a slight modification of Theorem 3.36 in [4].

Theorem 3.5. *Let X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation, taking values in an open interval $\mathcal{O}$. Let $v \in C^{0,1}([0,T] \times \mathcal{O})$. Then $Y_t = v(t, X_t)$ is an $(\mathcal{F}_t)$–weak Dirichlet with continuous martingale component*

$$Y^c = Y_0 + \int_0^\cdot \partial_x v(s, X_s)\, dX_s^c. \tag{3.14}$$

Proof. The proof follows the same lines of the one of Theorem 3.36 in [4], taking into account that the set $\mathcal{O}$ is open and convex.

4 Main result

4.1 Characteristics of a weak Dirichlet process

We denote by $\check{\Omega}$ the canonical space of all càdlàg functions $\check{\omega} : [0,T] \to \mathbb{R}$, and by $\check{X}$ the canonical process defined by $\check{X}_t(\omega) = \check{\omega}(t)$. We also set $\check{\mathcal{F}} = \sigma(\check{X})$, and $\check{\mathbb{F}} = (\check{\mathcal{F}}_t)_{t \in [0,T]}$. We suppose below that X is an $(\mathcal{F}_t^X)$-weak Dirichlet process. Let μ be the jump measure of $\check{X}$ and ν the compensator of μ under the law $\mathcal{L}(X)$ of X.

Definition 4.1. *We call* characteristics *of X, associated with $k \in \mathcal{K}$, the triplet $(B^k, C, \check{X}^c\rangle, \nu) = (B^{k,\check{X}}, C^{\check{X}}, \nu^{\check{X}})$ on $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}})$ obtained from the unique decomposition (3.4) in Remark 3.1 for $\check{X}$ under $\mathcal{L}(X)$. In particular,* *(i)* B^k *is a predictable and* $\check{\mathbb{F}}$*-martingale orthogonal process, with* $B^k_0 = 0$*;*

(ii) C *is an* $\check{\mathbb{F}}$*-predictable and increasing process, with* $C_0 = 0$*;*

(iii) ν *is an* $\check{\mathbb{F}}$*-predictable random measure on* $[0,T] \times \mathbb{R}$*.*

Let $X = (X_t)_{t\in[0,T]}$ be an $(\mathcal{F}^X_t)$-weak Dirichlet process with finite quadratic variation with characteristics (B^k, C, ν). By Remark 3.26 in [4], we have $B^{k,X} = B^k \circ X$, $\nu^X = \nu \circ X$ and $C^X = C \circ X$. Therefore in this case (3.2)-(3.3) read

$$M := X^c + k(x) \star (\mu^X - (\nu \circ X)), \tag{4.1}$$

$$A := B^k \circ X + (x - k(x)) \star \mu^X. \tag{4.2}$$

4.2 The equivalence theorem

We provide an equivalence result for càdlàg weak Dirichlet processes that extends the analogous one for càdlàg semimartingales, see Theorem A.1.

Theorem 4.2. *Let* X *be a càdlàg process with finite quadratic variation. Let* B^k *be an* $(\check{\mathcal{F}}_t)$*-predictable process such that* $B^k_0 = 0$ *and* $B^k \circ X$ *has finite quadratic variation,* C *be an* $(\check{\mathcal{F}}_t)$*-adapted continuous process such that* $C_0 = 0$ *and* $C \circ X$ *has finite variation, and* ν *be an* $(\check{\mathcal{F}}_t)$*-predictable random measure on* $[0,T] \times \mathbb{R}$*.*

There is equivalence between the two following statements.

(i) X *is an* $(\mathcal{F}^X_t)$*-weak Dirichlet process with local characteristics* (B^k, C, ν)*.*

(ii) For each bounded function F *of class* $C^{1,2}$*,* $\int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s$ *is an* $(\mathcal{F}^X_t)$*-martingale orthogonal process, and the process*

$$F(\cdot, X_\cdot) - F(0, X_0) - \int_0^\cdot \partial_s F(s, X_s) ds - \frac{1}{2}\int_0^\cdot \partial^2_{xx} F(s, X_s)\,(d(C\circ X)_s + d[B^k \circ X, B^k \circ X]^c_s)$$
$$- \int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s \tag{4.3}$$

is an $(\mathcal{F}^X_t)$*-local martingale.*

Remark 4.3.

1. The stochastic integral $\int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s$ is a predictable process, being its jump process predictable. Indeed, by $(1.15)_-$ in [14], it is given by

   $$\partial_x F(t, X_{t-})\, \Delta(B^k \circ X)_t, \quad t \in [0,T].$$

   The first term is càglàd, therefore predictable, the second one is predictable since $B^k \circ X$ is predictable.

2. If X is a semimartingale, then $B^{k,X}$ has bounded variation, so that $\int_0^\cdot \partial_x F(s, X_s)\, d^-(B^k \circ X)_s = \int_0^\cdot \partial_x F(s, X_{s-})\, d(B^k \circ X)_s$ has bounded variation, therefore it is an $(\mathcal{F}_t)$-martingale orthogonal process. Moreover, $[B^k \circ X, B^k \circ X]^c_s = 0$, see Remark 3.4-(i), and we retrieve the result of Jacod and Shiryaev, see Theorem A.1 (and (1.1) in the non-homogeneous form).

Proof. *(i)* $\Rightarrow$ *(ii)*. Let X be an $(\mathcal{F}^X_t)$-weak Dirichlet process with finite quadratic variation with characteristics (B^k, C, ν). Let $F \in C^{1,2}$. We recall that $X = M + A$ as in (4.1) and (4.2). By Theorem 5.15 in [1] we have

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_x F(s, X_{s-})\, dM_s$$

$$+\int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-}+x) - F(s, X_{s-}))\frac{k(x)}{x}(\mu^X-\nu^X)(ds\,dx)$$
$$-\int_{]0,\,t]\times\mathbb{R}} x\,\partial_x F(s, X_{s-})\,\frac{k(x)}{x}\,(\mu^X-\nu^X)(ds\,dx)$$
$$+\int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-}+x) - F(s, X_{s-}) - x\,\partial_x F(s, X_{s-}))\,\frac{x-k(x)}{x}\mu^X(ds\,dx)$$
$$+\int_0^t \partial_s F(s, X_s)\,ds + \int_0^t \partial_x F(s, X_s)\,d^- A_s + \frac{1}{2}\int_0^t \partial^2_{xx} F(s, X_s)\,d[X,X]^c_s$$
$$+\int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-}+x) - F(s, X_{s-}) - x\,\partial_x F(s, X_{s-}))\frac{k(x)}{x}\nu^X(ds\,dx). \tag{4.4}$$

In fact Theorem 5.15 in [1] was written for $k(x) = x\mathbb{1}_{\{|x|\leq 1\}}$, but it naturally extends to a generic truncation function k.

By (4.2) and using Proposition 2.5, we get

$$\int_0^t \partial_x F(s, X_s)\,d^- A_s$$
$$= \int_0^t \partial_x F(s, X_s)\,d^- B^{k,X}_s + \int_{]0,\,t]\times\mathbb{R}} \partial_x F(s, X_{s-})\,(x-k(x))\mu^X(ds\,dx). \tag{4.5}$$

On the other hand, by (4.1)

$$\int_0^t \partial_x F(s, X_{s-})\,dM_s = \int_0^t \partial_x F(s, X_{s-})\,dX^c_s + \int_0^t \partial_x F(s, X_{s-})\,dM^{d,k}_s,$$

where $M^{d,k} = k(x)\star(\mu^X-\nu^X)$. We notice that

$$\Delta\left(\int_0^t \partial_x F(s, X_{s-})\,dM^{d,k}_s\right) = \partial_x F(t, X_{t-})\,\Delta M^{d,k}_t$$
$$= \partial_x F(t, X_{t-})\int_{\mathbb{R}} k(x)(\mu^X-\nu^X)(\{t\}\times dx)$$
$$= \Delta\left(\int_{]0,t]\times\mathbb{R}} \partial_x F(s, X_{s-})\,k(x)(\mu^X-\nu^X)(ds\,dx)\right).$$

We remind that, for any $Y(\cdot)$ predictable random field, $Y(x)\star(\mu^X-\nu^X)$ is the unique purely discontinuous martingale orthogonal process whose jumps are indistinguishable from

$$\Delta\left(\int_{]0,t]\times\mathbb{R}} Y_s(x)(\mu^X-\nu^X)(ds\,dx)\right),$$

see Corollary 4.19, Section I, in [12]. We conclude that

$$\int_0^t \partial_x F(s, X_{s-})\,dM_s$$
$$= \int_0^t \partial_x F(s, X_{s-})\,dX^c_s + \int_{]0,t]\times\mathbb{R}} \partial_x F(s, X_{s-})\,k(x)(\mu^X-\nu^X)(ds\,dx). \tag{4.6}$$

Plugging (4.5) and (4.6) in (4.4), we get

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_x F(s, X_{s-})\,dX^c_s + \int_{]0,\,t]\times\mathbb{R}} \partial_x F(s, X_{s-})\,k(x)(\mu^X-\nu^X)(ds\,dx)$$

$$
\begin{aligned}
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \frac{k(x)}{x}\, (\mu^X - \nu^X)(ds\, dx) \\
&- \int_{]0,\,t]\times\mathbb{R}} \partial_x F(s, X_{s-})\, k(x)\, (\mu^X - \nu^X)(ds\, dx) \\
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \frac{x - k(x)}{x}\, \mu^X(ds\, dx) \\
&+ \int_0^t \partial_s F(s, X_s)\, ds + \frac{1}{2} \int_0^t \partial^2_{xx} F(s, X_s)\, d[X, X]^c_s \\
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\frac{k(x)}{x}\, \nu^X(ds\, dx) \\
&+ \int_0^t \partial_x F(s, X_s)\, d^- B^{k,X}_s + \int_{]0,\,t]\times\mathbb{R}} \partial_x F(s, X_{s-})\, (x - k(x))\, \mu^X(ds dx),
\end{aligned}
$$

that reads

$$
\begin{aligned}
F(t, X_t) = F(0, X_0) &+ \int_0^t \partial_x F(s, X_{s-})\, dX^c_s \\
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \frac{k(x)}{x}\, (\mu^X - \nu^X)(ds\, dx) \\
&+ \int_0^t \partial_s F(s, X_s)\, ds + \frac{1}{2} \int_0^t \partial^2_{xx} F(s, X_s)\, d[X, X]^c_s + \int_0^t \partial_x F(s, X_s)\, d^- B^{k,X}_s \\
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\frac{k(x)}{x}\, \nu^X(ds\, dx) \\
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \frac{x - k(x)}{x}\, \mu^X(ds\, dx).
\end{aligned} \tag{4.7}
$$

At this point we make use of the fact that F is bounded. Indeed, being $F \in C^{0,1}_b$, by Theorem 3.16 and Remark 3.17 in [4],

$$
\forall\, a \in \mathbb{R}_+ \text{ s.t. } (F(s, X_{s-} + x) - F(s, X_{s-}))\, \mathbb{1}_{\{|x|>a\}} \star \mu^X \in \mathcal{A}_{\mathrm{loc}}. \tag{4.8}
$$

Therefore, by Lemma C.3 in [4],

$$
(F(s, X_{s-} + x) - F(s, X_{s-}))\, \frac{x - k(x)}{x} \star \mu^X \in \mathcal{A}_{\mathrm{loc}},
$$

so that

$$
(F(s, X_{s-} + x) - F(s, X_{s-}))\, \frac{x - k(x)}{x} \star \nu^X_t
$$

is well-defined for every $t \in \mathbb{R}_+$. Adding and subtracting the above mentioned term in (4.7), and using Corollary 3.3 (recalling that $C^X = [X^c, X^c]$), we get

$$
\begin{aligned}
F(t, X_t) = F(0, X_0) &+ \int_0^t \partial_x F(s, X_{s-})\, dX^c_s \\
&+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx) \\
&+ \int_0^t \partial_s F(s, X_s)\, ds + \frac{1}{2} \int_0^t \partial^2_{xx} F(s, X_s)\, (dC^X_s + d[B^{k,X}, B^{k,X}]^c_s) + \int_0^t \partial_x F(s, X_s)\, d^- B^{k,X}_s
\end{aligned}
$$

$$+\int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-}+x) - F(s, X_{s-}) - k(x)\,\partial_x F(s, X_{s-}))\,\nu^X(ds\,dx), \tag{4.9}$$

which implies that (4.3) is an $(\mathcal{F}_t^X)$-local martingale.

It remains to prove that $\int_0^\cdot \partial_x F(s, X_s)\,d^- B_s^{k,X}$ is martingale orthogonal. By Theorem 3.36 in [4], $F(t, X_t)$ is an $(\mathcal{F}_t^X)$-weak Dirichlet process with continuous martingale component

$$F(0, X_0) + \int_0^\cdot \partial_x F(s, X_{s-})\,dX_s^c.$$

By the uniqueness of the decomposition of weak Dirichlet processes (see Proposition 3.2 in [4]), we get from (4.9) that

$$\int_0^\cdot \partial_x F(s, X_s)\,d^- B_s^{k,X} + \Gamma \tag{4.10}$$

is a martingale orthogonal process, where

$$\begin{aligned}\Gamma :=& \int_{]0,\,\cdot]\times\mathbb{R}} (F(s, X_{s-}+x) - F(s, X_{s-}))\,(\mu^X - \nu^X)(ds\,dx)\\ &+ \int_0^\cdot \partial_s F(s, X_s)\,ds + \frac{1}{2}\int_0^t \partial_{xx}^2 F(s, X_s)\,(dC_s^X + d[B^{k,X}, B^{k,X}]_s^c)\\ &+ \int_{]0,\,\cdot]\times\mathbb{R}} (F(s, X_{s-}+x) - F(s, X_{s-}) - k(x)\,\partial_x F(s, X_{s-}))\,\nu^X(ds\,dx).\end{aligned}$$

Here Γ is a martingale orthogonal process since is the sum of a purely discontinuous martingale and bounded variation processes. This finally allows to conclude that the first term in (4.10) is martingale orthogonal.

$(ii) \Rightarrow (i)$. We apply (ii) with F time-homogeneous. Then, for each function F of class C_b^2,

$$\int_0^\cdot \partial_x F(X_s)\,d^-(B^k \circ X)_s$$

is an $(\mathcal{F}_t)$-martingale orthogonal process, and the process

$$\begin{aligned}M^F :=& F(X_\cdot) - F(X_0) - \frac{1}{2}\int_0^\cdot F''(X_s)\,(d(C\circ X)_s + d[B^k\circ X, B^k \circ X]_s^c)\\ &- \int_0^\cdot F'(X_s)\,d^-(B^k\circ X)_s\\ &- \int_{]0,\,\cdot]\times\mathbb{R}} (F(X_{s-}+x) - F(X_{s-}) - k(x)\,F'(X_{s-}))\,(\nu\circ X)(ds\,dx),\end{aligned} \tag{4.11}$$

is an $(\mathcal{F}_t^X)$-local martingale.

Step 1: X is an $(\mathcal{F}_t^X)$-weak Dirichlet process. Let us now set $\tilde F(x) = \arctan x$. Being $\tilde F \in C_b^2$, by Theorem 3.16 and Remark 3.17 in [4], $Y = \tilde F(X)$ is an $(\mathcal{F}_t^X)$-special weak Dirichlet process. On the other hand, by Lemma 2.7, Y has finite quadratic variation. Since $X_s = \tilde F^{-1}(Y_s)$ and $\tilde F^{-1} \in C^1(\mathcal{O})$ with $\mathcal{O} = (-\frac{\pi}{2}, \frac{\pi}{2})$, we can apply Theorem 3.5 to $X_s = \tilde F^{-1}(Y_s)$, getting that X is a weak Dirichlet process.

Step 2: if (B^k, C, ν) and $(\bar B^k, \bar C, \bar\nu)$ both verify (4.11), then

$$(B^k, C, \nu) = (\bar B^k, \bar C, \bar \nu) \tag{4.12}$$

in the $\mathcal{L}(X)$-sense.

Step 2a: $B^k - \bar{B}^k$ has finite variation. We start by noticing that, for every $F \in C_b^2$, by uniqueness of decomposition of the special weak Dirichlet process $F(X)$, by (4.11) we have

$$\frac{1}{2}\int_0^\cdot F''(X_s)\,(d(C\circ X - \bar{C}\circ X)_s + d([B^k\circ X, B^k\circ X]^c - [\bar{B}^k\circ X, \bar{B}^k\circ X]^c)_s)$$
$$+ \int_0^\cdot F'(X_s)\, d^-(B^k\circ X - \bar{B}^k\circ X)_s$$
$$+ \int_{]0,\,\cdot]\times\mathbb{R}} (F(X_{s-}+x) - F(X_{s-}) - k(x)\,F'(X_{s-}))\,(\nu\circ X - \bar{\nu}\circ X)(ds\,dx) = 0.$$

By Remark 3.26-c) in [4], $[B^k\circ X, B^k\circ X] = [B^k, B^k]\circ X$ almost surely with respect to $\mathcal{L}(X)$. So, again almost surely with respect to $\mathcal{L}(X)$, $[B^k\circ X, B^k\circ X]^c = [B^k, B^k]^c\circ X$, since $\Delta(B^k\circ X) = (\Delta B^k)\circ X$, taking into account (2.3). Therefore,

$$\frac{1}{2}\int_0^\cdot F''(\check{X}_s)\,(d(C-\bar{C})_s + d([B^k,B^k]^c - [\bar{B}^k,\bar{B}^k]^c)_s) + \int_0^\cdot F'(\check{X}_s)\,d^-(B^k-\bar{B}^k)_s$$
$$+ \int_{]0,\,\cdot]\times\mathbb{R}} (F(\check{X}_{s-}+x) - F(\check{X}_{s-}) - k(x)\,F'(\check{X}_{s-}))\,(\nu-\bar{\nu})(ds\,dx) = 0. \tag{4.13}$$

It will enough to prove that $B^k - \bar{B}^k$ has finite variation on $\check{\Omega}_n := \{\omega\in\check{\Omega}:\ \sup_{t\le T}|\check{X}_t| \le n\}$, being $\check{\Omega} = \cup_n\check{\Omega}_n$ up to a null set, with respect to $\mathcal{L}(X)$. We apply (4.13) with $F_n(x) = x\,\chi_n(x)$, where $\chi_n \in C_b^\infty$, $|\chi_n|\le 1$ and

$$\chi_n(x) = \begin{cases} 1 & \text{if} \quad |x|\le n,\\ 0 & \text{if} \quad |x| > n+1,\end{cases}$$

getting

$$\frac{1}{2}\int_0^\cdot F_n''(\check{X}_s)\,(d(C-\bar{C})_s + d([B^k,B^k]^c - [\bar{B}^k,\bar{B}^k]^c)_s) + \int_0^\cdot F_n'(\check{X}_s)\,d^-(B^k-\bar{B}^k)_s$$
$$+ \int_{]0,\,\cdot]\times\mathbb{R}} (F_n(\check{X}_{s-}+x) - F_n(\check{X}_{s-}) - k(x)\,F_n'(\check{X}_{s-}))\,(\nu-\bar{\nu})(ds\,dx) = 0. \tag{4.14}$$

By Remark 2.2,

$$\mathbb{1}_{\check{\Omega}_n}\int_0^\cdot F_n'(\check{X}_s)d^-(B^k-\bar{B}^k)_s = \mathbb{1}_{\Omega_n}\int_0^\cdot d^-(B^k-\bar{B}^k)_s = \mathbb{1}_{\Omega_N}(B_t^k - \bar{B}_t^k),$$
$$\mathbb{1}_{\check{\Omega}_n}\int_0^\cdot F_n''(\check{X}_s)(d(C-\bar{C})_s + d([B^k,B^k]^c - [\bar{B}^k,\bar{B}^k]^c)_s) = 0.$$

Consequently, (4.14) yields

$$\mathbb{1}_{\check{\Omega}_n}(B^k-\bar{B}^k) = -\mathbb{1}_{\check{\Omega}_n}\int_{]0,\,\cdot]\times\mathbb{R}} (F_n(\check{X}_{s-}+x) - F_n(\check{X}_{s-}) - k(x)\,F_n'(\check{X}_{s-}))\,(\nu-\bar{\nu})(ds\,dx),$$

which implies that $\mathbb{1}_{\Omega_n}(B^k-\bar{B}^k)$ has finite variation.

Step 2b: (4.12) holds true. Let $u\in\mathbb{R}$. Notice that $(e^{iux} - 1 - iuk(x))\star\nu \in \mathcal{A}_{\rm loc}$. Indeed, $|e^{iux} - 1 - iuk(x)| \le \alpha(1\wedge|x|^2)$ for some constant α, and $\sum_{s\le\cdot}|\Delta X_s|^2 < +\infty$ a.s., hence $(1\wedge|x|^2)\star\nu\in\mathcal{A}_{\rm loc}$, see Proposition C.1 in [4].

We remark that (4.13) extends for a complex valued function F such that $Re(F)$ and $Im(F)$ belong to C_b^2. We can then apply (4.13) with $F(x) = e^{iux}$. We have $F'(x) = iuF(x)$, $F''(x) = -u^2 F(x)$, and

$$F(\check{X}_{s-} + x) - F(\check{X}_{s-}) - k(x)F'(\check{X}_{s-}) = e^{iu\check{X}_{s-}}(e^{iux} - 1 - iuk(x)).$$

Since by Step 2a the process $B^k - \bar{B}^k$ has bounded variation, by Remark 2.4-(i)

$$\int_0^{\cdot} F'(\check{X}_s)d^-(B^k - \bar{B}^k)_s = \int_0^{\cdot} F'(\check{X}_{s-})d(B^k - \bar{B}^k)_s,$$

and we get that

$$\int_{[0,t]} e^{iu\check{X}_{s-}}\Big[iu\, d(B^k - \bar{B}^k)_s - \frac{1}{2}u^2(d(C - \bar{C})_s + d([B^k, B^k]^c - [\bar{B}^k, \bar{B}^k]^c)_s)$$
$$+ \int_{\mathbb{R}} (e^{iux} - 1 - iuk(x))(\nu - \bar{\nu})(ds\, dx)\Big] = 0 \quad \text{up to an evanescent set,}$$

or, equivalently,

$$\int_{[0,t]} e^{iu\check{X}_{s-}} dH(u)_s = 0 \quad \text{up to an evanescent set (with respect to } \mathcal{L}(X)), \tag{4.15}$$

with

$$H(u)_t := iu(B^k - \bar{B}^k)_t - \frac{1}{2}u^2((C - \bar{C})_t + ([B^k, B^k]^c - [\bar{B}^k, \bar{B}^k]^c)_t)$$
$$+ \int_{\mathbb{R}} (e^{iux} - 1 - iu\, k(x))(\nu - \bar{\nu})([0,t] \times dx). \tag{4.16}$$

Notice that, since $B^k - \bar{B}^k$ has finite variation, the same property holds for $H(u)$.

In particular, there is a null set $\mathcal{N}(u)$ for which (4.15) holds for every t. Differentiating (4.15) in t for every $\omega \notin \mathcal{N}(u)$, we get that $H(u)_t = 0$ for every $\omega \notin \mathcal{N}(u)$, for every t. We define $\mathcal{N} = \cup_{u\in\mathbb{Q}} \mathcal{N}(u)$. Since the left-hand side of (4.15) is continuous in u (uniformly in t), $H(u)_t = 0$ for every $\omega \notin \mathcal{N}$, for every t and u.

Now let us fix $\omega \notin \mathcal{N}$ and $t \in [0,T]$. We set $b_t := (B^k - \bar{B}^k)_t$, $c_t := (C - \bar{C})_t$, $\Lambda_t(dx) := (\nu - \bar{\nu})([0,t] \times dx)$, and $\bar{b} = \bar{c} = \bar{\Lambda} = 0$. Then, applying Lemma 4.4 below, we conclude that $b = 0$, $c = 0$ and $\Lambda = 0$ and this concludes the proof of Step 2.

Step 3: X is an $(\mathcal{F}_t^X)$-weak Dirichlet process with local characteristics (B^k, C, ν). By Step 1, X is an $(\mathcal{F}_t^X)$-weak Dirichlet process. Let $(\bar{B}^k, \bar{C}, \bar{\nu})$ be the characteristics of X, see Definition 4.1. We conclude by applying $(i) \Rightarrow (ii)$ together with Step 2. □

The following lemma is the extension of Lemma 2.44, Chapter II, in [12] in the case of signed measures Λ and symmetric (not necessarily positive) matrices c.

Lemma 4.4. *Let $b, \bar{b} \in \mathbb{R}^d$, let $c, \bar{c}$ symmetric $d \times d$ matrices, and $\Lambda, \bar{\Lambda}$ signed measures on $\mathbb{R}^d$ that satisfy $\Lambda(\{0\}) = 0$, $\bar{\Lambda}(\{0\}) = 0$ and whose total variation measure integrate $(1 \wedge |x|^2)$. Let*

$$\psi(u) = iub - \frac{1}{2}u^T cu + \int_{\mathbb{R}^d} (e^{iux} - 1 - iuk(x))\Lambda(dx), \quad u \in \mathbb{R}^d. \tag{4.17}$$

If ψ satisfies (4.17) with $(\bar{b}, \bar{c}, \bar{\Lambda})$ also, then $b = \bar{b}$, $c = \bar{c}$ and $\Lambda = \bar{\Lambda}$.

Proof. Let $w \in \mathbb{R}^d \setminus \{0\}$ and define the function

$$\varphi_w(u) := \psi(u) - \frac{1}{2}\int_{-1}^{1} \psi(u+sw)ds.$$

One can easily prove that

$$\varphi_w(u) = \frac{1}{6}w^T cw + \int_{\mathbb{R}^d} \left(1 - \frac{sin(wx)}{wx}\right) e^{iux}\Lambda(dx).$$

Therefore, the function $\varphi_w(u)$ is the Fourier transform of the measure

$$G_w(dx) = \frac{1}{6}w^T cw\, \delta_0(dx) + \left(1 - \frac{sin(wx)}{wx}\right)\Lambda(dx), \tag{4.18}$$

where δ_0 denoted the Dirac measure concentrated in $x = 0$. It follows that each measure G_w is uniquely determined by the function φ_w, or equivalently by the function ψ.

By subtraction, we can suppose $\bar{b} = 0$, $\bar{c} = 0$ and $\bar{\Lambda} = 0$, so that $\psi = 0$ and therefore $\varphi_w = 0$ and $G_w(dx) = 0$ for every $w \in \mathbb{R}^d$. Evaluating the right-hand side of (4.18) in the singleton $\{0\}$ we get

$$w^T cw = 0, \quad w \in \mathbb{R}^d.$$

By the spectral theorem, being c a symmetric matrix it is (orthogonally) diagonalizable, so $c = p^T Dp$ with D diagonal and p orthogonal matrix. Therefore, setting $\tilde{w} = pw$,

$$\tilde{w}^T D\tilde{w} = 0, \quad \tilde{w} \in \mathbb{R}^d,$$

therefore $D = 0$ and $c = 0$ as well.

Going back to (4.18), we have that

$$\left(1 - \frac{sin(wx)}{wx}\right)\Lambda(dx) \equiv 0, \quad w \in \mathbb{R}^d.$$

Since $1 - \frac{sin(a)}{a} > 0$ for all $a \neq 0$, and recalling that $\Lambda(\{0\}) = 0$, we get $\Lambda = 0$. Finally, from (4.17), we also have $b = 0$. □

Appendix

A Jacod-Shiryaev framework

For the sake of the reader, we recall below the equivalence result for càdlàg semimartingales stated in Theorem 2.42, Chapter II, of [12].

Theorem A.1. *Let X be an adapted càdlàg process.*

Let B^k be an $\check{\mathbb{F}}$-predictable process, with finite variation on finite intervals, and $B^k_0 = 0$, C be an $(\check{\mathcal{F}}_t)$-adapted continuous process of finite variation with $C_0 = 0$, and ν be an $(\check{\mathcal{F}}_t)$-predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$.

There is equivalence between the two following statements.

(i) *X is a semimartingale with characteristics (B^k, C, ν).*

(ii) For each bounded function F of class C_b^2, the process

$$F(X_\cdot) - F(X_0) - \int_0^\cdot \partial_s F(s, X_s) ds - \frac{1}{2}\int_0^\cdot F''(X_s)\, d(C \circ X)_s - \int_0^\cdot F'(X_s)\, d(B^k \circ X)$$
$$- \int_{]0,\,\cdot] \times \mathbb{R}} (F(X_{s-} + x) - F(X_{s-}) - k(x)\, F'(X_{s-}))\, (\nu \circ X)(ds\, dx)$$

is a local martingale.

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